Probability in the Engineering and Informational Sciences, **13**, 1999, 293–308. Printed in the U.S.A.

INEQUALITIES BETWEEN EVENT AND TIME AVERAGES

J. GEORGE SHANTHIKUMAR
Department of IEOR and
Walter A. Haas School of Business
University of California
Berkeley, California 94720

MICHAEL A. ZAZANIS
Department of IEOR
University of Massachusetts
Amherst, Massachusetts 01003

We consider a stochastic process with an embedded point process in a stationary and ergodic context. Under a "lack of anticipation" assumption for the evolution of the process vis-à-vis the point process, a new better (worse) than used expectation property for the point process, and a monotonicity assumption for the behavior of the process between points, inequalities between event and time averages are obtained. Sample path monotonicity between points is not required (as is the case with existing approaches) and can be replaced with a simple monotonicity requirement for the expected value of the process between points. Inequalities between conditional event and time averages are also examined via a novel argument involving a conditional version of the Palm inversion formula.

1. INTRODUCTION

The connection between event and time averages has received a great deal of attention in the literature. Wolff [27] established PASTA as a general property of Poisson events for nonanticipating systems (i.e., for systems whose present state does not depend on future Poisson arrivals). Among the more recent papers that have appeared on this subject we mention Melamed and Whitt [14,15], Brémaud [3,5], Stidham and El Taha [24], and Köning and Schmidt [12,13]. The converse

© 1999 Cambridge University Press 0269-9648/99 $12.50

problem to PASTA, namely the problem of whether and under what conditions equality between event and time averages implies Poisson events, was also investigated in the above papers and is known as ANTIPASTA (see Miyazawa and Wolff [17]; for a survey, see Brémaud, Kannurpatti, and Mazumdar [6]).

An important problem, related to PASTA, is to identify sufficient conditions under which event averages are larger (or smaller) than time averages. Köning and Schmidt [10,11] have provided an answer for stationary and ergodic G/G/c/m queueing systems. Additional results along the same lines, and further references, are contained in Franken, Köning, Arndt, and Schmidt [9] and Brandt, Franken, and Lisek [2]. The same problem was addressed in Niu [20], following the martingale approach in Wolff [27].

The main contributions of this paper can be summarized. A new approach to the Lack of Anticipation (LA) property is presented. Roughly speaking, we examine a process X_t with an embedded point process T_n and causality is expressed as a property of the Palm expectation of functions of the system conditional on the time of occurrence of the next event. This more general definition becomes necessary since, for non-Poisson streams, future events will not generally be independent of the history of X_t, regardless of whether the system is causal.

The analysis is carried out in the stationary and ergodic framework, which implies that both event and time averages exist and they are equal to the time-stationary and event-stationary (Palm) expectations, respectively. The issue then becomes one of comparison of these two expectations. This is in contrast to most of the PASTA literature where, typically, stationarity and ergodicity are not assumed. In this more general framework it is typically shown that, if the time average (over a finite horizon) converges to a number (or a random variable) as the time horizon goes to infinity, the event average must also converge and the limit is the same number (or random variable), provided that the events form a Poisson stream and the Lack of Anticipation property [27] (or, more generally, the Lack of Bias property [14,15]) is satisfied.

Instead of requiring monotonicity of the sample paths between events, we simply require monotonicity of certain conditional expectations as functions of time, a condition which is satisfied when the paths are stochastically monotonic between events. This significantly increases the range of applicability of our results.

No renewal assumptions for the point process are made. It need only be stationary and ergodic and possess the new better (worse) than used expectation (NB(W)UE) property (in contrast with [20]). This extension is far from straightforward in view of the difficulties involved in appropriately generalizing the LA assumption.

Inequalities between conditional event and time stationary expectations are given, paralleling results on conditional PASTA (Rosenkrantz and Simha [21]; van Doorn and Regterschot [26]). Besides their intrinsic interest, inequalities between conditional time-stationary and event-stationary expectations sometimes comprise a necessary intermediate step toward inequalities between unconditional expectations. This is done via a conditional version of the Palm inversion formula, presented in the Appendix.

2. EVENT AND TIME AVERAGES

Suppose $\{X_t; t \in \mathbf{R}\}$ is a real valued stochastic process with *left continuous* sample paths (which will often be referred to as "the system") and $\{T_n; n \in \mathbf{Z}\}$ a point process (the "events"). Let $\{\mathcal{F}_t; -\infty < t < \infty\}$ be the filtration generated by the history of $\{X_t; t \in \mathbf{R}\}$ and $\{T_n; n \in \mathbf{Z}\}$. These processes are assumed jointly stationary and ergodic under the probability measure P. Throughout the paper we denote by P^0 the Palm transformation of P with respect to the points of $\{T_n\}$, and as usual, E^0 denotes expectation with respect to P^0. (Intuitively, P^0 is the conditional probability measure, given that we have a point at the origin; see, e.g., Franken et al. [9] and Baccelli and Brémaud [1].) We shall follow the standard convention $P^0(T_0 = 0) = 1$, $P(T_0 \leq 0 < T_1) = 1$. Define the distribution function for the interevent times by $P^0(T_1 \leq x) = F^0(x)$ and assume that its expectation is finite: $E^0 T_1 = 1/\lambda < \infty$. To simplify the exposition, assume also that $F^0(0) = 0$. Finally, N is the counting measure associated with $\{T_n\}$, that is, for any Borel subset B of $\mathbf{R}$, $N(B) = \sum_{n \in \mathbf{Z}} \mathbf{1}(T_n \in B)$, and N_C the restriction of N on C for any Borel set C (thus, $N_C(B) = N(B \cap C)$).

