Smoothed Monte Carlo Estimators for the Time-in-the-Red in Risk Processes

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Abstract

We consider a modified version of the de Finetti model in insurance risk theory in which, when surpluses become negative the company has the possibility of borrowing, and thus continue its operation. For this model we examine the problem of estimating the "time-in-the red" over a finite horizon via simulation. We propose a *smoothed estimator* based on a conditioning argument which is very simple to implement as well as particularly efficient, especially when the claim distribution is heavy tailed. We establish unbiasedness for this estimator and show that its variance is lower than the naïve estimator based on counts. Finally we present a number of simulation results showing that the smoothed estimator has variance which is often significantly lower than that of the naïve Monte-Carlo estimator.

Keywords: Risk theory, de Finetti model, Smoothed Monte Carlo, Variance reduction.
Short title: Smoothed Monte Carlo for Risk Processes.

1 Introduction

We consider the classical continuous time claim process in insurance risk theory which has the following structure: Claims occur at times $\{t_n; n \in \mathbb{N}\}$ which form a Poisson process with rate λ and corresponding counting process $\{N(t); t \geq 0\}$ where $N(t) = \sum_{k=1}^{\infty} \mathbf{1}(t_k \leq t)$. The claim sizes $\{Z_k; k \in \mathbb{N}\}$ are independent, identically distributed random variables, having common distribution function F with $F(x) = 0$ when $x < 0$ and finite mean μ. Furthermore, N and $\{Z_k\}$ are assumed to be independent. Then, $\sum_{i=1}^{N(t)} Z_i$ represents the accumulated claims up to time t.

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In the Cramér–Lundberg model (e.g. see [5]) the free reserves process X is defined as

$$X_t = u + ct - \sum_{i=1}^{N(t)} Z_i$$

where u is the initial capital and $c > 0$ the premium income per unit time. Then, the profit over the interval $[0, t]$ is $S_t = ct - \sum_{i=1}^{N(t)} Z_i$. The *relative safety loading* ρ is defined by

$$\rho = \frac{c}{\lambda\mu} - 1$$

and gives the expected profit rate per unit time as a percentage of the expected claims that have occurred up to that point. We assume that $\rho > 0$, (positive safety loading). Under this assumption, as $t \to \infty$, X_t almost surely will drift to $+\infty$. The typical measure for the long-term financial stability of the risk business in the Cramér–Lundberg model is the *ruin probability*, which is usually expressed as a function of the initial capital u, and defined as $P\{X_t < 0 \text{ , for some } t > 0\}$. If one only considers the operation of the company over a finite horizon, say the interval $[0, T]$, then the finite horizon ruin probability, $P\{X(t) < 0, \text{ for some } t \leq T\}$ may be the relevant performance criterion. For further details we refer the reader to Grandell [5].

While the Cramér–Lundberg model has played a central role in the development of risk theory and of actuarial techniques for the analysis of the long-run stability of an insurance company, it has also been criticized, particularly in connection with the feature of the model that free reserves accumulate without bound as $t \to \infty$. In the 1950's de Finetti [1] proposed an alternative point of view which places in the center of the economic argument not the long run stability of the company but the present value of the dividend stream the firm's operation generates for shareholders. Arguing along these lines he analyzed a simple model and showed that the optimal dividend strategy before the inevitable ruin occurs is a *barrier strategy*. This means that there exists a level $L > 0$ such that, as long as the free reserves are below it all premium income is added to them, whereas as soon as free reserves exceed it, all additional income from premiums is distributed to the shareholders as dividends. In de Finetti's model, the ruin of the insurance company (i.e. the event that the free reserves will become at some time negative) is a certain event. In his original formulation de Finetti argued that the objective of the insurance company would be to maximize *the present value* of the total amount of dividends distributed to shareholders as opposed to keeping the ruin probability below a given value.

We consider a variation of de Finetti's model where, when free reserves become negative the company is not ruined but is instead allowed to continue its operation by borrowing. The period of time during which the free reserves are negative is usually referred to as *time in the red.*

Time in the red has been studied before, in the context of the classical Cramér–Lundberg model based on the following considerations. As argued in Gerber [4], sometimes the event of ruin has a very small probability and the portfolio is just one out of many in the company. The company may thus have enough funds available to support some negative surplus for some time (or secure support from outside sources) in the hope that the portfolio will recover in the future, allowing the company to keep this business alive. This can be regarded as an investment, since the process will recover in the future. The problem of finding if this recovery is quick enough or not, giving good value or not for the money invested, was studied by dos Reis [3] where the distribution of the number of occasions on which the surplus falls below zero is given and results for the moments of the duration of a single period of negative surplus and the total duration of negative surplus are obtained. Also, Dickson and dos Reis [2] consider the distributions of the duration of a single period of negative surplus and of the total duration of negative surplus. They derive explicit results in some cases and show how to approximate these distributions through the use of a discrete time risk model. A markovian analysis of a discrete model with recursive formulas for computing the time in the red is presented in Wagner [10]. The above analytic results have been obtained under the assumption that the claim occurrence process is Poisson. In general, analytic results are not available and one would have to resort to simulation experiments in order to estimate the time in the red.

