Short communication

Sample path analysis of level crossings for the workload process*

Michael A. Zazanis
Department of Industrial Engineering and Management Sciences, Northwestern University, Evanston, IL 60208, USA

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We examine level crossings of sample paths of queueing processes and investigate the conditions under which the limiting empirical distribution for the workload process exists and is absolutely continuous. The connection between the density of the workload distribution and the rate of downcrossings is established as a sample path result that does not depend on any stochastic assumptions. As a corollary, we obtain the sample path version of the Takács formula connecting the time and customer stationary distributions in a queue. Defective limiting empirical distributions are considered and an expression for the mass at infinity is derived.

Keywords: Level crossings, sample path analysis, empirical distributions, Takács formula.

1. Introduction

The investigation of relationships between time-stationary characteristics of the workload process and rates of downcrossings has a long history. We refer the reader to the monograph of Franken et al. [6, pp. 57 and 142]. Among the early results on level crossing methods for queues, we mention Brill and Posner [1,2], who examined queues with Poisson arrivals, Köning et al. [6], Rolski [10], and Schmidt [11], who investigated level crossings in a stationary and ergodic context, and Cohen [4] and Shanthikumar [13], who examined regenerative queues using level crossing methods. Miyazawa [9] developed the general form of the Rate Conservation Law and used it to derive the connection between level crossings and the density of the time stationary workload. In the same vein is the paper by Ferrandiz and Lazar [5]. Besides their intrinsic interest, level crossings have been used in the analysis and control of priority and vacation queues and queues in a random environment. We refer the reader to Shanthikumar [13,14], Miyazawa [9], and the references therein.

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In this paper, we take a sample path approach in the spirit of Stidham [15], and Stidham and El Taha [16], which does not require stochastic assumptions and which leads to a simpler and more direct proof. Our results are related to the deterministic version of Miyazawa's Rate Conservation Law developed independently by Sigman [12], and to the sample path approach in Wolff [18] for the distribution of excess life and age in point processes.

In the absence of any stochastic assumptions, we define "waiting" and "sojourn" time distributions as limiting empirical distributions and show that, under appropriate conditions, their existence implies the existence and absolute continuity of the limiting empirical distribution for the proportion of time the workload spends below a certain level. The connection between the empirical density of the workload process and the rate of downcrossings at a given level is established. Since downcrossing and upcrossing rates are equal, this leads to the sample path version of the Takács formula connecting time and customer stationary workload distributions in queues.

In the stationary and ergodic context, the steady-state distribution for the workload process is either honest or defective with mass 1 at infinity w.p.1. Empirical distributions derived from arbitrary sample paths may exhibit more complicated limiting behavior. In particular, the limiting empirical distribution for the workload process may be defective, with only part of its mass at infinity. These issues are investigated in section 4 where, among other results, an expression for the mass at infinity is given.

2. Piecewise continuous sample paths and associated empirical distributions

Let $\{T_n\}_{n\in\mathbb{N}_0}$ be a strictly increasing sequence of points such that $T_0 \geq 0$, $\lim_n T_n = \infty$, and $\{V_t\}_{t\geq 0}$ be a nonnegative, right continuous real function with left limits defined on $[0, \infty)$. Let $W_n \stackrel{\text{def}}{=} \lim_{t\uparrow T_n} V_t$ and $B_n \stackrel{\text{def}}{=} V_{T_n} - W_n \geq 0$. We assume V_t to be *strictly decreasing* in the interval $[T_n, T_{n+1})$, $n = 0, 1, \dots$, except when it is equal to zero.

ASSUMPTION 1

Throughout the paper, we assume that the "arrival rate" λ, defined by

$$\lim_{n\to\infty} \frac{1}{n} T_n = \lambda^{-1},$$

exists and $0 < \lambda < \infty$.