2.1. The Lack of Anticipation Property

Our first task is to extend the LA assumption to encompass non-Poisson streams of events: When N is a Poisson counting measure, one can simply require that, for all t, the history of the process $\{X_s; s \leq t\}$ be independent of the restriction of N on (t,∞), $N_{(t,\infty)}$ (i.e., independent of future events). *When N is not a Poisson steam, however, the history of the system $\{X_s; s \leq t\}$ may depend on the future of the point process, not necessarily because the system is noncausal and anticipates future events but because of the statistical dependence between $N_{(t,\infty)}$ and $N_{(-\infty,t]}$ on which $\{X_s; s \leq t\}$ depends.* The following assumption (which is equivalent to that in [20]) circumvents this problem. We will call it Strict Lack of Anticipation to distinguish it from related assumptions that follow.

Assumption SLA (Strict Lack of Anticipation): For any $y \geq 0$, $s \geq 0$, $n \in \mathbf{N}$, any bounded measurable $h : \mathbf{R}^n \to \mathbf{R}$, and any $A \in \mathcal{F}_0$,

$$E^0[h(X_{t_1}, \ldots, X_{t_n}) | A \cap \{T_1 > s + y\}] = E^0[h(X_{t_1}, \ldots, X_{t_n}) | A \cap \{T_1 > s\}], \quad \mathbf{(1)}$$

for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq s$.

The above assumption is unnecessarily strict for our purposes and will be relaxed. In this paper we will use instead the following special case of SLA:

Assumption LA (Lack of Anticipation): For any positive, bounded, real function h,

$$E^0[h(X_t) | T_1 > t] = E^0[h(X_t) | T_1 = t], \quad \text{for } t > 0. \quad \mathbf{(2)}$$

Remark 1: It is easy to see that (2) is equivalent to the condition

$$E^0[h(X_t) | T_1 \geq t] = E^0[h(X_t) | T_1 = t], \quad \text{for } t > 0, \quad \mathbf{(3)}$$

whether t is an atom of F^0 or not.

Remark 2: To see that SLA implies LA, notice that with $A = \Omega$, $n = 1$, (1) becomes

$$E^0[h(X_t)|T_1 > t + y] = E^0[h(X_t)|T_1 > t], \quad \text{for all } y > 0,\ t > 0, \tag{4}$$

which is equivalent to

$$E^0[h(X_t)|t \le T_1 \le t + y] = E^0[h(X_t)|T_1 > t + y]\left(1 - \frac{P^0(T_1 = t)}{P^0(t \le T_1 \le t + y)}\right) + E^0[h(X_t)|T_1 = t]\,\frac{P^0(T_1 = t)}{P^0(t \le T_1 \le t + y)}. \tag{5}$$

Assuming that t is not an atom of F^0 (i.e., $P^0(T_1 = t) = 0$), (5) becomes

$$E^0[h(X_t)|t \le T_1 \le t + y] = E^0[h(X_t)|T_1 > t + y]$$

which, as $y \to 0$, reduces to (2) using a Dominated Convergence argument.

When t is an atom of F^0, as $y \to 0$, (5) reduces to a tautology. From SLA it follows that, for $t > 0$, $0 \le y \le t$,

$$E^0[h(X_{t-y})|T_1 > t - y] = E^0[h(X_{t-y})|T_1 > t]. \tag{6}$$

Assume without loss of generality h to be continuous. Since X has left-continuous paths a.s. and h is bounded,

$$\lim_{y\downarrow 0} E^0[h(X_{t-y})|T_1 > t - y] = \frac{\lim_{y\downarrow 0} E^0[h(X_{t-y})\mathbf{1}(T_1 > t - y)]}{\lim_{y\downarrow 0} E^0[\mathbf{1}(T_1 > t - y)]} = E^0[h(X_t)|T_1 \ge t]$$

from the bounded convergence theorem. As a consequence, (6) implies (3), which is equivalent to LA.

2.2. Stochastic Ordering and Monotonicity Assumptions

Consider now a real, measurable function h, such that $E[h(X_0)] \le \infty$. We will restrict our attention to systems that satisfy the LA assumption and we will introduce stochastic ordering and monotonicity assumptions under which the event average $E^0[h(X_0)]$ and the time average $E[h(X_0)]$ satisfy the inequality $E^0[h(X_0)] \le (\ge) E[h(X_0)]$.

Assumption NBUE (NWUE): The point process $\{T_n\}$ has the NBUE (NWUE) property, that is, $P^0(T_1 > x) \ge P(T_1 > x)$, $(P^0(T_1 > x) \le P(T_1 > x))$ for all $x > 0$.

Remark 3: Let $F^e(x) = \lambda \int_0^x [1 - F^0(s)]\,ds$. The NBUE (NWUE) assumption is equivalent to $F^e \le_{\text{st}} (\ge_{\text{st}})\ F$.

Finally a monotonicity assumption is needed that involves both the system (X_t) and the function h. Let $\text{supp}(F^0)$ denote the support of F^0:

Assumption MD (MI) (Monotonicity): Under P^0 (i.e., conditional on $T_0 = 0$),

$$\psi(t) = E^0[h(X_t)|T_1 > t]$$

is a decreasing (MD) (increasing (MI)) function of t on $\text{supp}(F^0)$.

Sufficient conditions that imply assumption MD (MI) are (in order of increasing generality):

i. $\{X_t\}$ has paths that are monotone decreasing (increasing) P^0-a.s. on $T_0 \le t < T_1$ and h is a monotone increasing function.

ii. $\{X_t\}$ is stochastically monotone decreasing (increasing) on $T_0 \le t < T_1$ and h is a monotone increasing function.

iii. $h(X_t)$ is stochastically monotone decreasing (increasing) on $T_0 \le t < T_1$.

