Paired Type-II Shot-Noise Counters and Infinite Server Queues: Analytic Results and Sensitivity Analysis Estimates

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Abstract

We consider counters consisting of two or more parallel subsystems. Pulses (e.g. particles) to be counted arrive according to a Poisson process and are counted, provided that all subsystems are operational. Two related models are examined. In the first, an arriving pulse causes all subsystems to become inoperative for a period of time. Any pulse arriving during this "dead time" is not counted but nontheless causes an additional (overlapping) dead time. This model is analyzed by means of a Poisson point process in an appropriate Euclidean space and the probability of registering a typical pulse is determined using renewal theoretic techniques. The second type of multi-component counter examined consists of two counters in parallel subject to a saturation effect that follows a shot-noise process. A pulse is registered provided that the saturation level is below a given value. The system is analyzed by considering the underlying two-dimensional shot noise process and the joint Laplace transform of saturation levels is determined. Finally, sensitivity analysis estimates are developed based on integration-by-parts methods in Poisson space using the Malliavin calculus.

1 Introduction and model description

In this paper we examine generalizations of stochastic models known in the literature as type II counters. Interest in these models originated with particle counters such as the Geiger-Mueller counter counting neutrons arising from the disintegration of radioactive material. Typically, after registering a particle, such a counter becomes momentarily inoperative. During this

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"dead time", a particle hitting the counter is not registered. In counters of type I particles that are not register do not initiate dead times. In type II counters even particles that are not registered initiate dead times. One of the main problems that arise then is to estimate the number of unregistered particles based on the number of registered particles and the statistics of dead time duration. We refer the reader to the classic papers [21], [11], [19], and to [17, p.49].

The first type of model we will examine here is a variation of the counter model corresponding to exponential attenuation instead of a dead time. When a particle arrives, whether it is registered or not, it causes a pulse of random magnitude which in turn decreases exponentially in magnitude with time. The overall "saturation process" is the superposition of the effects of these pulses. This model is known as the (Poisson) shot noise process. It originated in the context of the analysis of noise in vacuum tube electrical circuits but it has become a fundamental stochastic model with applications in many areas, including Insurance and Financial Mathematics [18]. We examine a system where a particle is registered when at its arrival epoch the saturation level is below a given threshold and we analyze this system using level crossing methods [3], [13]. We also analyze such a system with two counters in parallel in which a particle is counted when both saturation levels are below a given threshold. As an interesting by-product of this analysis a novel type of bivariate Gamma distribution is obtained.

The second type of model, examined in section 4 is a classical counter of type II consisting of M components in parallel. Each component in such a counter operates essentially as an infinite server queue and thus we have parallel $M/G/\infty$ queues driven by the same Poisson input process and thus correlated. The performance criterion of interest in this model is the probability that the ith arrival finds *all systems idle*. This probability is determined by a representation of the vector with components equal to the number of customers in each system at time t in terms of a Poisson process in a Euclidean space of dimension $M+1$. A renewal theoretic argument is used to obtain the distribution of the number of customers in a busy period and thence the long term probability of registering an arriving particle.

In the final section of the paper the basic ideas of Malliavin calculus in Poisson space are briefly sketched and used in order to obtain efficient sensitivity analysis estimates for performance criteria of these Poisson drive systems. These techniques have been used for over two decades in systems driven by Brownian motion in the context of problems in Mathematical Finance (see for instance [7]). Following the approach of Privault [14], [15] we develop a family of sensitivity analysis estimators for these systems.

2 Single component type-II counter with exponential attenuation

Pulses arrive according to a Poisson process with rate λ. The ith arrival, occurring at T_i is characterized by a shock of size ξ_i. Suppose that $\{\xi_i\}$ is a sequence of positive random variables with distribution function G, independent of the Poisson process $\{T_i\}$. Each shock causes a "saturation effect" which decays exponentially with rate $\alpha > 0$. Thus the saturation effect of the ith shock at time $t > T_i$ is $\xi_i e^{-\alpha(t-T_i)}$. Assuming these effects to be additive, the overall saturation of the counter at time t is

$$X_t \ := \ \sum_{i=1}^{\infty} \xi_i e^{-\alpha(t-T_i)^+}, \qquad t \geq 0. \tag{1}$$

The ith pulse arriving at time T_i is registered provided that the saturation process at time T_i- is less than a fixed threshold $u > 0$. Whether it is registered or not the pulse contributes to the saturation process. This behaves as a counter of type II with the difference that in the classical counter models arriving pulses cause complete paralyzation for a (typically random) period of time and pulses arriving during this "dead time" are not registered. For a description of the classical counter models see for instance [16], [21], [22], and [17, p.49].