To the best of our knowledge, the time in the red process for the de Finetti model has not so far been studied. In this paper we propose a smoothed Monte Carlo estimator which has of course the advantage of being applicable under general stochastic assumptions regarding the claim process. This estimator is related to the estimator proposed by Ross and Schechner [8] for the estimation of the mean passage time and the distribution of the passage time in stochastic simulations of discrete Markov chains.

2 Model description and Monte Carlo estimators

To describe the evolution of the system let, as before, $\{t_n; n = 1, 2, \ldots\}$ denote the epochs when claims occur, Z_n the size of the nth claim, and X_t the size of the free reserves at time t. Also set $t_0 = 0$ and suppose that the initial value of the free reserves process is u. Then the process $\{X_t; t \geq 0\}$ has piece–wise continuous paths, which we will assume to be *right–continuous* with probability one. Between claim occurrences the sample paths of $\{X_t\}$ increase with rate c until the level L is reached and then they remain constantly equal to L (since additional income from premiums is distributed to the shareholders) until the next claim occurs. When this happens, the free reserves are decreased by the amount of the claim (see figure 1). The evolution of the process $\{X_t\}$ can be described heuristically by the following equations

$$\begin{aligned} \frac{d}{dt}X_t &= c\mathbf{1}(X_t < L), \quad t \in (t_n, t_{n+1}),\ n = 0, 1, 2, \ldots \\ X_{t_n} &= X_{t_n-} - Z_n, \quad n = 1, 2, \ldots, \end{aligned}$$

together with the initial condition $X_0 = u$. In the above equations, as usual, X_{t_n-} denotes the value of free reserves just before the nth claim occurs while X_{t_n} the corresponding value just after the claim. From a mathematical standpoint, $\{X_t\}$ is defined (pathwise) as the unique solution of the integral equation

$$X_t \;=\; u + c\int_0^t \mathbf{1}(X_s < L)ds \;-\; \sum_{k=1}^{N(t)} Z_k.$$

An explicit solution to the above equation is provided by equation (13) of the Appendix.

If the operating horizon is t, the total amount of money given to the shareholders is equal to

$$c\int_0^t \mathbf{1}(X_s = L)ds,$$

while the total time in the red is equal to

$$\int_0^t \mathbf{1}(X_s < 0)ds.$$

A discretized performance criterion which is essentially equivalent to the total time in the red over the horizon $[0, t]$ would be

$$M(t) = \sum_{i=1}^{N(t)} \mathbf{1}\left(X_{t_i} < 0\right), \tag{1}$$

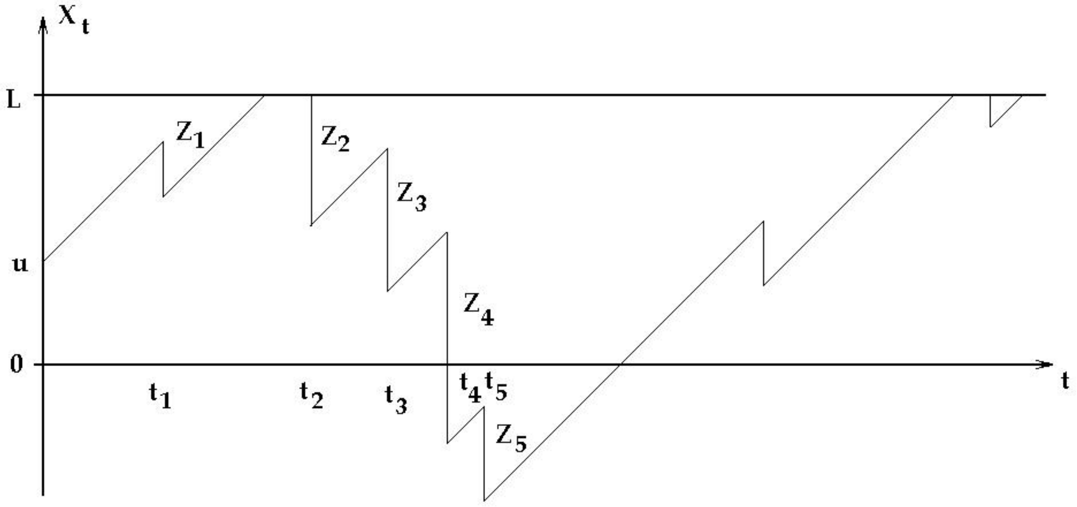

Figure 1: The free reserves process X_t

the total number of claims up to time t that result in negative free reserves. By an abuse of terminology *we will also be calling $M(t)$ henceforth for convenience "time in the red".*

Since we assume that the net profit condition holds, once the process falls below zero it will remain negative only for a finite period of time before becoming positive again. If we call such periods of negative surpluses *red periods*, then, as long as we have positive loading, red periods are random variables that are finite with probability 1.