We point out, however, that in some cases, the stochastic monotonicity of the process X_t between events may be hard to establish or may not be true, and nonetheless assumption MD (MI) may be valid. Such an example will be provided later.

2.3. Inequalities Between Event and Time Averages

THEOREM 1: *Let X_t be a process with an embedded point process T_n satisfying assumptions LA and NBUE (NWUE), and let h be a real, measurable function, for which $Eh(X_0) < \infty$ and assumption MD holds. Then*

$$E^0h(X_0) \le(\ge)\ Eh(X_0). \tag{7}$$

If $Eh(X_0) < \infty$ and assumption MI holds, then

$$E^0h(X_0) \ge(\le)\ Eh(X_0). \tag{8}$$

PROOF: By stationarity,

$$E^0h(X_0) = E^0h(X_{T_1}) = \int_0^\infty E^0[h(X_s)|T_1 = s]\,dF^0(s). \tag{9}$$

The above equation together with LA, our NBUE (NWUE) and MD assumptions give

$$E^0h(X_0) \le(\ge) \int_0^\infty E^0[h(X_s)|T_1 > s]\lambda[1 - F^0(s)]\,ds. \tag{10}$$

From the above it follows that

$$\begin{aligned} E^0h(X_0) &\le(\ge)\ \lambda \int_0^\infty E^0[h(X_s)|T_1 > s]P^0(T_1 > s)\,ds \\ &= \lambda \int_0^\infty E^0[h(X_s)\mathbf{1}(T_1 > s)]\,ds \\ &= \lambda E^0 \int_0^{T_1} h(X_s)\,ds \\ &= Eh(X_0), \end{aligned} \tag{11}$$

where, in the last two equations we have used Fubini's theorem and the Palm inversion formula, respectively [1]. ■

Assumption MD (MI) was stated in its most general form. In the following corollaries special cases of systems satisfying MD (MI) are given.

Let $\mathcal{C}_{\mathcal{I}}$ denote the set of increasing functions $\mathcal{I} :\to \mathbf{R}$. Let $\{\widetilde{X}(t)\}$, $t \in \text{supp}(F^0)$ be a collection of random variables with distribution $P(\widetilde{X}(t) \in B) = E^0(X_t \in B | T_1 = t)$. Suppose that $\widetilde{X}(t)$ is a family of stochastically decreasing r.v.'s (in the usual stochastic order). Then, for any real function h,

$$h \in \mathcal{C}_{\mathbf{R}} \quad \text{implies} \quad Eh(\widetilde{X}(t))$$

is a decreasing function of t. (Similar remarks obviously hold for stochastically decreasing in the convex or concave and stochastically increasing in the usual, convex, or concave stochastic orders [22,23,25].)

Remark 4: Together with the LA assumption, this implies that

$$E^0[h(X_u)|T_1 > t] \geq E^0[h(X_s)|T_1 > t] \qquad \forall 0 < u \leq s \leq t.$$

This condition is obviously satisfied when the sample paths of X_t are decreasing in $[T_n, T_{n+1})$ for all $n \in Z$ a.s. Our assumption is, however, much weaker, and this allows one to apply the above results to a considerably larger number of systems.

The main point to notice is that we do not require the sample paths of the stochastic process X_t to be decreasing as in Niu [20]. We simply require the stochastic monotonicity of X_t for $t \in [T_n, T_{n+1})$. Also, the event sequence need not be renewal.

COROLLARY 1: *Consider a system satisfying NBUE (NWUE) and LA which, furthermore, has sample paths decreasing (increasing) w.p.1 between events, that is, under P^0, $X_s \geq X_t$ P^0-a.s., for $T_0 = 0 \leq s \leq t \leq T_1$. Then (7) holds for any monotone increasing h.*

PROOF: Obviously the monotonicity of the sample paths between events implies that MD (MI) assumption holds; therefore, Theorem 1 applies. ■

Remark 5: The above result (Corollary 1) applied to the workload process of GI/G/c/m queues first appeared in [10]. (For further details we refer the reader to Franken et al. [9].) Niu [20] obtained a result similar to Corollary 1, though his lack of anticipation assumption is more restrictive than the one presented here.

COROLLARY 2: *Suppose that a system satisfies NBUE (NWUE) and LA and has sample paths that are stochastically decreasing (increasing) between events in the usual stochastic order, that is, under P^0, for any nondecreasing h, $E^0[h(X_t)|T_1 > t]$ is a decreasing (increasing) function. Then (7) holds.*

Remark 6: This significantly generalizes the results in [20], since we only require stochastic monotonicity.

3. EXAMPLES AND APPLICATIONS

In the following examples we consider systems which, between events, have stochastic monotonicity properties (either in the usual stochastic ordering or in the increasing convex ordering) while the sample paths themselves are not monotonic.