DEFINITIONS

u^x is a downcrossing epoch at level $x > 0$ if there exists $\varepsilon > 0$ such that $V_{u^x-\delta} > x > V_{u^x+\delta}$ for all $0 < \delta < \varepsilon$. Let D_t^x be the downcrossing counting process at level x. Let $\{u_i^x\}$ be the sequence of downcrossing epochs at level x and $\{r_i^x\}$ the corresponding sequence of upcrossings defined by $r_i^x = \sup\{s : s > u_{i-1}^x, V_s \le x\}$. Let $\{m_i^x\}$ be the sequence of occupation times above x defined by $m_i^x = u_i^x - r_i^x$. Define the mean occupation time in $[x, \infty)$, m^x, by the following limit when its exists:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} m_i^x = m^x.$$

For every $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, define the *empirical distributions* $F_{W,n}$, $F_{W+B,n}$, and $F_{V,t}$.

$$F_{W,n}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(W_i \le x)}, \tag{1}$$

$$F_{W+B,n}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(W_i + B_i \le x)}, \tag{2}$$

$$F_{V,t}(x) = \frac{1}{t} \int_0^t \mathbf{1}_{(V_s \le x)} \mathrm{d}s. \tag{3}$$

Define also the *limiting empirical distributions*, whenever they exist, by means of the limits:

$$F_W(x) = \lim_{n\to\infty} F_{W,n}(x), \quad F_{W+B}(x) = \lim_{n\to\infty} F_{W+B,n}(x) \quad \text{and} \quad F_V(x) = \lim_{n\to\infty} F_{V,t}(x).$$

Finally, define $F_{W,n}(x-)$ as in (1) with a strict inequality and

$$F_W(x-) = \lim_{n\to\infty} F_{W,n}(x-).$$

3. Rates of downcrossings and empirical densities

In the above context, we show that the existence of the "arrival rate" λ and the limiting empirical distributions F_W and F_{W+B} at a single point x are enough to guarantee the existence of the rate of downcrossings λ^x.

THEOREM 1

Assume that, for some $x > 0$, the limiting empirical distributions $F_W(x)$ and $F_{W+B}(x)$ exist. Then

(i) The rate of x-downcrossings $\lambda^x \stackrel{\text{def}}{=} \lim_{t\to\infty} (1/t) D_t^x$, exists and is given by

$$\lambda^x = \lambda[F_W(x-) - F_{W+B}(x)]. \tag{4}$$

(ii) If the limiting empirical distribution at x, $F_{W+B}(x)$, exists, then the average occupation time m^z exists and the following relation holds:

$$\lambda^x m^x = 1 - F_V(x). \tag{5}$$

Proof

(i) We first show that the limit exists and is equal to the expression in (4) along the sequence of "arrival points" $\{T_n\}$. If an x-downcrossing occurs in (T_n, T_{n+1}), then $\mathbf{1}_{(W_i+B_i>x)} - \mathbf{1}_{(W_{i+1}\geq x)} = 1$, else the difference of the indicator functions in 0. Therefore,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{T_n} D^x_{T_n} &= \lim_{n\to\infty} \frac{1}{T_n} \sum_{i=1}^{n} \mathbf{1}_{(W_i+B_i>x)} - \mathbf{1}_{(W_{i+1}\geq x)} \\ &= \lim_{n\to\infty} \frac{n}{T_n} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(W_i+B_i>x)} - \mathbf{1}_{(W_{i+1}\geq x)}.\end{aligned}$$

An appeal to "$X = \lambda Y$" (Stidham and El Taha [16]) completes the proof.

(ii) The second part follows from a direct application of the sample path version of Little's law (Stidham [15]). □

In general, unless one makes additional assumptions, there is no guarantee that the empirical distribution $F_V(x)$ will exist. However, the situation changes if one assumes that $F_W(x)$ and $F_{W+B}(x)$ exist for all x in some dense subset of the reals, J. Next, we establish the absolute continuity of F_V and the connection between the corresponding density and the downcrossing rate λ^x for systems satisfying

