Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Operations Research Letters 32 (2004) 422–430
Operations Research Letters
www.elsevier.com/locate/dsw

Cycle times in single server cyclic Jackson networks

Michael A. Zazanis

Department of Statistics, Athens University of Economics and Business, Patision 76, Athens 104 34, Greece

Received 18 April 2003; received in revised form 4 September 2003; accepted 5 December 2003

Abstract

Using the arrival theorem together with elementary facts regarding integrated tail distributions and length-biased sampling we obtain closed form expressions for the inter-output time and the cycle time distribution in cyclic, single-class Jackson networks. Corresponding expressions for the normalization constant and the throughput are also obtained.
© 2003 Elsevier B.V. All rights reserved.

Keywords: Jackson networks; Output process; Cycle time distribution; Cyclic networks; Arrival theorem

1. Repeated integrated tails

Let F_1 be a distribution function on $\mathbb{R}^+$ with finite mean $m_1 := \int_0^\infty \bar{F}_1(x)\,\mathrm{d}x$, where $\bar{F}_1(x) := 1 - F_1(x)$ denotes the tail function, and assume for simplicity that $F_1(0) = 0$. If in addition the distribution F_1 has a finite moment of order $N-1$ then we can define recursively distribution functions F_n, $n = 2, 3, \ldots, N$, on $\mathbb{R}^+$ via the relationship

$$\bar{F}_{n+1}(x) = \frac{1}{m_n}\int_x^\infty \bar{F}_n(y)\,\mathrm{d}y,$$

$$m_n = \int_0^\infty \bar{F}_n(y)\,\mathrm{d}y, \quad n = 1, 2, \ldots, N-1. \tag{1}$$

It is easy to show that $m_n < \infty$ if and only if $\int_0^\infty x^n\,\mathrm{d}F_1(x) < \infty$. Iterating (1) we obtain

$$\bar{F}_n(x_n) = C_n^{-1}\int_{x_n}^\infty \int_{x_{n-1}}^\infty \cdots \int_{x_2}^\infty \bar{F}_1(x_1) \times \mathrm{d}x_1, \ldots, \mathrm{d}x_{n-2}\,\mathrm{d}x_{n-1}$$

$$= C_n^{-1}\int_{x_n}^\infty \frac{(x_1 - x_n)^{n-2}}{(n-2)!}\bar{F}_1(x_1)\,\mathrm{d}x_1, \quad n = 2, 3, \ldots,$$

where

$$C_n = \prod_{k=1}^{n-1} m_k = \int_0^\infty \frac{x_1^{n-2}}{(n-2)!}\bar{F}_1(x_1)\,\mathrm{d}x_1 = \frac{1}{(n-1)!}\int_0^\infty x_1^{n-1}\,\mathrm{d}F_1(x_1), \quad n = 2, 3, \ldots,$$

the last expression resulting from integration by parts. Denote the corresponding Laplace transforms by $\phi_n(s) := \int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}F_n(x)$. Then (1) translates into

$$\phi_n(s) = \frac{1 - \phi_{n-1}(s)}{s m_{n-1}}, \quad n = 2, 3, \ldots . \tag{2}$$

From this recursive relationship we obtain

$$\phi_n(s) = (-1)^n \frac{\sum_{k=0}^{n-2} s^k (-1)^k \prod_{j=1}^k m_j - \phi_1(s)}{s^{n-1}\prod_{j=1}^{n-1} m_j} \tag{3}$$

E-mail address: zazanis@aueb.gr (M.A. Zazanis).

0167-6377/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/j.orl.2003.12.001

(with the empty sum equal to zero and the empty product equal to 1). Without loss of generality for our purposes we will further assume that F_1 is light tailed, i.e. that $\int_0^\infty e^{\epsilon x}\,dF_1(x) < \infty$ for some $\epsilon > 0$ and thus possesses moments of all orders. We will denote the kth moment of F_1 by $v_k := \int_0^\infty x^k\,dF_1(x)$, $k = 0, 1, 2, \ldots$. In view of the light-tailed nature of F_1

$$\phi_1(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} s^k v_k,$$

the power series expansion being valid at least in the interval $(-\epsilon, \epsilon)$. Upon substituting the above into (3) we obtain

$$\phi_n(s) = (-1)^n \frac{\sum_{k=0}^{n-2} (-1)^k s^k \left(\prod_{j=1}^{k} m_j - (1/k!) v_k\right) - \sum_{k=n-1}^{\infty} ((-1)^k/k!) s^k v_k}{s^{n-1} \prod_{j=1}^{n-1} m_j}. \qquad (4)$$

Since $\phi_n(s)$ cannot have poles at $s = 0$ we conclude from (4) that

$$\frac{1}{k!} v_k = m_1, \ldots, m_k, \quad k = 1, 2, \ldots, n-2, \qquad (5)$$

and hence that $m_k = v_k/(k v_{k-1})$. (n being arbitrary, the above holds of course for any $k \in \mathbb{N}$). Taking into account (5), (4) becomes

$$\phi_n(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} s^k \frac{1}{\binom{n+k-1}{n-1}} \frac{v_{n+k-1}}{v_{n-1}}.$$