3.1. Age Process for the Superposition of Two Independent Point Processes

As before, let $\{T_n^i; n \in \mathbf{Z}\}$, $i = 0,1$, be the arrival epochs of two point processes, assumed jointly stationary and ergodic under the probability measure P. We further assume that under P the two point processes are independent. Let P^i denote the Palm probability measure with respect to the point process $\{T_n^i\}$, and E^i the corresponding expectation. It is not hard to see that the two point processes are also independent under P^i [1]. We define the distributions $F_i(x) = P^i(T_1^i \leq x)$, $i = 0,1$, and

$$F_i^e(x) \stackrel{\text{def}}{=} P(T_1^i \leq x) = \frac{\displaystyle\int_0^x \bar{F}_i(u)\, du}{\displaystyle\int_0^\infty \bar{F}_i(u)\, du},$$

with $\bar{F}_i = 1 - F_i$. Denote the superposition of the two processes by $\{R_n\}$, and define the process X_t with *left continuous* sample paths by means of

$$X_t = \sum_{n=-\infty}^{\infty} \mathbf{1}(R_n < t \leq R_{n+1})(t - R_n).$$

X_t is of course the "age process" associated with $\{R_n\}$. As a consequence of the independence of the two point processes under P^0, it is not hard to see that

$$P^0(X_t > x | T_1^0 = t) = P^0(X_t > x | T_1^0 > t) = \bar{F}_1^e(x)\mathbf{1}(x \leq t).$$

This, in turn, implies, via a standard approximation argument, that

$$\begin{aligned} E^0[h(X_t)|T_1^0 = t] &= E^0[h(X_t)|T_1^0 > t] \\ &= \int_0^t h(x)\, dF_1^e(x) + h(t)\bar{F}_1^e(t). \end{aligned} \tag{12}$$

The first equation in (12) shows that the LA assumption is satisfied. If, in addition, h is increasing, then the RHS of (12) is increasing in t; hence, MI is satisfied. Thus, from Theorem 1 it follows that if $\{T_n^0\}$ satisfies the NBUE (NWUE) condition, then

$$E^0[h(X_0)] \geq (\leq)\, E[h(X_0)].$$

3.2. The Workload Process in a GI + G/GI/1 Queue

Consider a single server queue with an arrival process consisting of the superposition of two stationary and ergodic arrival processes, $S = \{S_n; n \in \mathbf{Z}\}$ and $T =$

$\{T_n; n \in \mathbf{Z}\}$. The two arrival processes are independent. The service process is i.i.d. with (possibly) different distributions for the two arrival streams. Assume the "T" component of the arrival process to be renewal with NBUE (NWUE) interarrival distribution. We can use Theorem 1 to show that the expected workload as seen by the renewal stream of arrivals "T" is smaller (larger) than the time average. We state this formally as

THEOREM 2: *Let X_t denote the workload process* (*defined to be* left-continuous) *and E^0 denote the expectation with respect to the Palm measure corresponding to the renewal arrival point process T. If the arrival process T satisfies the NBUE assumption then*

$$E^0 X_0 \leq E X_0,$$

with the inequality reversed in the NWUE case.

PROOF: While our theorem concerns the total workload in the system, it will be useful to distinguish between the workload due to the S arrival stream and that due to the renewal stream T. In fact we will assume that S customers have *preemptive priority* over T customers. With the above assumption, let X_t^S be the workload due to S customers and X_t^T that due to T customers. Clearly X_t^S and X_t^T are not independent. Under the assumption that S customers have preemptive priority over T customers, however, the workload process X_t^S is independent of the arrival process $\{T_n\}$.

Let P^0 denote the Palm measure with respect to the point process T. It is easy to see intuitively that the statistics of X_t^S under P^0 and under P should be the same because of the fact that $\{X_t^S; t \in \mathbf{R}\}$ and $\{T_n; n \in \mathbf{Z}\}$ are independent. This is, in fact, the case provided that the distribution function $P^0(T_1 - T_0 \leq x)$ is spread out. A rigorous proof of this can be obtained using the Choquet-Dény theorem (Zazanis [28]).

We now show that $E^0[X_t | T_1 = t]$ is a monotone decreasing function of t. Indeed, due to the independence of X_t^S and $\{T_n\}$, $E^0[X_t^S | T_1 = t] = EX_0^S$. On the other hand, X_t^T is a decreasing function of t on $\{T_1 > t\}$ P^0-a.s. Bearing in mind that $X_t = X_t^S + X_t^T$, we see that $E^0[X_t | T_1 = t]$ is a decreasing function of t; hence the monotonicity assumption MD is satisfied.

Finally, to show that the LA assumption is satisfied, notice that for $0 = T_0 \leq t \leq T_1$, $X_t^T = X_0^T - \int_0^t 1(X_u^S = 0, X_u^T > 0)\, du$. X_0^T, in turn, is independent of $\{T_n; n \geq 1\}$. Thus, bearing in mind that X_t^T is left-continuous, $E^0[X_t^T | T_1 = t] = E^0[X_t^T | T_1 > t]$. Turning our attention now to the part of the workload due to S, we see that $E^0[X_t^S | T_1 = t] = E^0[X_t^S | T_1 > t] = EX_0^S$ because of the independence of X^S and T. Hence, $E^0[X_t^S + X_t^T | T_1 = t] = E^0[X_t^S + X_t^T | T_1 > t]$. ■

3.3. The Number of Customers in a $(\mathrm{GI} + \mathrm{G})^B/\mathrm{M}/1$ Queue

Consider a single server queue with two independent stationary and ergodic streams of arrivals. Each arrival corresponds to a batch of customers requiring independent, exponentially distributed services times with rate μ, the same for both streams. As in the previous example, let $\{T_n\}$ denote the arrival epochs of batches from the first stream, which we will assume to be renewal with NBUE (NWUE) interarrival times, and let $\{S_n\}$ denote those of the second stream. Under the above assumptions, the statistics of the number of customers in the system will not change if we suppose that customers from stream S have preemptive priority over those of stream T. Letting $N_t^T(N_t^S)$ be the number of customers from stream $T(S)$, $N_t = N_t^T + N_t^S$ be the total number of customers, and arguing as in Section 3.2, we can show that $E^0 N_0 \le EN_0$ (with the inequality reversed in the NWUE case).