ASSUMPTION 2

$\{V_t\}$ is continuous and right-differentiable in (T_n, T_{n+1}) for all $n \in \mathbb{N}$ and there exists a "processing rate" function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(y) \geq c > 0$, for $y > 0$, and $g(0) = 0$ such that the right derivative D^+V_t exists and

$$D^+V_t = -g(V_t).$$

This implies, of course, that V_t is *strictly decreasing* in $[T_n, T_{n+1})$, except when $V_t = 0$. Let us also recall the following lemma from real analysis:

LEMMA 1 (see Chung [3, pp. 133–134])

Assume that F_W and F_{W+B} exist for a dense countable subset of the reals, say J, which contains all discontinuity points of F_W and F_{W+B}. Then they exist for all $x \in \mathbb{R}$ and $F_{W,n}$ (respectively, $F_{W+B,n}$) converges to F_W (respectively, F_{W+B}) uniformly.

THEOREM 2

Assume that the limiting empirical distributions defined above exist for all $x \in J$. Then, for any interval $(a, b]$, $a > 0$, the limiting empirical distribution $F_V(a, b] \stackrel{\text{def}}{=} F_V(b) - F_V(a)$ exists and is given by

$$F_V(a, b] = \lambda \int_a^b [F_W(y) - F_{W+B}(y)] g(y)^{-1} \mathrm{d}y. \tag{6}$$

Proof

We compute again the limit along the sequence of "arrival points" $\{T_n\}$. Using again "$Y = \lambda X$", we obtain

$$F_V(a,b] = \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \mathbf{1}_{(a<V_s\le b)} \mathrm{d}s. \tag{7}$$

Since between two successive arrivals, the process decreases with rate $g(V_s)$, we have

$$\int_{T_i}^{T_{i+1}} \mathbf{1}_{(a<V_s\le b)} \mathrm{d}s = \int_a^b \mathbf{1}_{(W_i+B_i>y)} g(y)^{-1} \mathrm{d}y - \int_a^b \mathbf{1}_{(W_{i+1}\ge y)} g(y)^{-1} \mathrm{d}y. \tag{8}$$

Expressing the integral in (7) as a sum of integrals with the aid of (8) and letting $n \to \infty$, we obtain the expression in (6) for $F_V(a, b]$:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n}{T_n} \lim_{n\to\infty} \int_a^b \left[\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(W_i+B_i>y)} - \frac{1}{n} \sum_{i=2}^{n+1} \mathbf{1}_{(W_i\ge y)} \right] g(y)^{-1} \mathrm{d}y \\
&\quad = \lambda \int_a^b \lim_{n\to\infty} [F_{W,n}(y-) - F_{W+B,n}(y)] g(y)^{-1} \mathrm{d}y \\
&\quad = \lambda \int_a^b [F_W(y-) - F_{W+B}(y)] g(y)^{-1} \mathrm{d}y.
\end{aligned} \tag{9}$$

Passing the limit inside the integral in the above derivation is justified by the fact that $F_{W,n}$ and $F_{W+B,n}$ converge uniformly to F_W and F_{W+B} according to lemma 1. Since $F_W(y-) = F_W(y)$, except possibly at a countable set of points, we obtain (6). □

Using queueing theoretic language, (6) gives the connection between the time stationary and customer stationary distributions of the workload in a queueing system. It is in fact the *sample path version* of the Takács formula [17] which has been obtained under a variety of stochastic assumptions by a number of authors. We mention in particular Lemoine [8], Köning et al. [7], and Schmidt [11]. (For a complete account, see [6].)

COROLLARY 1

Under the assumption of theorem 2 and the additional assumption that $F_V(0) \stackrel{\text{def}}{=} \lim_{t\to\infty} F_{V,t}(0)$ exists, $F_V(x)$ is absolutely continuous, except for possible atoms at 0 and ∞, with density $(\lambda/g(x))[F_W(x) - F_{W+B}(x)]$.

COROLLARY 2

From theorem 1 and corollary 1 follows that for any level $x > 0$, F_V has density $f_V(x) = \lambda^x/g(x)$. In particular, when the processing rate $g(x) = \mathbf{1}_{(x>0)}$, the density at level x is equal to the rate of downcrossings at that level.