The above power series displays explicitly the moments of the integrated tail distributions in terms of the moments of the original distribution. In fact, for the kth moment of F_n we have

$$(-1)^k \phi_n^{(k)}(0) = \frac{1}{\binom{n+k-1}{n-1}} \frac{v_{n+k-1}}{v_{n-1}}. \qquad (6)$$

2. Stationary and synchronous output times and the arrival theorem

Here we show how the above ideas, together with the arrival theorem, can be used to obtain a recursion on the number of customers for the Laplace transform of the *inter-output* time distribution in a closed, single server, cyclic Jackson network. From this recursion closed form expressions for the inter-output time distribution will be derived. The results presented here provide a simple, probabilistic derivation of the representation of inter-output times in cyclic networks given in [14].

Consider a closed, cyclic Jackson network with M stations and c customers and denote by $\mu_1, \ldots, \mu_M$ the service rates of the M exponential servers. The state of this Markovian system can be described by the vector $\mathbf{n} = (n_1, n_2, \ldots, n_M)$, where n_m is the number of customers in station $m = 1, 2, \ldots, M$. Thus the state space is the simplex

$$\mathscr{S}_c := \left\{ \mathbf{n} = (n_1, \ldots, n_M) : n_m \in \mathbb{N} \cup \{0\}, \; m = 1, 2, \ldots, M, \sum_{m=1}^{M} n_m = c \right\}.$$

It is well known that the stationary distribution assumes the product form

$$p(\mathbf{n}) = \frac{1}{G(c)} \prod_{m=1}^{M} \left(\frac{1}{\mu_m}\right)^{n_m}, \quad \mathbf{n} \in \mathscr{S}_c, \qquad (7)$$

where the normalization constant is given by

$$G(c) = \sum_{\mathbf{n} \in \mathscr{S}_c} \prod_{m=1}^{M} \left(\frac{1}{\mu_m}\right)^{n_m}.$$

(When $c = 0$, $G(0) = 1$.)

Let us denote by $\{\tilde{\mathbf{X}}^c(t); t \in \mathbb{R}\}$ the *time-stationary* version of the above cyclic network with c customers where $\tilde{\mathbf{X}}^c(t) = (\tilde{X}_1^c(t), \ldots, \tilde{X}_M^c(t))$ is the state of the system at time t, $\tilde{X}_m^c(t)$ denotes the number of customers in station m at time t, and $P(\tilde{\mathbf{X}}^c(t) = \mathbf{n})$ is given by (7) for all t. Also let $\{\tilde{D}_n^c; n \in \mathbb{Z}\}$ denote the point process of departures from node M that corresponds to $\tilde{\mathbf{X}}^c$ and which is of course a *time-stationary* point process. We will adopt the standard labelling convention regarding the points of $\{\tilde{D}_n^c\}$ according to which $\tilde{D}_1^c$ is the first point strictly to the right of the time origin. Thus $P(\tilde{D}_0^c \leqslant 0 < \tilde{D}_1^c) = 1$ (but of course $\tilde{D}_0^c < 0$ w.p. 1). If we assume the process $\tilde{\mathbf{X}}^c$ to have right-continuous sample paths with probability 1 and set $\tilde{\mathbf{Y}}_n^c := \tilde{\mathbf{X}}^c(\tilde{D}_n^c)$, then $\{(\tilde{D}_n^c, \tilde{\mathbf{Y}}_n^c); n \in \mathbb{Z}\}$ is a time-stationary Markov-renewal process. In particular, the embedded Markov chain $\{\tilde{\mathbf{Y}}_n^c\}$ has state space $\mathscr{S}'_c := \{\mathbf{k} : \mathbf{k} = \mathbf{n} + \mathbf{e}_1; \mathbf{n} \in \mathscr{S}_{c-1}\}$, where $\mathbf{e}_m := (0, \ldots, 1, \ldots, 0)$ denotes the unit vector in the mth direction of $\mathbb{R}^M$. We will also denote by $\{Q_{\mathbf{kl}}(t); \mathbf{k}, \mathbf{l} \in \mathscr{S}'_c, t \geqslant 0\}$ the corresponding Markov-renewal transition kernel, i.e. $Q_{\mathbf{kl}}(t) := P(\tilde{D}_{n+1}^c - \tilde{D}_n^c \leqslant t, \tilde{\mathbf{Y}}_{n+1}^c = \mathbf{l} \mid \tilde{\mathbf{Y}}_n^c = \mathbf{k}\}$.