4. INEQUALITIES BETWEEN CONDITIONAL EVENT AND TIME AVERAGES

Here we will broaden the above framework by adjoining to the original process, X_t, a "mark" or "environment" process Y_t. The combined process (X_t, Y_t) together with the point process T_n are assumed jointly stationary and ergodic. This framework does not differ mathematically from that of Section 2. Rather, it simply represents a shift of emphasis which, as we shall see, can be quite fruitful in applications.

For instance, suppose that with each one of the "events" T_n, we associate a mark K_n taking values in a mark space $(\mathbf{K}, \mathcal{K})$. While $\mathbf{K}$ could be in general any complete separable metric space and $\mathcal{K}$ its Borel sets, in the examples we will present it is typically a countable set or a subset of $\mathbf{R}^n$. Define the right-continuous process $\{Y_t; t \in \mathbf{R}\}$ by means of $Y_t = \sum_{n \in Z} K_n \mathbf{1}(T_n \le t < T_{n+1})$. Let $\{\mathcal{F}_t; -\infty < t < \infty\}$ be the filtration generated by the internal history of $\{X_t; t \in \mathbf{R}\}$ and $\{(T_n, K_n); n \in Z\}$. Suppose that these processes are jointly stationary and ergodic under the probability measure P.

A simple example would be that of a Markov Renewal Process (T_n, K_n) driving a stochastic system. Here the environment process is the process $Y_t = K_n$ on $T_n \le t < T_{n+1}$. Another example often occurring in applications is that of a doubly stochastic Poisson process. Here, the environment process would be the right-continuous modification of the predictable version of the $\mathcal{F}_t$-predictable process Y_t with *nonnegative* sample paths. We will assume that $Y_t \le a$ w.p.1 for some positive constant a, but we will not require that $P(Y_t = 0) = 0$.

Markov-Modulated Poisson Processes are, of course, the simplest such example. In this framework, the relationship between conditional event and time averages was investigated [21,26], and a conditional PASTA result was obtained under Wolff's LAA assumption.

We next present conditional versions of the Lack of Anticipation, NBUE (NWUE), and Monotonicity assumptions of the previous section.

Assumption CLA (Conditional Lack of Anticipation): For any positive, bounded, real function h,

$$E^0[h(X_s) | Y_0 = k, T_1 > s] = E^0[h(X_s) | Y_0 = k, T_1 = s], \quad \text{for all } k \in \mathbf{K}. \tag{13}$$

Assumption C-NBUE (C-NWUE) [Conditional NBUE (NWUE)]: The point process $\{T_n\}$ has the conditional NBUE (NWUE) property; that is,

$$P^0(T_1 > x | Y_0 = k) \ge P(T_1 > x | Y_0 = k), \tag{14}$$

$(P^0(T_1 > x | Y_0 = k) \le P(T_1 > x | Y_0 = k))$, for all $x > 0$ and all $k \in \mathbf{K}$.

Let $\text{supp}(F^0)$ denote the support of F^0. Our last assumption involves both the system (i.e., X_t) and the function h.

Assumption CMD (CMI) (Conditional Monotonicity): Under P^0 (i.e., conditional on $T_0 = 0$),

$$\psi(s,k) = E^0[h(X_s)|Y_0 = k, T_1 > s] \tag{15}$$

is a decreasing (CMD) (increasing (CMI)) function of s on $\text{supp}(F^0)$ for all $k \in \mathbf{K}$.

THEOREM 3: *Let X_t be a process with an embedded point process T_n satisfying assumptions CLA and C-NBUE (C-NWUE), and let h be a real, measurable function which satisfies $Eh(X_0) < \infty$ and CMD. Then*

$$E^0[h(X_0)|Y_{0-} = k] \leq (\geq)\, E[h(X_0)|Y_0 = k]. \tag{16}$$

If $Eh(X_0) < \infty$ and CMI assumption holds, then

$$E^0[h(X_0)|Y_{0-} = k] \geq (\leq)\, E[h(X_0)|Y_0 = k]. \tag{17}$$

PROOF: Let $F^0(s|k) = P^0(T_1 \leq s|Y_0 = k)$. By stationarity, and the fact that X_t is left-continuous and Y_t is right-continuous and constant on the interval $[0, T_1)$,

$$\begin{aligned} E^0[h(X_0)|Y_{0-} = k] &= E^0[h(X_{T_1})|Y_{T_1-} = k] = E^0[h(X_{T_1})|Y_0 = k] \\ &= \int_0^\infty E^0[h(X_s)|Y_0 = k, T_1 = s]\, dF^0(s|k). \end{aligned} \tag{18}$$

From (18), and CLA, C-NBUE (C-NWUE), and CMD assumptions, we obtain

$$\begin{aligned} &\int_0^\infty E^0[h(X_s)|Y_0 = k, T_1 = s]\, dF^0(s|k) \\ &\quad \leq (\geq) \int_0^\infty E^0[h(X_s)|Y_0 = k, T_1 > s]\, \frac{P^0(T_1 > s|Y_0 = k)}{E^0[T_1|Y_0 = k]}\, ds. \end{aligned} \tag{19}$$

Hence, from (18), (19), and (13), it follows that

$$\begin{aligned} E^0[h(X_0)|Y_{0-} = k] &\leq (\geq) \frac{1}{E^0[T_1|Y_0 = k]} \int_0^\infty E^0[h(X_s)\mathbf{1}(T_1 > s)|Y_0 = k]\, ds \\ &= \frac{1}{E^0[T_1|Y_0 = k]}\, E^0\left[\int_0^{T_1} h(X_s)\, ds|Y_0 = k\right] \\ &= E[h(X_0)|Y_0 = k], \end{aligned} \tag{20}$$

where the last two equalities follow from Fubini's theorem and the conditional version of the Palm inversion formula, respectively (see Lemma 1 of the Appendix). ■