COROLLARY 3

Under the assumptions of theorem 2, from theorem 1 and corollary 2 follows that, for any level $x > 0$, the mean occupation time above x exists and is given by

$$(m^x)^{-1} = g(x)\,\frac{f_V(x)}{1 - F_V(x)}.$$

4. Existence of $F_V(0)$ and $F_V(\infty)$

From theorem 2, with $a = 0$ and $b \to \infty$, we obtain

$$F_V(\infty) - F_V(0) = \lambda \int_0^\infty [F_W(y) - F_{W+B}(y)]g(y)^{-1}\,\mathrm{d}y, \tag{10}$$

provided that the limiting empirical distributions F_W and F_{W+B} exist on a dense subset of $(0,\infty)$. If F_V is honest (i.e. if $F_V(\infty) = 1$) or, more generally, if $F_V(\infty)$ exists, the above argument would guarantee the existence of $F_V(0)$. If the processing rate function g is such that $\int_0^\infty g(y)^{-1}\mathrm{d}y < \infty$, then using (9) with $a = 0$ and $b = \infty$ and appealing to the dominated convergence theorem, we obtain

$$1 - F_V(0) = \lambda \int_0^\infty [F_W(y) - F_{W+B}(y)]g(y)^{-1}\,\mathrm{d}y. \tag{11}$$

From (10) and (11), we conclude that $F_V(\infty) = 1$ regardless of whether F_{W+B} and F_W are defective or not.

In the more interesting case where $\int g^{-1} = \infty$, additional conditions on $\{F_{W+B,n}\}$ are needed to guarantee the existence of $F_V(0)$. Theorem 3 provides a sufficient condition for $F_V(0)$ to exist and for F_V to be honest. Theorem 4 provides sufficient conditions for the existence of $F_V(0)$ and an expression for the "mass at infinity", $F_V\{\infty\}$.

THEOREM 3

Suppose that the family of empirical distributions $\{F_{W+B,n}\}_{n \in \mathbb{N}}$ is uniformly integrable. Then $F_V(0)$ exists and F_V is honest.

Proof

Our uniform integrability assumption is equivalent to the statement

$$\int_x^\infty [1 - F_{W+B,n}(y)]\,\mathrm{d}y \to 0 \quad \text{as } x \to \infty, \text{ uniformly in } n. \tag{12}$$

Since $F_{W+B,n}(y) \le F_{W,n}(y)$, it follows that, for the family $\{F_{W,n}\}$ a statement similar to (12) holds (and therefore that $\{F_{W,n}\}$ is also uniformly integrable). From the above and the inequality $g(y)^{-1} \le c^{-1}$ (see assumption 2), it follows that

$$\int_x^\infty [F_{W,n}(y) - F_{W+B,n}(y)]\,g(y)^{-1}\,\mathrm{d}y \to 0 \quad \text{as } x \to \infty, \text{ uniformly in } n. \tag{13}$$

An argument similar to (9) in the proof of theorem 2 shows that

$$\begin{aligned} 1 - F_v(\infty) &= \lim_{x\to\infty} 1 - F_V(x) \\ &= \lambda \lim_{x\to\infty} \lim_{n\to\infty} \int_x^\infty [F_{W,n}(y) - F_{W+B,n}(y)]\,g(y)^{-1}\,\mathrm{d}y. \end{aligned} \tag{14}$$

In view of (13), we can change the order of the limits in (14) to conclude that $1 - F_V(\infty) = 0$. This immediately implies the existence of $F_V(0)$ (cf. remark 3). □

As we will see, it is possible for F_V to be defective even though F_{W+B} (and a fortiori F_W) is not. Here, we will make the simplifying assumption that the processing rate g is constant which, without loss of generality, we will assume equal to 1. Define the *limiting empirical mean jump* by

$$\overline{B} = \lim_{n\to\infty} \sum_{i=1}^{n} B_i,$$

whenever this limit exists.