Note that $\{\tilde{\mathbf{Y}}_n^c; n \in \mathbb{Z}\}$ is in general *not* a stationary Markov chain. The equilibrium distribution of this embedded chain is in fact given by

$$\pi(\mathbf{n} + \mathbf{e}_1) = \frac{1}{G(c-1)} \prod_{m=1}^{M} \left(\frac{1}{\mu_m}\right)^{n_m}, \quad \mathbf{n} \in \mathscr{S}_{c-1}. \tag{8}$$

The above follows from the arrival theorem which expresses the stationary distribution of the embedded Markov chain at the points of departure from a node to the time-stationary distribution of the network (7) and is usually expressed by the statement that "the jumping customer sees the rest of the network in equilibrium with one less customer". We refer the reader to [13, p. 123].

Let us now denote by $\mathbf{X}^c(t) = (X_1^c(t), \ldots, X_M^c(t))$ the *Palm* (or synchronous, or event-stationary) version of the process with respect to service completions at node M. In the general setting of a stationary marked point process, the Palm version of the marked point process is constructed using a transformation of the original probability measure (see [1]). In this paper, however, we will adopt an elementary viewpoint based on the arrival theorem and the Markovian nature of the system. We will construct the synchronous (or Palm, as we will be referring to it) version of the Markov-renewal process of departure times from node M, $\{(D_n^c, \mathbf{Y}_n^c); n \in \mathbb{Z}\}$, as follows. Set $D_0^c = 0$ w.p. 1 and $P(\mathbf{Y}_0^c = \mathbf{k}) = \pi(\mathbf{k})$ for all $\mathbf{k} \in \mathscr{S}'_c$ where π is given by (8), then use the same Markov-renewal kernel, $Q_{\mathbf{kl}}(t)$, to construct the point process. Note that the Palm version of the Markov-renewal process constructed above is *point-shift invariant*. This means that the time origin coincides with a ("random") point of $\{D_n^c; n \in \mathbb{Z}\}$ and the process is *invariant under shifts* of the time origin along its points in the sense that $(D_{i_1}^c, \ldots, D_{i_n}^c) \stackrel{\mathrm{d}}{=} (D_{i_1+k}^c - D_k^c, \ldots, D_{i_n+k}^c - D_k^c)$, for any $n \in \mathbb{N}$, $k, i_1, \ldots, i_n \in \mathbb{Z}$. Define also the corresponding sequence of inter-output times, $\sigma_n^c := D_{n+1}^c - D_n^c$ and note that $\{\sigma_n^c; n \in \mathbb{Z}\}$ is a stationary sequence of random variables. Also, let $\Psi_c(s) := E\mathrm{e}^{-s\sigma_0^c} = E\mathrm{e}^{-sD_1^c}$ denote the Laplace transform of the inter-output times and γ_c the mean inter-output time when there are c customers in the system. We are now ready to state.

Theorem 1. *The distribution between two consecutive departures from node M in the Palm version of the system with c customers is the same as the distribution of the forward recurrence time of the point process of departures from node M in the stationary version of the system with $c-1$ customers, i.e.*

$$E\mathrm{e}^{-sD_1^c} = E\mathrm{e}^{-s\tilde{D}_1^{c-1}}. \tag{9}$$

Furthermore,

$$\Psi_c(s) = \frac{1 - \Psi_{c-1}(s)}{s\gamma_{c-1}}. \tag{10}$$

Proof. Consider the Palm version of the system with c customers. This means that $t = 0$ coincides with a departure from node M. In a cyclic, single server network with population $c \geqslant 2$ *the next departure cannot be affected by the customer who has just left* since that customer cannot overtake (or influence in any way) the remaining customers. Thus, the next departure depends solely on the configuration of the $c-1$ remaining customers in the network. However, the distribution of $\mathbf{X}^c(0)$ is given by (8) which means that the distribution of the remaining customers is the stationary distribution in a network with $c-1$ customers. Thus we obtain (9).

To establish (10), note that

$$E\mathrm{e}^{-s\tilde{D}_1^c} = \frac{1 - \Psi_c(s)}{s\gamma_c} \tag{11}$$

which expresses the Laplace transform of the forward recurrence time of the time-stationary point process $\{\tilde{D}_n^c\}$ in terms of the Laplace transform of the inter-arrival times of the Palm version of the same point process. (The fact that the validity of (11) extends beyond the realm of renewal processes hinges upon standard results regarding stationary point processes [1] or, more specifically in this case, Markov-renewal processes.) From (9) and (11) we readily obtain (10). □

The recursion on c in (10) will be the basis for our subsequent analysis (cf. Eq. (2)). To underscore the essential simplicity of the argument, we will assume throughout the paper that *the M service rates μ_m are all different*.