We illustrate the necessity for conditioning by means of the following example:

Example (The MR/G/c/m queue): Suppose that $\{(T_n, K_n);\ n \in \mathbf{Z}\}$ is a stationary Markov Renewal Process under the probability measure P, with $K_n \in \mathbf{K}$, where $\mathbf{K}$ is a countable set. This stream constitutes the input to a queueing system with c servers and waiting buffer of size m. The service requirements of the customers constitute a stationary and ergodic stream of nonnegative random variables, independent of the arrival stream. If upon arrival all buffer spaces are full, the customer leaves never to return. It is known (e.g., see [1]) that this system always possesses a stationary regime (though not necessarily unique). Denote by P^0 the Palm transformation of the probability measure P with respect to the point process $\{T_n\}$. Let X_t be the workload at time t. Clearly, $X_s \geq X_t$ w.p.1 on $T_n < s \leq t \leq T_{n+1}$. Obviously, the CLA assumption is satisfied. The LA assumption, however, is not. Indeed, unlike the system considered in Theorem 2, here it is not necessarily true that $E^0[h(X_t)|T_1 > t] = E^0[h(X_t)|T_1 = t]$: knowing that $T_1 = t$ affects the conditional statistics of the arrival process before $T_0 = 0$, in particular the length of the interarrival time $T_0 - T_{-1}$ on which T_0, and therefore X_t, obviously depends. On the other hand, the CLA assumption is satisfied:

$$E^0[h(X_t)|Y_0 = k, T_1 > t] = E^0[h(X_t)|Y_0 = k, T_1 = t].$$

Since $\{T_n\}$ is a Markov Renewal Process, conditional on Y_0, the statistics of the arrival process before T_0 does not depend on T_1; that is,

$$P^0(T_i \in B_i;\ i = -1, \ldots, -n|Y_0 = k, T_1 > t)$$
$$= P^0(T_i \in B_i;\ i = -1, \ldots, -n|Y_0 = k, T_1 = t).$$

Also, since X_t is nonincreasing for $t \in [T_0, T_1)$, for $s > 0$,

$$E^0[h(X_t)|Y_0 = k, T_1 > t] \geq E^0[h(X_{t+s})|Y_0 = k, T_1 > t + s],$$

for any nondecreasing function h.

5. FROM CONDITIONAL TO UNCONDITIONAL INEQUALITIES

5.1. General Remarks

Equation (20) can be written as

$$E^0[h(X_{T_1})|Y_0]E^0[T_1|Y_0] \leq (\geq)\ E^0\left[\int_0^{T_1} h(X_s)\,ds|Y_0\right]. \tag{21}$$

Taking expectations with respect to Y_0, dividing both sides with $E^0[T_1]$, and using again the Palm inversion formula we have

$$\frac{E^0[E^0[h(X_{T_1})|Y_0]E^0[T_1|Y_0]]}{E^0[T_1]} \leq (\geq)\ Eh(X_0). \tag{22}$$

Thus, we have the following:

THEOREM 4: *Suppose that the assumptions of the previous theorem are satisfied and, in addition,* $\Phi^0(Y_0) \stackrel{\text{def}}{=} E^0[h(X_{T_1})|Y_0]$, $\Lambda^0(Y_0) \stackrel{\text{def}}{=} E^0[T_1|Y_0]$, *are positively correlated*; *that is*, $\text{Cov}(\Phi^0(Y_0), \Lambda^0(Y_0)) \geq 0$. *Then*,

$$E^0[h(X_0)] \leq E[h(X_0)].$$

The same result holds with the inequality reversed in the NWUE case, *when the covariance* $\text{Cov}(\Phi^0(Y_0), \Lambda^0(Y_0))$, *is negative*.

PROOF: It follows immediately from (21) and (22). ■

5.2. Countable Mark Space

Suppose that **K** is countable and let

$$\phi^0(k) \stackrel{\text{def}}{=} E^0[h(X_0)|Y_{0-} = k] \quad \text{and} \quad \phi(k) \stackrel{\text{def}}{=} E[h(X_0)|Y_0 = k].$$

We will also denote by $p^0(k)$ the Palm probability $P^0(Y_{0-} = k)$ and by $p(k)$ the time-stationary probability $P(Y_0 = k)$. In this section we will examine sufficient conditions under which

$$E^0[h(X_0)] = \sum_{k \in \mathbf{K}} \phi^0(k) p^0(k) \leq (\geq) \sum_{k \in \mathbf{K}} \phi(k) p(k) = E[h(X_0)]. \tag{23}$$

Using the Palm inversion formula we can obtain the following expression for the likelihood ratio between the Palm and time-stationary probabilities for Y:

$$L(k) \stackrel{\text{def}}{=} \frac{P(Y_0 = k)}{P^0(Y_{0-} = k)} = \frac{E^0[T_1 1(Y_{0-} = k)]}{E^0[T_1] P^0(Y_{0-} = k)}$$
$$= \frac{E^0[T_1|Y_{0-} = k]}{E^0[T_1]}. \tag{24}$$

We will assume without loss of generality that $p(k)/p^0(k)$ is increasing in i. (If this is not the case we can always relabel the elements of K.) Hence, $p^0 \leq_{LR} p$.