The *naïve Monte-Carlo estimator* for the time in the red based on counts is $M(t)$, as given in (1). In this paper we propose the following *smoothed estimator*

$$K(t) = \sum_{i=1}^{N(t)} \overline{F}(X_{t_i-}). \tag{2}$$

where $\overline{F}(x) := 1 - F(x)$ for all $x \in \mathbb{R}$.

Remark: Recall that X_{t_i-} is the size of the reserves, *just prior to the occurrence of the ith claim* and $\overline{F}(X_{t_i-})$ is the conditional probability that the process will fall below level zero after the occurrence of the ith claim, given the size of the free reserves just before the claim occurs. Clearly, while the naïve estimator assigns to each claim a value of 1 or 0 according to whether it results to a negative value for the free reserves or not, the smoothed estimator assigns a value *between 0 and 1* equal to the conditional probability that the claim results to a negative value for the free reserves process, given the value of that process just prior to the claim. In particular we note that the contribution of the ith claim in the naïve estimator is based on the value of the free reserves process at time t_i, X_{t_i}, while the smoothed estimator *uses the value just prior to* t_i, X_{t_i-}. If $X_{t_i-} < 0$ then $X_{t_i} < 0$ *a fortiori* and such claims contribute 1 to both the naïve and the smoothed estimator since $\overline{F}(x) = 1$ when $x < 0$.

The statistical properties of the smoothed estimator will be examined in the next section where it will be shown that it is superior to the naïve estimator.

3 Statistical Properties of the Smoothed Estimator $K(t)$

Here we formulate and prove our main result, namely that the smoothed estimator is unbiased and has lower variance than the naïve estimator for all t. Of course, the fact that the smoothed estimator has lower variance is not surprising since, as a general principle, conditioning reduces variance. A proof is necessary nonetheless and is provided here. What is, perhaps, surprising is the extent to which variance is reduced by the simple form of conditioning proposed. This is shown clearly in the experimental results in section 4.

Denote by $(\Omega, \mathscr{F}, P)$ the probability space on which the free reserves process has been defined and by $\mathscr{F}_t = \sigma - \{X_s; s \leq t\}$ the σ–field generated by the process X up to time t. The filtration $\{\mathscr{F}_t; t \geq 0\}$ represents thus the history of the process. For background on the theory of processes we refer the reader to Métivier [6]. We recall that $\mathscr{F}_{t+} = \bigcap_{t'>t} \mathscr{F}_{t'}$ and that $\mathscr{F}_{t-} = \bigvee_{t'<t} \mathscr{F}_{t'}$, the σ–field generated by all $\mathscr{F}_{t'}$ with $t' < t$. In accordance with the "usual assumptions" the filtration $\{\mathscr{F}_t\}$ is right–continuous and hence $\mathscr{F}_{t+} = \mathscr{F}_t$.

Recall that, if T is an $\mathscr{F}_t$–stopping time then the stopped σ–field, $\mathscr{F}_T$, is defined as $\mathscr{F}_T := \{A \in \mathscr{F} : A \cap \{T \leq t\} \in \mathscr{F}_t$ for all $t \geq 0\}$. We also define $\mathscr{F}_{T-}$ as the σ–field generated by the collection of sets $\{A \in \mathscr{F} : A \cap \{T < t\} \in \mathscr{F}_t$ for all $t \geq 0\}$.

In our case, the times when claims occur, $\{t_n\}$, form an increasing sequence of stopping times with respect to the filtration $\{\mathscr{F}_t\}$. The corresponding σ–fields $\mathscr{F}_{t_i}$ represent the information available up to the epoch of occurrence of the ith claim, *including the size of the ith claim, Z_i*. Because of the simple structure of our process which evolves according to a deterministic law between claim occurrences and our assumption of right–continuity for the sample paths, it is easy to see that the corresponding σ–fields $\mathscr{F}_{t_i-}$ contain all the information up to the epoch of occurrence of the ith claim, *excluding the size of the ith claim*.

We are now ready to state our main result.

Theorem 1. *The smoothed estimator for the time in the red, given by (2) is unbiased and has lower variance than the naïve estimator (1) i.e. $EK(t) = EM(t)$ for all $t \geq 0$ and*

$$\mathsf{Var}(K(t)) \leq \mathsf{Var}(M(t)) \quad \textit{for all } t \geq 0. \tag{3}$$

Proof: We first establish the unbiasedness of the smoothed estimator by showing that its expectation is equal to that of the naïve estimator which is obviously unbiased. Indeed we have

$$\begin{aligned} EM(t) &= E\left[\sum_{i=1}^{\infty} \mathbf{1}(X_{t_i} < 0, t_i \leq t)\right] \\ &= \sum_{i=1}^{\infty} E\left[E\left[\mathbf{1}(t_i \leq t)\mathbf{1}(X_{t_i} < 0)\right] \mid \mathscr{F}_{t_i-}\right] \\ &= \sum_{i=1}^{\infty} E\left[E\left[\mathbf{1}(X_{t_i} < 0) \mid \mathscr{F}_{t_i-}\right]\mathbf{1}(t_i \leq t)\right] \\ &= \sum_{i=1}^{\infty} E\left[\mathbf{1}(t_i \leq t)\overline{F}(X_{t_i-})\right] \\ &= E\left[\sum_{i=1}^{\infty} \overline{F}(X_{t_i-})\mathbf{1}(t_i \leq t)\right] = EK(t) \end{aligned}$$