3. Inter-output times and throughput

From the basic recursion (10), setting

$$\Gamma_k = \prod_{i=1}^{k} \gamma_i, \quad k = 1, 2, \ldots, \quad \Gamma_0 = 1, \tag{12}$$

we obtain (cf. Eq. (3))

$$\Psi_c(s) = (-1)^c \frac{\sum_{k=0}^{c-2} (-1)^k s^k \Gamma_k - \Psi_1(s)}{s^{c-1}\Gamma_{c-1}}, \tag{13}$$

where $\Psi_1(s) = \prod_{m=1}^{M} \mu_m/(\mu_m + s)$. The last equation follows from the fact that the inter-output time in a network with a single customer is the sum of M independent exponential random variables, $Y := Y_1 + \cdots + Y_M$, where $Y_m \sim \exp(\mu_m)$, $m = 1, 2, \ldots, M$. A straightforward partial fractions expansion gives $\Psi_1(s) = \sum_{m=1}^{M} \alpha_m \mu_m/(\mu_m + s)$ where

$$\alpha_m = \prod_{\substack{l=1 \\ l \neq m}}^{M} \frac{\mu_l}{\mu_l - \mu_m}. \tag{14}$$

When $|s| < \min\{\mu_1, \ldots, \mu_M\}$, a series expansion gives

$$\Psi_1(s) = \sum_{k=0}^{\infty} (-1)^k s^k \sum_{m=1}^{M} \alpha_m \mu_m^{-k}$$

and hence

$$E[Y^k] = k! \sum_{m=1}^{M} \alpha_m \mu_m^{-k}. \tag{15}$$

Since $\Psi_1(s) = \sum_{i=0}^{\infty} (-1)^i 1/i! E[Y^i] s^i$, from (13) we obtain

$$\Psi_c(s) = (-1)^c \frac{\sum_{k=0}^{c-2} (-1)^k s^k \left(\Gamma_k - \frac{1}{k!} E[Y^k]\right) - \sum_{k=c-1}^{\infty} (-1)^k s^k \frac{1}{k!} E[Y^k]}{s^{c-1}\Gamma_{c-1}}. \tag{16}$$

Again, we note that $\Psi_c(s)$ cannot have poles at zero and, since c is an arbitrary natural number,

$$\Gamma_k = \frac{1}{k!} E[Y^k], \quad k = 0, 1, 2, \ldots. \tag{17}$$

Taking into account (12), this gives

$$\gamma_k = \frac{E[Y^k]}{kE[Y^{k-1}]} \quad \forall k \in \mathbb{N}.$$

Let λ_k denote the *throughput* of the system with k customers. Clearly $\lambda_k = \gamma_k^{-1}$ and thus we obtain the following expression for the system throughput with c customers:

$$\lambda_c = c \frac{E[Y^{c-1}]}{E[Y^c]} = \frac{\sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)}}{\sum_{m=1}^{M} \alpha_m \mu_m^{-c}}, \tag{18}$$

the second equation above following from (15), provided that all the node rates are different. Define now the quantities

$$\beta_m(c) = \frac{\alpha_m \mu_m^{-c}}{\sum_{m=1}^{M} \alpha_m \mu_m^{-c}}.$$

Using them we can express the system throughput as a weighted average of the rates of the individual stations. Then (18) can be rewritten as

$$\lambda_c = \sum_{m=1}^{M} \beta_m(c) \mu_m.$$

Returning to (16) and (17) we see that

$$\begin{aligned}
\Psi_c(s) &= \frac{(c-1)!}{E[Y^{c-1}]} \sum_{k=0}^{\infty} (-1)^k s^k \\
&\quad \times \frac{1}{(c+k-1)!} E[Y^{c+k-1}] \\
&= \frac{\sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)} \mu_m/(\mu_m + s)}{\sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)}} \\
&= \sum_{m=1}^{M} \beta_m(c-1) \frac{\mu_m}{\mu_m + s}.
\end{aligned}$$

The last equation expresses the Laplace transform of the inter-output time as a weighted sum of the Laplace transforms of the service times at the M nodes provided that the μ_m's are distinct). These three equivalent expressions for the inter-output time distribution give three corresponding expressions for its kth moment:

$$(-1)^k \Psi_c^{(k)}(0) = \frac{1}{\binom{c+k-1}{c-1}} \frac{E[Y^{c+k-1}]}{E[Y^{c-1}]}$$

$$= k! \frac{\sum_{m=1}^{M} \alpha_m \mu_m^{-(c+k-1)}}{\sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)}}$$

$$= k! \sum_{m=1}^{M} \beta_m(c-1)\mu_m^{-k}$$

(cf. Eq. (6)).

4. The normalization constant

Using the well-known expression for the throughput

$$\lambda_c = \frac{G(c-1)}{G(c)}$$

we obtain the relationship $\lambda_1, \lambda_2, \ldots, \lambda_c = G(0)/G(c)$. Taking also into account (18) we have

$$G(c) = \frac{\sum_{m=1}^{M} \mu_m^{-1}}{\sum_{m=1}^{M} \alpha_m \mu_m^{-1}} \sum_{m=1}^{M} \alpha_m \mu_m^{-c}.$$

However, $\sum_{m=1}^{M} \mu_m^{-1} = \sum_{m=1}^{M} \alpha_m \mu_m^{-1}$ as can be seen either directly from (14), or from (15) with $k = 1$. Thus,

$$G(c) = \sum_{m=1}^{M} \alpha_m \mu_m^{-c}.$$

The above closed form expression for the normalization constant was first obtained by different means in [10], see also [8]. While the above expressions for the normalization constant and the throughput also hold for closed networks with arbitrary Markovian routing, the results concerning the distribution of cycle times in the next section require that these networks be cyclic. Also, the normalization constant, the throughput, and the Laplace transform of the inter-output times, are *symmetric* functions of the μ_m's (i.e. invariant under permutations of the μ_m's). Therefore, we can relabel, if necessary, the service rates so that $\mu_1 < \mu_2 < \cdots < \mu_M$. In that case the signs of the α_m's alternate.