THEOREM 5: *Suppose that* $\phi^0(k)$ *is increasing* (*decreasing*) *in k. If the results of Section* 4 *apply*, *then*

$$E^0 h(X_0) \leq (\geq) Eh(X_0).$$

PROOF: Under the assumptions of Section 4 we have seen that

$$\phi^0(k) \leq (\geq) \phi(k). \tag{25}$$

Hence,

$$Eh(X_0) = \sum_{k\in\mathbf{K}} \phi(k)p(k) \geq(\leq) \sum_{k\in\mathbf{K}} \phi^0(k)\frac{p(k)}{p^0(k)}p^0(k)$$
$$\geq(\leq) \sum_{k\in\mathbf{K}} \phi^0(k)p^0(k) \sum_{k\in\mathbf{K}} p(k) = E^0h(X_0), \quad \textbf{(26)}$$

where the last inequality following from the fact that $\phi^0(k)$ is increasing and $p^0 \leq_{LR} p$. Similar arguments can be used if $\phi(k)$ is increasing (resp. decreasing) in k. ■

6. EXTENSIONS

A distribution F^0 is a-MRLA (a-MRLB) iff

$$\bar{F}^0(x) \leq(\geq) \frac{1}{a}\int_x^\infty \bar{F}^0(y)\,dy \quad \text{for all } x \geq 0. \quad \textbf{(27)}$$

When $a = \int_0^\infty \bar{F}^0(y)\,dy$ the above definition reduces to the NBUE (NWUE) property.

Assumption C-MRLA (C-MRLB) [Conditional MRLA (MRLB)]: The point process $\{T_n\}$ has the conditional a-MRLA (a-MRLB) property; that is,

$$P^0(T_1 > x|Y_0 = y) \geq \frac{1}{a}P(T_1 > x|Y_0 = y), \quad \textbf{(28)}$$

$(P^0(T_1 > x|Y_0 = y) \leq 1/a\,P(T_1 > x|Y_0))$ for all $x > 0$ and all $y \in \mathbf{K}$.

THEOREM 6: *Suppose that the system satisfies assumptions CLA, CMD, and C-MRLA (C-MRLB) for some constant a. Then*

$$E^0h(X_0) \leq(\geq) \frac{1}{\lambda a}Eh(X_0), \quad \textbf{(29)}$$

where $\lambda = 1/(E^0[T_1])$ *is the rate of the point process* $\{T_n\}$.

PROOF: From (18) and (28) we have

$$\begin{aligned} E^0[h(X_0)|Y_0] &= \int_0^\infty E^0[h(X_s)|Y_0,T_1 = s]\,dF^0(s|Y_0) \\ &= \int_0^\infty E^0[h(X_s)|Y_0,T_1 > s]\,dF^0(s|Y_0) \\ &\leq(\geq) \frac{1}{a}\int_0^\infty E^0[h(X_s)|Y_0,T_1 > s]P^0(T_1 > s|Y_0)\,ds \\ &= \frac{1}{a}\int_0^\infty E^0[h(X_s)\mathbf{1}(T_1 > s)|Y_0]\,ds \\ &= \frac{1}{a}E^0\left[\int_0^{T_1} h(X_s)\,ds|Y_0\right], \end{aligned}$$

where the second equality above follows from the CLA assumption. Taking expectation w.r.t. Y_0 above we obtain

$$E^0[h(X_0)] \leq (\geq) \frac{1}{a} E^0\left[\int_0^{T_1} h(X_s)\, ds\right]. \tag{30}$$

Invoking once more the Palm inversion formula we can write the expectation in the RHS of (30) as $(1/\lambda)Eh(X_0)$ which establishes (29). ■

Example (The MR/G/c/∞ Queue): Suppose that the arrival process is Markov Renewal with conditional interarrival distribution satisfying the MRLA(MRLB) $-(1/\lambda)$ property [9, p. 148]. Then Theorem 4 can be applied to the workload process.

References

1. Baccelli, F. & Brémaud, P. (1986). *Palm probabilities.* Lecture Notes in Statistics, Springer.
2. Brandt, A., Franken, P., & Lisek, B. (1990). *Stationary stochastic models.* Chichester: John Wiley.
3. Brémaud, P. (1989). Characteristics of queueing systems observed at events and the connection between stochastic intensity and Palm probability. *QUESTA* 5: 99–112.
4. Brémaud, P. (1991). An elementary proof of Sengupta's invariance relation and a remark on Miyazawa's conservation principle. *Journal of Applied Probability* 28: 950–954.
5. Brémaud, P. (1993). A Swiss army formula for Palm calculus. *Journal of Applied Probability* 30: 40–51.
6. Brémaud, P., Kannurpatti, R., & Mazumdar, R. (1991). Event and time averages: A review. *Advances in Applied Probability* 24: 377–411.
7. Daley, D.J. & Vere-Jones, D. (1988). *An introduction to the theory of point processes.* Springer-Verlag.
8. El-Taha, M. & Stidham, S. (1992). A filtered ASTA property. *QUESTA* 11: 211–222.
9. Franken, P., Köning, D., Arndt, U., & Schmidt, V. (1981). *Queues and point processes.* Chichester: Wiley.
10. Köning, D. & Schmidt, V. (1978). Relationships between time- and customer-stationary characteristics of service systems. In P. Bártfai & J. Tomkó (eds.), *Point processes and queueing problems.* Colloquia Mathematica Societatis János Bolyai, 24. New York: North-Holland.
11. Köning, D. & Schmidt, V. (1980). Stochastic inequalities between customer-stationary and time-stationary characteristic of queueing systems with point processes. *Journal of Applied Probability* 17: 768–777.
12. Köning, D. & Schmidt, V. (1989). EPSTA: The coincidence of time stationary and customer stationary distributions. *QUESTA* 5: 247–264.
13. Köning, D. & Schmidt, V. (1990). Extended and conditional versions of the PASTA property. *Advances in Applied Probability* 22: 510–512.
14. Melamed, B. & Whitt, W. (1990). On arrivals that see time averages. *Operations Research* 38: 156–172.
15. Melamed, B. & Whitt, W. (1990). On arrivals that see time averages: A Martingale approach. *Journal of Applied Probability* 27: 376–384.
16. Miyazawa, M. (1976). Stochastic order relations among GI/G/1 queues with a common traffic intensity. *Journal of the Operational Research Society of Japan* 19: 193–208.
17. Miyazawa, M. & Wolff, R.W. (1990). Further results on ASTA for general stationary processes and related problems. *Journal of Applied Probability* 27: 376–384.
18. Neveu, J. (1976). Sur les mesures de Palm de deux processus ponctuels stationnaires. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwarde Gebiete* 34: 199–203.
19. Neveu, J. (1976). Processus ponctuels. In *Ecole d'été de probabilités de St. Flour VI, Lect. Notes Math.* 598: 249–447, Springer-Verlag.