The interchange between the sum and the expectation in the second and in the last equality can be justified easily using an argument based on the monotone convergence theorem. The fourth equality holds because

$$\begin{aligned} E\left[\mathbf{1}(X_{t_i} < 0) \mid \mathscr{F}_{t_i-}\right] &= E\left[\mathbf{1}(X_{t_i-} - Z_i < 0) \mid \mathscr{F}_{t_i-}\right] = E\left[\mathbf{1}(Z_i > X_{t_i-}) \mid \mathscr{F}_{t_i-}\right] \\ &= \overline{F}(X_{t_i-}). \end{aligned}$$

This establishes the unbiasedness of the smoothed estimator.

Next we will establish (3). We should point out that when the claim distribution is deterministic the smoothed estimator becomes the same as the naïve one. Thus the inequality above cannot be strict in all cases. It is however possible to show that, if the deterministic claim case is excluded, then the inequality (3) becomes strict. We begin with the relationship

$$\begin{aligned} E[M^2(t)] &= E\left[\sum_{i,j} \mathbf{1}(X_{t_i} < 0)\mathbf{1}(X_{t_j} < 0)\mathbf{1}(t_i \le t, t_j \le t)\right] \\ &= EM(t) + 2E\left[\sum_{i<j} \mathbf{1}(X_{t_i} < 0)\mathbf{1}(X_{t_j} < 0)\mathbf{1}(t_j \le t)\right]. \end{aligned}$$

(Both indices, i and j here and in the sequel range of course from 1 to infinity.) Write also the corresponding relationship

$$\begin{aligned} E[K^2(t)] &= E\left[\sum_{i,j} \overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\mathbf{1}(t_i \le t, t_j \le t)\right] \\ &= E\left[\sum_{i} \overline{F}^2(X_{t_i-})\mathbf{1}(t_i \le t)\right] + 2E\left[\sum_{i<j} \overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\mathbf{1}(t_j \le t)\right] \\ &= EK(t) + E\left[\sum_{i} \left(\overline{F}^2(X_{t_i-}) - \overline{F}(X_{t_i-})\right)\mathbf{1}(t_i \le t)\right] \\ &\quad +2E\left[\sum_{i<j} \overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\mathbf{1}(t_j \le t)\right]. \end{aligned}$$

However, we have already established that $EM(t) = EK(t)$ and

$$E\left[\sum_{i} \left(\overline{F}^2(X_{t_i-}) - \overline{F}(X_{t_i-})\right)\mathbf{1}(t_i \le t)\right] \le 0$$

since each one of the terms inside the sum is negative or zero. Thus, in order to establish (3), it suffices to prove that

$$E\left[\sum_{i<j} \mathbf{1}(X_{t_i} < 0)\mathbf{1}(X_{t_j} < 0)\mathbf{1}(t_j \le t)\right] \ge E\left[\sum_{i<j} \overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\mathbf{1}(t_j \le t)\right]$$

or equivalently

$$E\left[\sum_{i<j} \left[\mathbf{1}(X_{t_i} < 0)\mathbf{1}(X_{t_j} < 0) - \overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\right]\mathbf{1}(t_j \le t)\right] \ge 0$$

or

$$\sum_{i<j} E\left[E\left[\left(\mathbf{1}(X_{t_i}<0)\mathbf{1}(X_{t_j}<0)-\overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\right)\mathbf{1}(t_j\le t)\mid \mathscr{F}_{t_i}\right]\right]\ge 0. \qquad (4)$$

Since $X_{t_i-},\ X_{t_i}\in\mathscr{F}_{t_i}$, the inner expectation of the typical term in the above summation can be expressed as

$$\begin{aligned} & E\left[\mathbf{1}(X_{t_i}<0)\mathbf{1}(X_{t_j}<0)\mathbf{1}(t_j\le t)-\overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right] \\ &\quad = E\left[\mathbf{1}(X_{t_i}<0)\mathbf{1}(X_{t_j}<0)\mathbf{1}(t_j\le t)-\mathbf{1}(X_{t_i}<0)\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)\right. \\ &\qquad \left.+\mathbf{1}(X_{t_i}<0)\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)-\overline{F}(X_{t_i-})\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right] \\ &\quad = \mathbf{1}(X_{t_i}<0)E\left[\left(\mathbf{1}(X_{t_j}<0)-\overline{F}(X_{t_j-})\right)\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right] \\ &\qquad +\left(\mathbf{1}(X_{t_i}<0)-\overline{F}(X_{t_i-})\right)E\left[\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right]. \end{aligned}$$