5. The cycle time distribution

We now extend the analysis of Section 2 to cycle times of customers and to this end we consider the Palm version of the system with c customers. The cycle time of the customer who jumps from node M to node 0 at time $D_0^c = 0$ is equal to $W_0^c := D_c^c - D_0^c$. This is equal in distribution to $W_{-j}^c := D_{c-j}^c - D_{-j}^c$ for any $j \in \mathbb{Z}$ by the point-shift invariance of the Palm version of the process. On the other hand, $\tilde{W}_0^c := \tilde{D}_c^c - \tilde{D}_0^c$ does not have the same distribution as W_0^c. The reason for this discrepancy lies in a manifestation of the so-called "renewal paradox" since larger intervals are more likely to contain 0 than smaller intervals. The situation here is however complicated by the fact that the intervals $\{[\tilde{D}_n^c, \tilde{D}_{c+n}^c); n \in \mathbb{Z}\}$ are *overlapping*. An analysis is presented in Proposition 1 of the Appendix from which we see that

$$\frac{1}{c} \sum_{j=0}^{c-1} P(\tilde{W}_{-j}^c \in \mathrm{d}x) = \frac{x}{E[W_0^c]} P(W_0^c \in \mathrm{d}x). \tag{19}$$

(It is perhaps worth pointing out that, while the right hand side of (19) is precisely what we would expect from the "renewal paradox", the left hand side is an *average* of the probabilities for the c overlapping intervals that contain 0.) We also point out that (19) is equivalent to the following statement regarding the corresponding Laplace transforms:

$$\frac{1}{c} \sum_{j=0}^{c-1} E[\mathrm{e}^{-s\tilde{W}_{-j}^c}]$$

$$= -\frac{1}{E[W_0^c]} \frac{\mathrm{d}}{\mathrm{d}s} E[\mathrm{e}^{-sW_0^c}]. \tag{20}$$

We are now ready to state

Theorem 2. *In the above framework, the Laplace transform of the cycle time of a typical customer in a network with c customers, which we will denote by* $\Phi_c(s) := E\mathrm{e}^{-sW_0^c}$, *satisfies the following recursive relationship*:

$$\Phi_{c+1}(s) = -\frac{1}{m_c} \frac{\mathrm{d}}{\mathrm{d}s} \Phi_c(s), \tag{21}$$

where $m_c := -\Phi_c'(0)$ *denotes the corresponding mean cycle time.*

Proof. Our point of departure is Eq. (20). The typical term of the sum is

$$E[e^{-s\tilde{W}^c_{-j}}]$$

$$= \sum_{\mathbf{n}\in\mathscr{S}_c} E[e^{-s(\tilde{D}^c_{c-j}-\tilde{D}^c_{-j})} \mid \tilde{\mathbf{X}}^c_0 = \mathbf{n}] \cdot P(\tilde{\mathbf{X}}^c_0 = \mathbf{n})$$

$$= \sum_{\mathbf{n}\in\mathscr{S}_c} E[e^{-s\tilde{D}^c_{c-j}} \mid \tilde{\mathbf{X}}^c_0 = \mathbf{n}]$$

$$\times E\,[e^{-s(-\tilde{D}^c_{-j})} \mid \tilde{\mathbf{X}}^c_0 = \mathbf{n}]P(\tilde{\mathbf{X}}^c_0 = \mathbf{n}), \tag{22}$$

where, in the second equation, we have used the conditional independence of past and future given the present state of the Markov process $\tilde{\mathbf{X}}^c$. Now, as a consequence of the arrival theorem,

$$P(\tilde{\mathbf{X}}^c_0 = \mathbf{n}) = P(\mathbf{X}^{c+1}_0 = \mathbf{n} + \mathbf{e}_1) \tag{23}$$

for all $\mathbf{n} \in \mathscr{S}_c$. Furthermore, we claim that

$$E[e^{-s\tilde{D}^c_{c-j}} \mid \tilde{\mathbf{X}}^c_0 = \mathbf{n}]$$

$$= E[e^{-sD^{c+1}_{c-j}} \mid \mathbf{X}^{c+1}_0 = \mathbf{n} + \mathbf{e}_1], \tag{24}$$

$$E[e^{s\tilde{D}^c_{-j}} \mid \tilde{\mathbf{X}}^c_0 = \mathbf{n}]$$

$$= E[e^{sD^{c+1}_{-j-1}} \mid \mathbf{X}^{c+1}_0 = \mathbf{n} + \mathbf{e}_1]. \tag{25}$$