THEOREM 4

Assume that $g(x) = \mathbf{1}_{(x>0)}$, that the limiting empirical mean jump $\overline{B}$ exists, and that $\lim_{n\to\infty} n^{-1}W_n = 0$. Then $F_V(0)$ exists and is equal to $1 - \lambda\,\overline{B}$. Furthermore, F_V is defective iff $\overline{B} > \int_0^\infty [F_W(x) - F_{W+B}(x)]\mathrm{d}x$ and the mass at infinity is

$$F_V\{\infty\} \stackrel{\text{def}}{=} 1 - F_V(\infty) = \lambda\,\overline{B} - \lambda \int_0^\infty [F_W(x) - F_{W+B}(x)]\,\mathrm{d}x. \tag{15}$$

Proof

We examine the existence of the limit:

$$\overline{F}_V(0) \stackrel{\text{def}}{=} 1 - F_V(0) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{(V_s>0)}\,\mathrm{d}s. \tag{16}$$

Since λ exists, it is easy to see that we can let $t \to \infty$ along the sequence $\{T_n\}$. The existence of the limit in (16) is then tantamount to the existence of

$$\lim_{n\to\infty} \frac{n}{T_n} \frac{1}{n} \int_0^{T_n} \mathbf{1}_{(V_s>0)}\,\mathrm{d}s = \lambda \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (W_i + B_i - W_{i+1}). \tag{17}$$

We thus have

$$1 - F_V(0) = \lambda \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} B_i - \frac{1}{n} W_{n+1} \tag{18}$$

$$= \lambda \lim_{n\to\infty} \int_0^\infty [F_{W,n}(y) - F_{W+B,n}(y)]\,\mathrm{d}y. \tag{19}$$

Equation (18) is obtained by telescoping the sum in (17) and using the fact that $\lim n^{-1}W_1 = 0$. Equation (19) is obtained from (17) using an argument similar to (8). From (18) and our assumptions follows that $F_V(0)$ exists and

$$F_V(0) = 1 - \lambda\,\overline{B}. \tag{20}$$

(Since $W_{n+1} \geq W_1 + \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} A_i$, dividing by n and letting $n \to \infty$ gives $\lambda\,\overline{B} \leq 1$.) Using Fatou's lemma in (19), we can pass the limit inside the integral to obtain

$$1-F_V(0)=F_V\{\infty\}+F_V(\infty)-F_V(0)\geq\lambda\int_0^\infty[F_W(x)-F_{W+B}(x)]\,\mathrm{d}x, \tag{21}$$

Comparing (10), (20), and (21) concludes the proof. □

COROLLARY 4

For a system satisfying the assumptions of theorem 4, if F_{W+B} is defective, F_W must also be defective and $F_W\{\infty\}=F_{W+B}\{\infty\}$.

Proof

By contradiction. Suppose $F_{W+B}\{\infty\}>F_W\{\infty\}$. Then

$$\infty=\lambda\int_0^\infty[F_W(x)-F_{W+B}(x)]\,\mathrm{d}x\leq 1-F_V(0).$$ □

Example

We illustrate the case where F_V is defective even though F_{W+B} (and F_W) is not. Define $\mathrm{S}=\{n^2;\,n\in\mathbb{N}\}$, and consider a single-server queueing system with processing rate g identically equal to 1; interarrival sequence defined by $A_n=a$ if $n\notin\mathrm{S}$, and $A_n=a\sqrt{n}$ if $n\in\mathrm{S}$; service sequence given by $B_n=b$ if $n\notin\mathrm{S}$, and $B_n=b\sqrt{n}$ if $n\in\mathrm{S}$, where $0<b<a$. It is easy to check that λ exists and equals $2/3a$, that F_W exists and has an atom of size 1 at 0, and that F_{W+B} exists and has an atom of size 1 at b. Also, the empirical mean service time exists and $\overline{B}=3b/2$. One can also check that F_V exists and is defective, i.e. has $b/3a$ of its mass at infinity, in agreement with (15). Finally, notice that $F_V(0)=1-b/a=1-\lambda\overline{B}$ $=1-(2/3a)(3b/2)$.

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