However, $i<j$ implies that $\mathscr{F}_{t_i}\subset\mathscr{F}_{t_j-}$ and hence, taking into consideration that $X_{t_j}=X_{t_j-}-Z_j$,

$$\begin{aligned} & E\left[\left(\mathbf{1}(X_{t_j}<0)-\overline{F}(X_{t_j-})\right)\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right] \\ &\quad = E\left[E\left[\left(\mathbf{1}(X_{t_j-}<Z_j)-\overline{F}(X_{t_j-})\right)\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_j-}\right]\middle|\,\mathscr{F}_{t_i}\right] \\ &\quad = E\left[E\left[\mathbf{1}(X_{t_j-}<Z_j)\middle|\,\mathscr{F}_{t_j-}\right]\mathbf{1}(t_j\le t)-\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right] \\ &\quad = 0 \end{aligned}$$

where, in the next to the last equation we have used the fact that $\mathbf{1}(t_j\le t)\in\mathscr{F}_{t_j-}$. In the last equation we have also used the fact that $E\left[\mathbf{1}(X_{t_j-}<Z_j)\middle|\,\mathscr{F}_{t_j-}\right]=\overline{F}(X_{t_j-})$. So, in order to establish (4), it suffices to show that

$$\sum_{i<j} E\left[\left(\mathbf{1}(X_{t_i}<0)-\overline{F}(X_{t_i-})\right)E\left[\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)\mid\mathscr{F}_{t_i}\right]\right]\ge 0$$

or equivalently

$$\sum_{i<j} E\left[\left(\mathbf{1}(X_{t_i}<0)-\overline{F}(X_{t_i-})\right)\overline{F}(X_{t_j-})\mathbf{1}(t_j\le t)\right]\ge 0. \qquad (5)$$

But $X_{t_i}=X_{t_i-}-Z_i$ and $X_{t_j-}=X_{t_i}+Y_{i,j}$ with

$$Y_{i,j}:=-\sum_{k=i+1}^{j-1}Z_k+c\int_{t_i}^{t_j}\mathbf{1}(X_s<L)ds,$$

where the first term in the above sum is the amount paid due to claims and the second is the total income from premiums that is added to the free reserves. With this notation the typical term in the sum (5) can be written as

$$E\left[\left(\mathbf{1}(X_{t_i-}-Z_i<0)-\overline{F}(X_{t_i-})\right)\overline{F}(X_{t_i}+Y_{i,j})\mathbf{1}(t_j\le t)\right]. \qquad (6)$$

We will show that (6) is non–negative and this will establish (5) and thus the second part of the theorem. To this end it is enough to show that the following conditional expectation is non–negative:

$$E\left[\left(\mathbf{1}(X_{t_i-}-Z_i<0)-\overline{F}(X_{t_i})\right)\overline{F}(X_{t_i}+Y_{i,j})\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i-}\right]\ge 0. \tag{7}$$

In order to prove (7) we have to check two cases:

1. $X_{t_i-}\le 0$. Then $\mathbf{1}(X_{t_i-}-Z_i<0)=1$ and $\overline{F}(X_{t_i-})=1$ with probability 1, so the left hand side of (7) vanishes.

2. $X_{t_i-}>0$. In this case, write the left hand side of (7) as an iterated expectation

$$E\left[E\left[\left(\mathbf{1}(X_{t_i-}-Z_i<0)-\overline{F}(X_{t_i-})\right)\overline{F}(X_{t_i}+Y_{i,j})\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right]\middle|\,\mathscr{F}_{t_i-}\right] \tag{8}$$

(remember that $\mathscr{F}_{t_i-}\subset\mathscr{F}_{t_i}$). Using the Strong Markov property, the inner expectation in (8) can be written as

$$\begin{aligned}&\left(\mathbf{1}(X_{t_i-}-Z_i<0)-\overline{F}(X_{t_i-})\right)E\left[\overline{F}(X_{t_i}+Y_{i,j})\mathbf{1}(t_j\le t)\middle|\,\mathscr{F}_{t_i}\right]\\&\qquad=\left(\mathbf{1}(X_{t_i-}-Z_i<0)-\overline{F}(X_{t_i-})\right)E\left[\overline{F}(X_{t_i}+Y_{i,j})\mathbf{1}(t_j\le t)\middle|\,X_{t_i}\right].\end{aligned} \tag{9}$$

Let

$$\varphi(\overline{u}):=E\left[\overline{F}(X_{t_i}+Y_{i,j})\mathbf{1}(t_j\le t)\middle|\,X_{t_i}=\overline{u}\right]. \tag{10}$$