To justify (24) note that the conditional expectation on the left depends only on the configuration $\mathbf{n}$ of the c customers due to the Markovian nature of the system. For the same reason, the conditional expectation on the right depends only on the configuration $\mathbf{n}+\mathbf{e}_1$ of the $c+1$ customers. The two configurations are the same with the exception that in the network with $c+1$ customers we have one additional customer in the first station. Since, the network is cyclic and has single exponential servers with rates that do not depend on the number of customers in the station, the extra customer who is present in the first station in the system with $c+1$ customers cannot overtake the other customers or influence in any way the first c output times after time 0. Thus the distribution of the $(c-j)$th output time $(j = 0, 1, \ldots, c-1)$ *given the configurations* $\mathbf{n}$ and $\mathbf{n}+\mathbf{e}_1$, respectively, is the same for the two systems.

The validity of (25) follows by an identical argument if we examine the same process backwards in time. (The apparent discrepancy in the subscripts $-j$ on the left hand side of (25) and $-j-1$ on the right hand side is due to the numbering convention.) The first point strictly to the left of 0 in $\{\tilde{D}^c_n\}$ is $\tilde{D}^c_0$ w.p.1 while the first point strictly to the left of 0 in $\{D^{c+1}_n\}$ is D^{c+1}_{-1}. Thus from (22)–(25), we have that

$$E[e^{-s\tilde{W}^c_{-j}}]$$

$$= \sum_{\mathbf{n}\in\mathscr{S}_c} E[e^{-sD^{c+1}_{c-j}} \mid \mathbf{X}^{c+1}_0 = \mathbf{n} + \mathbf{e}_1]$$

$$\times E[e^{-s(-D^{c+1}_{-j-1})} \mid \mathbf{X}^{c+1}_0 = \mathbf{n}+\mathbf{e}_1]P(\mathbf{X}^{c+1}_0 = \mathbf{n}+\mathbf{e}_1)$$

$$= \sum_{\mathbf{n}\in\mathscr{S}_c} E[e^{-s(D^{c+1}_{c-j}-D^{c+1}_{-j-1})} \mid \mathbf{X}^{c+1}_0 = \mathbf{n} + \mathbf{e}_1]$$

$$\times P(\mathbf{X}^{c+1}_0 = \mathbf{n} + \mathbf{e}_1)$$

$$= E[e^{-sW^{c+1}_{-j-1}}],$$

where in the second equation above we have used once more the Markov property. However,

$$E[e^{-sW^{c+1}_{-j-1}}] = E[e^{-s(D^{c+1}_{c-j}-D^{c+1}_{-j-1})}]$$

$$= E[e^{-s(D^{c+1}_{c+1}-D^{c+1}_0)}] = E[e^{-sW^{c+1}_0}]$$

$$= \Phi_{c+1}(s)$$

for all $j \in \mathbb{Z}$ due to the point-shift invariance property of $\{D^{c+1}_n\}$. Thus from the above we have

$$\frac{1}{c}\sum_{j=0}^{c-1} E[e^{-s\tilde{W}^c_{-j}}] = \Phi_{c+1}(s)$$

which, together with (20), establishes the theorem. □

Upon iterating (21) we obtain

$$\Phi_c(s) = \frac{(-1)^{c-1}}{m_{c-1}, \ldots, m_2 m_1} \frac{d^{c-1}}{ds^{c-1}} \Phi_1(s), \tag{26}$$

where $\Phi_1(s)$ is the cycle time distribution in a network with a single customer. Thus

$$\Phi_1(s) = \prod_{m=1}^{M} \frac{\mu_m}{\mu_m + s} = \sum_{m=1}^{M} \alpha_m \frac{\mu_m}{\mu_m + s}. \tag{27}$$

From (26) and (27) we obtain

$$\Phi_c(s) = \frac{(c-1)!}{m_{c-1},\ldots,m_2 m_1} \sum_{m=1}^{M} \alpha_m \frac{\mu_m}{(\mu_m+s)^c}$$
$$= \frac{(c-1)!}{m_{c-1},\ldots,m_2 m_1} \sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)} \left(\frac{\mu_m}{\mu_m+s}\right)^c.$$

Since $\Phi_c(1) = 1$ we conclude that $m_1 m_2 \cdots m_{c-1} = (c-1)! \sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)}$. Hence, we obtain the following expression for the Laplace transform of the cycle time:

$$\Phi_c(s) = \frac{\sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)} (\mu_m/(\mu_m+s))^c}{\sum_{m=1}^{M} \alpha_m \mu_m^{-(c-1)}}$$
$$= \sum_{m=1}^{M} \beta_m(c-1) \left(\frac{\mu_m}{\mu_m+s}\right)^c. \quad (28)$$