20. Niu, S. (1984). Inequalities between time and event averages. *Operations Research* 32: 785–795.
21. Rosenkrantz, W.A. & Simha, R. (1992). Some theorems on conditional PASTA: A stochastic integral approach. *Operations Research Letters* 11: 173–177.
22. Ross, S.M. (1996). *Stochastic processes*, 2nd ed. New York: Wiley.
23. Shaked, M. & Shanthikumar, J.G. (1994). *Stochastic orders and their applications.* New York: Academic Press.
24. Stidham, S. & El-Taha, M. (1989). Sample-path analysis of processes with embedded point processes. *QUESTA* 5: 5–36.
25. Stoyan, D. (1983). *Comparison methods for queues and other stochastic models.* New York: Wiley.
26. van Doorn, E.A. & Regterschot, G.J.K. (1988). Conditional PASTA. *Operations Research Letters* 7: 229–232.
27. Wolff, R.W. (1982). Poisson arrivals see time averages. *Operations Research* 30: 223–231.
28. Zazanis, M.A. (1993). The Choquet-Dény theorem and a relationship between time and event averages. Technical Report, IEOR Dept., Univ. of Massachusetts, Amherst, MA.

APPENDIX

Here we provide the proof of the conditional version of the Palm inversion formula which was used in Section 4. The process $\{Y_t\}$ is assumed to take values in a measurable space $(\mathbf{K},\mathcal{K})$ and is right-continuous and constant on each interval $[T_n,T_{n+1})$. ($\mathbf{K}$ is not assumed to be countable here.)

Lemma 1 (Conditional Palm inversion formula): *Suppose we are given a real, measurable function, h, such that $E[h(X_0)] \leq \infty$. Then the conditional expectation $E[h(X_0)|Y_0]$ is given by the expression*

$$E[h(X_0)|Y_0] = \frac{1}{E^0[T_1|Y_0]} E^0\left[\int_0^{T_1} h(X_s)\,ds|Y_0\right]. \tag{31}$$

Proof: It is enough to show that for any $B \in \mathcal{K}$

$$E[\mathbf{1}(Y_0 \in B)h(X_0)] = E[\mathbf{1}(Y_0 \in B)H(Y_0)], \tag{32}$$

where $H(Y_0)$ is the expression on the RHS of (31). Apply the Palm inversion formula on the RHS of (32), keeping in mind that $T_0 = 0$ P^0-a.s. to obtain

$$\begin{aligned} E[\mathbf{1}(Y_0 \in B)H(Y_0)] &= \frac{1}{E^0[T_1]} E^0\left[\int_0^{T_1} \mathbf{1}(Y_s \in B)H(Y_s)\,ds\right] \\ &= \frac{1}{E^0[T_1]} E^0[T_1\mathbf{1}(Y_0 \in B)H(Y_0)] \\ &= \frac{1}{E^0[T_1]} E^0[\mathbf{1}(Y_0 \in B)H(Y_0)E^0[T_1|Y_0]]. \end{aligned} \tag{33}$$

In the second equation above we have taken into account the fact that, for $0 = T_0 \leq s < T_1$, Y_s remains constant and equal to Y_0, and, similarly, $H(Y_s)$ remains equal to $H(Y_0)$. From the definition of $H(Y_0)$, $H(Y_0)E^0[T_0|Y_0]$, the last term of (33) can be written as

$$\frac{1}{E^0[T_1]} E^0\left[\mathbf{1}(Y_0 \in B) E^0\left[\int_0^{T_1} h(X_s)\, ds | Y_0\right]\right] = \frac{1}{E^0[T_1]} E^0\left[\mathbf{1}(Y_0 \in B) \int_0^{T_1} h(X_s)\, ds\right]$$
$$= \frac{1}{E^0[T_1]} E^0\left[\int_0^{T_1} h(X_s) \mathbf{1}(Y_s \in B)\, ds\right]$$
$$= E[\mathbf{1}(Y_0 \in B) h(X_0)]. \quad \textbf{(34)}$$

In the second equation we have again used the fact that Y_s is a piecewise constant process, changing only at the point of $\{T_n\}$, whereas the last equation is, once more, the Palm inversion formula.

This last series of equalities establishes (32) and completes the proof of the lemma. (If **K** is a countable set, a simpler proof of the theorem is possible. This simplifying assumption would, however, be too restrictive for our purposes.) ∎