In the Appendix (corollary 1) it is shown that φ is an increasing function of $\overline{u}$. But then, (8) can be written as

$$\begin{aligned}&E\left[\left(\mathbf{1}(X_{t_i-}-Z_i<0)-\overline{F}(X_{t_i-})\right)\varphi(X_{t_i-}-Z_i)\middle|\,\mathscr{F}_{t_i-}\right]\\&\quad\ge E\left[\mathbf{1}(X_{t_i-}-Z_i<0)-\overline{F}(X_{t_i-})\middle|\,\mathscr{F}_{t_i-}\right]E\left[\varphi(X_{t_i-}-Z_i)\middle|\,\mathscr{F}_{t_i-}\right]\\&\quad=0.\end{aligned} \tag{11}$$

The inequality above is a consequence of corollary 1 of the Appendix with

$$f(x):=\mathbf{1}(x>X_{t_i})-\overline{F}(X_{t_i})$$

and

$$g(x):=\varphi(X_{t_i-}-x)$$

being the corresponding increasing functions. Then the last equality in (11) follows immediately from the fact that $P(X_{t_i-} - Z_i < 0 \mid X_{t_i-}) = \overline{F}(X_{t_i-})$ This concludes the proof of (7) and hence the proof of the theorem. ∎

The fact that the smoothed estimator has lower variance is of course not surprising. What is interesting however, is the extent to which the simple type of smoothing we propose reduces variance. This is shown in the simulation results presented in the next section.

4 Simulation results

Simulation experiments were conducted in order to evaluate in practice the performance of the above algorithm. For different values of the initial capital u and the ceiling L, 10000 iterations were performed and 1000 claim epochs were created. Positive loading values of $\rho = 0.03$, 0.05, and 0.1 were considered. In all cases, the time and reserve axes where scaled so that $c = \lambda = 1$. Experiments were conducted for two different claim size distributions as follows

1. The exponential distribution with c.d.f

$$F(x) = 1 - e^{-x/\mu}, \;\; x > 0$$

   and mean $E(X) = \mu$, so that μ was set equal to 0.97, 0.952, and 0.909 respectively in order to have the above values for ρ. Results are shown in Table 1.

2. The Pareto distribution with c.d.f

$$F(x) = 1 - \left(1 + \frac{x}{b}\right)^{-a}, \;\; x > 0$$

   where $EX = \frac{b}{a-1}$ and $a > 1$. Using $a = 1.1$, 1.5, 2, 5, and 10 we obtain various values for b in order to have the above values for ρ. Results are tabulated in Tables 2,3, and 4.

It is worth noting that the smoothed estimator we propose outperforms the naïve Monte Carlo estimator often by an order of magnitude or more in terms of its variance, particularly in the case of the Pareto distribution which has heavy tails.

A Appendix - Stochastic Monotonicity

We begin with a lemma that establishes the stochastic monotonicity of the free reserves process with respect to the initial capital.

Lemma 1 (Stochastic Monotonicity). *If* $\{X_t(u)\}$ *is the risk process with initial capital* u*, then* $u_1 \leq u_2$ *implies* $X_t(u_1) \leq_{\mathrm{st}} X_t(u_2)$ *for all* t*.*

Proof: The lemma is an immediate consequence of the corresponding stochastic monotonicity result for queueing systems (see Stoyan [9]). With

$$S_t := ct - \sum_{k=1}^{N(t)} Z_k \tag{12}$$

we have the following representation for the free reserves process

$$X_t(u) = \min\left\{u + S_t, L + \inf_{0 \leq v \leq t}[S_t - S_v]\right\}. \tag{13}$$

From the above representation it is clear that $u_1 \leq u_2$ implies $X_t(u_1) \leq X_t(u_2)$ w.p.1. for all $t \geq 0$. ∎

Having established the stochastic monotonicity of the free reserves process as a function of the initial reserves, we can now use state the following corollary which establishes the monotonicity of φ whose definition is given in (10) and repeated here for convenience.

Corollary 1. *Suppose* $1 \leq i < j$*. The function*

$$\varphi(\overline{u}) := E[\overline{F}(X_{t_j-})\mathbf{1}(t_j \leq t) \big| X_{t_i} = \overline{u}]$$

is a decreasing function of $\overline{u}$*.*

Proof: Use the Strong Markov property to argue that

$$E[\overline{F}(X_{t_j-})\mathbf{1}(t_j \leq t) \big| X_{t_i} = \overline{u}] = E[\overline{F}(X_{t_{j-i}-})\mathbf{1}(t_{j-i} \leq t) \big| X_0 = \overline{u}]$$

and appeal to the stochastic monotonicity lemma above recalling that $\overline{F}$ is a decreasing function. ∎

We end this section with the following simple

Lemma 2. *Suppose that $f, g,$ are increasing functions, $\mathbb{R} \to \mathbb{R}$ and Z a real random variable. Then*

$$E[f(Z)g(Z)] \ \geq \ E[f(Z)]\, E[g(Z)]$$

provided the expectations exist.

For a proof see Ross [7].