An alternative expression for the cycle time distribution with a “change of measure” interpretation can be readily obtained from (26) and $\Phi_1(s) = E\mathrm{e}^{-sY}$. Indeed, since repeated differentiation inside the expectation is justified when Y is a sum of independent exponential random variables, we have $\Phi_c(s) = (m_{c-1},\ldots,m_1)^{-1} E[Y^{c-1}\mathrm{e}^{-sY}]$ which, together with $\Phi_c(0) = 1$, gives

$$\Phi_c(s) = \frac{E[Y^{c-1}\mathrm{e}^{-sY}]}{EY^{c-1}}.$$

The above expressions were obtained in [14] using the techniques of Palm Calculus. Of course, the study of flow times in cyclic networks has a long history. Among the early results we mention [4] which used a birth and death analysis to derive the cycle time in a two-station cyclic network and [2], where the joint distribution of the sojourn times in a two-station network is derived. In [11] the Laplace transform of the cycle time in a cyclic network is derived while in [3] a product form for the joint sojourn times in the stations of such a network is obtained using an elegant reversibility argument. See also [5,9], the extension to networks with multi-server stations in [12,6].

From (28) we can obtain readily in a concise form the probability that the cycle time exceeds a given threshold x, a quantity important in many applications:

$$P(W_0^c > x) = \sum_{m=1}^{M} \beta_m(c-1)\mathrm{e}^{-\mu_m x} \times \sum_{n=0}^{c-1} \frac{(\mu_m x)^n}{n!}.$$

This expression was first given in [7].

Acknowledgements

The author would like to thank an anonymous referee for numerous suggestions which have improved substantially the quality of the paper.

Appendix. Length biased sampling for overlapping intervals

Consider a Markov-renewal process (T_n, Y_n), $n \in \mathbb{N}$, where $\{Y_n\}$ is an irreducible, positive recurrent Markov chain on a countable state space $\mathbb{S}$ with probability transition matrix $[P_{ij}]$, $i, j \in \mathbb{S}$, and stationary distribution π. The corresponding transition kernel is

$$Q_{ij}(x) := P(T_{n+1} - T_n \leqslant x, Y_{n+1} = j \mid Y_n = i).$$

Suppose that $T_0 = 0$ w.p. 1 and $P(Y_0 = i) = \pi_i$, $i \in \mathbb{S}$. Thus (T_n, Y_n) is the so-called Palm version of the Markov-renewal process. Set also $\lambda^{-1} := E[T_1 - T_0] = \sum_{i,j\in\mathbb{S}} \pi_i \int_0^\infty y Q_{ij}(\mathrm{d}y)$. In what follows we will assume that λ is finite and strictly positive.

Suppose now that $(\tilde{T}_n, \tilde{Y}_n)$ is *another* Markov-renewal process with the same transition kernel but (*time-*) *stationary* under the probability measure P in the sense that $\{\tilde{X}(t); t \in \mathbb{R}\}$, where $\tilde{X}(t) := \sum_{n\in\mathbb{Z}} \tilde{Y}_n \mathbf{1}(\tilde{T}_n \leqslant t < \tilde{T}_{n+1})$, is a stationary process. We adopt the standard numbering convention regarding the points of the stationary process according to which $P(\tilde{T}_0 \leqslant 0 < \tilde{T}_1) = 1$ (cf. Section 2). It is a standard result in the theory of Markov-renewal processes that

$$P(\tilde{T}_1 - \tilde{T}_0 \in \mathrm{d}x, \tilde{Y}_0 = i, \tilde{Y}_1 = j)$$
$$= \lambda x P(T_1 - T_0 \in \mathrm{d}x, Y_0 = i, Y_1 = j)$$
$$= \lambda x \pi_i Q_{ij}(\mathrm{d}x). \quad (29)$$

A consequence of (29) is that

$$P(\tilde{T}_1 - \tilde{T}_0 \in \mathrm{d}x) = \lambda x P(T_1 - T_0 \in \mathrm{d}x). \quad (30)$$

The above expressions give a statement of the so-called “renewal paradox” in the Markov-renewal context.

Here we give the following generalization of the classic renewal paradox: with $(\tilde{T}_n, \tilde{Y}_n)$ the stationary Markov-renewal process defined above, let $c \geqslant 2$ be a natural number, and consider the *overlapping* collection of intervals $\{[\tilde{T}_n, \tilde{T}_{n+c}); n \in \mathbb{Z}\}$ (as opposed to the non-overlapping intervals $\{[\tilde{T}_n, \tilde{T}_{n+1}); n \in \mathbb{Z}\}$ of the previous section). In the proposition that follows we show that, if we choose one of the c such overlapping intervals that contain 0 at random, independently of everything else, its length has distribution given by the equivalent of (30).