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u	**L**	μ	$\mathbf{E[M(t)]}$	$\mathbf{E[K(t)]}$	$\mathbf{Var[M(t)]}$	$\mathbf{Var[K(t)]}$
5	**7**	0.97	0.034	0.0336	0.03554	0.00077
5	**8**	0.97	0.0112	0.0119	0.01167	0.0001
5	**10**	0.97	0.0015	0.0015	0.0014979	0.00000163
5	**11**	0.97	0.0007	0.00053	0.000699	0.0000002
8	**10**	0.97	0.0015	0.0015	0.0015	0.00000162
8	**11**	0.97	0.0006	0.000584	0.000599	0.00000021
10	**12**	0.97	0.0001	0.00016	0.0001	0.00000003
5	**7**	0.952	0.0232	0.0226	0.02386	0.00041
5	**8**	0.952	0.0031	0.0028	0.0033	0.00000625
5	**9**	0.952	0.0024	0.0027	0.00259	0.0000061
5	**10**	0.952	0.0009	0.0009	0.000899	0.00000075
5	**11**	0.952	0.0001	0.0003	0.0001	0.00000009
5	**12**	0.952	0.0001	0.00011	0.0001	0.00000001
8	**10**	0.952	0.0012	0.001	0.001198	0.00000077
8	**11**	0.952	0.0004	0.00034	0.00039988	0.00000009
10	**12**	0.952	0.0001	0.00011	0.0001	0.00000001
5	**7**	0.909	0.0093	0.0093	0.01	0.00007756
5	**8**	0.909	0.0032	0.00314	0.00319	0.00000854
5	**9**	0.909	0.0029	0.00276	0.002891	0.00000607
5	**10**	0.909	0.0002	0.0003	0.0001999	0.00000001
5	**11**	0.909	0.0002	0.0001	0.0001999	0.00000001
8	**10**	0.909	0.0001	0.0003	0.0001	0.00000011
8	**11**	0.909	0.0002	0.0001	0.0001998	0.00000001

Table 1: Exponential distribution.

u	L	a	b	$\mathbf{E[M(t)]}$	$\mathbf{E[K(t)]}$	$\mathbf{Var[M(t)]}$	$\mathbf{Var[K(t)]}$
10	**15**	1.1	0.097	0.3815	0.3733	0.4874	0.013724
		1.5	0.485	0.933	0.925	1.4146	0.177
		2	0.97	0.6716	0.6714	0.937	0.1187
		5	3.883	0.0381	0.0424	0.03925	0.0008676
		10	8.73	0.0039	0.0037	0.003885	0.00000798
10	**20**	1.1	0.097	0.338	0.335	0.438935	0.009738
		1.5	0.485	0.7236	0.7286	1.0321	0.09812
		2	0.97	0.4756	0.4693	0.57386	0.05045
		5	3.883	0.016	0.016	0.01673	0.0001
		10	8.73	0.0004	0.0006	0.00039988	0.0000002
10	**40**	1.1	0.097	0.25	0.25	0.31048	0.004027
		1.5	0.485	0.4135	0.4139	0.49916	0.020125
		2	0.97	0.18	0.18	0.19036	0.005117
		5	3.883	0.0011	0.0011	0.0011	0.00000044
		10	8.73	0	0.00000124	0	0
10	**50**	1.1	0.097	0.23	0.23	0.2625	0.00293
		1.5	0.485	0.334	0.333	0.37468	0.01107
		2	0.97	0.138	0.138	0.1446	0.00226
		5	3.883	0.0003	0.0004	0.0003	0.00000007
		10	8.73	0	0	0	0
20	**50**	1.1	0.097	0.229	0.229	0.256	0.003
		1.5	0.485	0.33	0.33	0.3825	0.011
		2	0.97	0.137	0.138	0.14317	0.00224
		5	3.883	0.0003	0.0004	0.0003	0.00000006
		10	8.73	0	0	0	0
20	**75**	1.1	0.097	0.188	0.188	0.207	0.00155
		1.5	0.485	0.23	0.227	0.25	0.0035
		2	0.97	0.075	0.076	0.0765	0.00048
		5	3.883	0	0	0	0
		10	8.73	0	0	0	0
50	**80**	1.1	0.097	0.18	0.18	0.2	0.00138
		1.5	0.485	0.2078	0.21	0.2244	0.003
		2	0.97	0.0702	0.069	0.072	0.00036
		5	3.883	0.0685	0.069	0.0686	0.00036
		10	8.73	0	0	0	0
50	**100**	1.1	0.097	0.16	0.16	0.187	0.00096
		1.5	0.485	0.1739	0.1688	0.1818	0.00153
		2	0.97	0.0509	0.0494	0.0513	0.00014
		5	3.883	0	0	0	0
		10	8.73	0	0	0	0