Proposition 1. *In the above Markov-renewal context, for any $c \in \mathbb{N}$ it holds that*

$$\frac{1}{c}\sum_{j=0}^{c-1} P(\tilde{T}_{c-j} - \tilde{T}_{-j} \in \mathrm{d}x)$$
$$= \frac{x}{E[T_c - T_0]} P(T_c - T_0 \in \mathrm{d}x). \quad (31)$$

Proof. Taking into account the defining properties of a Markov-renewal process and (29) we have

$$P(\tilde{T}_{c-j} - \tilde{T}_{-j} \in \mathrm{d}x)$$
$$= \sum_{i,k \in \mathbb{S}} \int_{u+v+w \in \mathrm{d}x} P(\tilde{T}_0 - \tilde{T}_{-j} \in \mathrm{d}u | \tilde{Y}_0 = i)$$
$$\times P(\tilde{T}_1 - \tilde{T}_0 \in \mathrm{d}v, \tilde{Y}_0 = i, \tilde{Y}_1 = k)$$
$$\times P(\tilde{T}_{c-j} - \tilde{T}_1 \in \mathrm{d}w | \tilde{Y}_1 = k)$$
$$= \lambda \sum_{i,k \in \mathbb{S}} \int_{u+v+w \in \mathrm{d}x} P(T_0 - T_{-j} \in \mathrm{d}u | Y_0 = i)$$
$$\times v P(T_1 - T_0 \in \mathrm{d}v, Y_0 = i, Y_1 = k)$$
$$\times P(T_{c-j} - T_1 \in \mathrm{d}w | Y_1 = k)$$
$$= \lambda E[(T_1 - T_0)\mathbf{1}(T_{c-j} - T_{-j} \in \mathrm{d}x)].$$

Using the above expression in the left hand side of (31) we express the probability that a randomly selected interval among those that cover 0 lies in $(x, x + \mathrm{d}x)$ as

$$\frac{1}{c}\sum_{j=0}^{c-1} \lambda E[(T_1 - T_0)\mathbf{1}(T_{c-j} - T_{-j} \in \mathrm{d}x)]$$
$$= \frac{1}{c}\sum_{j=0}^{c-1} \lambda E[(T_{j+1} - T_j)\mathbf{1}(T_c - T_0 \in \mathrm{d}x)]$$
$$= \frac{\lambda}{c} E\left[\mathbf{1}(T_c - T_0 \in \mathrm{d}x)\sum_{j=0}^{c-1}(T_{j+1} - T_j)\right]$$
$$= \frac{1}{E[T_c - T_0]} E[(T_c - T_0)\mathbf{1}(T_c - T_0 \in \mathrm{d}x)].$$

In the above string of equalities we have used the point-shift invariance of the Palm version of the process whence we have obtained the equality $E[(T_1 - T_0)\mathbf{1}(T_{c-j} - T_{-j} \in \mathrm{d}x)] = E[(T_{j+1} - T_j)\mathbf{1}(T_c - T_0 \in \mathrm{d}x)]$. We have also used the fact that $\lambda^{-1} = E[T_1 - T_0]$ and thus $c\lambda^{-1} = E[T_c - T_0]$ (again by point-shift invariance). This concludes the proof of the proposition. □

In conclusion we point out that proposition 1 can be shown to hold for any stationary point process.

References

[1] F. Baccelli, P. Brémaud, Elements of Queueing, Springer, New York, 1994.

[2] O.J. Boxma, P. Donk, On the response time and cycle time distributions in a two stage cyclic queue, Performance Evaluation 2 (1982) 181–194.

[3] O.J. Boxma, F.P. Kelly, A.G. Konheim, The product form for sojourn time distributions in cyclic exponential queues, J. Assoc. Comput. Mech. 31 (1) (1984) 128–133.

[4] W.M. Chow, The cycle time distribution of exponential cyclic queues, J. Assoc. Comput. Mach. 27 (1980) 281–286.

[5] H. Daduna, Passage times for overtake-free paths in Gordon-Newell networks, Adv. Appl. Probab. 14 (1982) 672–686.
[6] H. Daduna, R. Szekli, Conditional job-observer property for multitype closed queueing networks, J. Appl. Probab. 39 (4) (2002) 865–881.
[7] P.G. Harrison, A note on cycle times in tree-like networks, Adv. Appl. Probab. 16 (1984) 216–219.
[8] P.G. Harrison, On normalizing constants in queueing networks, Oper. Res. 33 (1985) 464–468.
[9] F.P. Kelly, P.K. Pollett, Sojourn times in closed queueing networks, Adv. Appl. Probab. 15 (1983) 638–656.
[10] E. Koenigsberg, Cyclic queues, Oper. Res. Quart. 9 (1958) 22–35.
[11] R. Schassberger, H. Daduna, The time for a round trip in a cycle of exponential queues, J. Assoc. Comput. Mach. 30 (1983) 146–150.
[12] R. Schassberger, H. Daduna, Sojourn times in queueing networks with multiserver nodes, J. Appl. Probab. 24 (2) (1987) 511–521.
[13] R.F. Serfozo, Introduction to Stochastic Networks, Springer, New York, 1999.
[14] M.A. Zazanis, Formulas and representations for cyclic Markovian networks via Palm calculus, Queueing Systems 26 (1–2) (1997) 151–167.