Table 2[1]: Pareto with $\rho = 0.03$

[1]Zero entries here and elsewhere for the naïve Monte Carlo estimator signify that in 10,000 iterations no claims resulted

u	**L**	**a**	**b**	$\mathbf{E[M(t)]}$	$\mathbf{E[K(t)]}$	$\mathbf{Var[M(t)]}$	$\mathbf{Var[K(t)]}$
10	**15**	1.1	0.095	0.36	0.36	0.453	0.0121
		1.5	0.476	0.83	0.84	1.2158	0.1534
		2	0.952	0.5797	0.5865	0.807	0.0979
		5	3.81	0.0329	0.0323	0.034	0.00058
		10	8.57	0.0027	0.0025	0.00269298	0.00000433
10	**20**	1.1	0.095	0.325	0.323	0.417	0.00913777
		1.5	0.476	0.68	0.68	0.9627	0.08529263
		2	0.952	0.4123	0.4098	0.50055	0.04075988
		5	3.81	0.0116	0.0119	0.0116	0.00007
		10	8.57	0.0004	0.0004	0.0003998	0.00000012
10	**40**	1.1	0.095	0.24	0.24	0.27988	0.00359
		1.5	0.476	0.38	0.38	0.461	0.017
		2	0.952	0.173	0.168	0.18575	0.0043
		5	3.81	0.0007	0.0009	0.0006995	0.0000003
		10	8.57	0	0	0	0
20	**50**	1.1	0.095	0.2198	0.22	0.251713	0.0026
		1.5	0.476	0.3008	0.31	0.34035	0.00996232
		2	0.952	0.126	0.124	0.13153	0.0019
		5	3.81	0.0003	0.0003	0.00029994	0.00000005
		10	8.57	0	0	0	0
50	**100**	1.1	0.095	0.1587	0.1572	0.16713	0.0008553
		1.5	0.476	0.1559	0.1585	0.159611	0.00137
		2	0.952	0.0457	0.045	0.04721	0.000132
		5	3.81	0.0001	0.00006	0.0001	0
		10	8.57	0	0	0	0

Table 3: Pareto with $\rho = 0.05$

in time in the red i.e. all sample paths were strictly positive. Thus both estimates for the mean $EM(t)$ and for the variance $\mathsf{Var}(M(t))$ are in these cases zero. For the smoothed Monte Carlo estimator zero entries mean that the corresponding values are equal to zero to eight significant digits.

u	**L**	**a**	**b**	$\mathbf{E[M(t)]}$	$\mathbf{E[K(t)]}$	$\mathbf{Var[M(t)]}$	$\mathbf{Var[K(t)]}$
10	**15**	1.1	0.0909	0.339	0.333	0.432	0.0103
		1.5	0.4545	0.692	0.6886	1.02624	0.1015
		2	0.909	0.416	0.416	0.5453	0.052055
		5	3.636	0.0153	0.0162	0.01506	0.00017677
		10	8.1818	0.0012	0.00118	0.0012	0.00000105
10	**20**	1.1	0.0909	0.295	0.297	0.357	0.00692
		1.5	0.4545	0.549	0.553	0.7316	0.05757
		2	0.909	0.2884	0.2922	0.34946	0.02306
		5	3.636	0.0057	0.0062	0.005668	0.0000215
		10	8.1818	0.0001	0.0001	0.0001	0.00000003
10	**40**	1.1	0.0909	0.231	0.227	0.26768	0.003
		1.5	0.4545	0.314	0.317	0.37496	0.0122
		2	0.909	0.134	0.128	0.14315	0.002733
		5	3.636	0.0003	0.0004	0.0003	0.0000001
		10	8.1818	0	0	0	0
10	**50**	1.1	0.0909	0.2048	0.2065	0.2246	0.002235
		1.5	0.4545	0.263	0.262	0.30026	0.007088
		2	0.909	0.095	0.095	0.09758	0.001275
		5	3.636	0.0001	0.0002	0.0001	0.00000002
		10	8.1818	0	0	0	0
20	**50**	1.1	0.0909	0.203	0.205	0.235	0.002084
		1.5	0.4545	0.255	0.262	0.2767	0.00707269
		2	0.909	0.0996	0.0954	0.1043	0.00125
		5	3.636	0.0002	0.00021	0.0002	0.00000002
		10	8.1818	0	0	0	0
20	**75**	1.1	0.0909	0.165	0.169	0.17986	0.0011758
		1.5	0.4545	0.188	0.182	0.19958	0.00241
		2	0.909	0.052	0.054	0.053	0.0003
		5	3.636	0	0	0	0
		10	8.1818	0	0	0	0
50	**80**	1.1	0.0909	0.167	0.164	0.17479	0.00103
		1.5	0.4545	0.163	0.169	0.17598	0.00199
		2	0.909	0.05	0.05	0.0528	0.00022
		5	3.636	0	0	0	0
		10	8.1818	0	0	0	0
50	**100**	1.1	0.0909	0.143	0.146	0.1527	0.000712
		1.5	0.4545	0.137	0.136	0.1473	0.00107
		2	0.909	0.033	0.035	0.0327	0.00009547
		5	3.636	0.0001	0.0001	0.0001	0
		10	8.1818	0	0	0	0

Table 4: Pareto with $\rho = 0.1$