Note that the process (1) can also be described by means of the stochastic differential equation

$$dX_t \ = \ -\alpha X_t dt + dZ_t, \quad X_0 \ \text{given}. \tag{2}$$

In the above, $\{Z_t; t \geq 0\}$ is the compound Poisson process defined by $Z_t = \sum_{i=1}^{N_t} \xi_i$ where $N_t := \sum_{i=1}^{\infty} \mathbf{1}(T_i \leq t)$ is the number of Poisson shocks in the interval $(0, t]$. The above equation (a special case of a Lévy driven Ornstein-Uhlenbeck equation) can be solved pathwise to obtain

$$X_t \ = \ e^{-at} X_0 + \int_0^t e^{-\alpha(t-s)} dZ_s. \tag{3}$$

The process $\{X_t; t \geq 0\}$ becomes stationary by choosing X_0, independent of the compound Poisson process $\{Z_t; t \geq 0\}$, and with Laplace transform

$$\mathbb{E}[e^{-sX_0}] \ = \ \exp\left(-\lambda \int_0^{\infty} [1 - G^*(se^{-\alpha u})] du\right). \tag{4}$$

In the above expression $G^*(s) := \int_0^{\infty} e^{-sx} dG(x)$ is the Laplace transform of the shock size ξ_1. We note that when the shock size distribution G is exponential with $G^*(s) = \frac{\mu}{\mu+s}$, then

$$\mathbb{E}[e^{-sX_0}] \ = \ \left(\frac{\mu}{\mu + s}\right)^{\lambda/\alpha}, \tag{5}$$

i.e. the stationary distribution is Gamma with shape parameter λ/α and rate μ.

In order to obtain the stationary probability that a pulse will be registered, say p_u can be obtained using the level crossing method (see Brill [3]). Let λ_u denote the level crossing rate at level u. This is the same whether we consider *upcrossings* or *downcrossings*. Applying the level crossing methodology one sees that the density of the stationary distribution $f(x)$ at level x is given by $f(x) = \lambda_x \mathbb{E}[|X_t'|^{-1}|X_t = x]$.Due to the dynamics of the system expressed by (3) $\mathbb{E}[|X_t'|^{-1}|X_t = x] = (\alpha x)^{-1}$. Thus

$$f(x) \;=\; \frac{\lambda_x}{\alpha x}.$$

On the other hand the rate of downcrossings is equal to the rate of upcrossings i.e. $\lambda_x = \int_0^x f(z)\bar{G}(x-z)dz$ where $\bar{G}(x) = 1 - G(x)$. Thus, the level crossing argument shows that the stationary density must satisfy the equation

$$f(x) \;=\; \frac{\lambda}{\alpha x}\int_0^x f(z)\bar{G}(x-z)dz \tag{6}$$

Proposition 1. *The Laplace transform of the stationary distribution of* $\{X_t\}$ *defined by* $F^*(s) = \int_0^\infty e^{-sx}f(x)dx$ *is given by*

$$F^*(s) \;=\; \exp\left(-\frac{\lambda}{\alpha}\int_0^s \frac{1-G^*(u)}{u}du\right). \tag{7}$$

Proof. Taking Laplace transforms on both sides of (6) we obtain

$$\begin{aligned}\int_0^\infty e^{-sx}xf(x)dx \;&=\; \frac{\lambda}{\alpha}\int_0^\infty e^{-sx}\int_0^x f(z)\bar{G}(x-z)dzdx \qquad \text{or}\\ -\frac{d}{ds}F^*(s) \;&=\; \frac{\lambda}{\alpha}\int_{z=0}^\infty e^{-sz}f(z)dz\int_{x=z}^\infty \bar{G}(x-z)dx \;=\; \frac{\lambda}{\alpha}\,F^*(s)\,\frac{1-G^*(s)}{s}.\end{aligned}$$

Thus the Laplace transform of the stationary density satisfies the differential equation

$$\frac{d}{ds}F^*(s) \;=\; -\frac{\lambda}{\alpha}\,F^*(s)\,\frac{1-G^*(s)}{s}$$

whose solution is (7). □

This in principle determines the probability that a pulse is registered since the distribution of the saturation level is obtained from the above theorem. For instance, when the pulse distribution is exponential the saturation level is Gamma-distributed according to (5).

3 Paired Shot-noise Type II Counters

Consider now two counters of the type considered in the previous section in parallel. Particles arrive according to a Poisson process $\{T_i\}$ with rate λ. The ith particle causes a pulse of size

ξ_i^1 to the first counter and ξ_i^2 to the second. We assume that $\{(\xi_i^1, \xi_i^2)\}$ is a sequence of i.i.d. vectors with joint distribution function $G(z_1, z_2) := \mathbb{P}(\xi^1 \le z_1, \xi^2 \le z_2)$ and corresponding Laplace transform $\phi(s_1, s_2) := \mathbb{E}[e^{-s_1\xi^1 + s_2\xi^2}]$. Again, the two saturation processes can be expressed as the solution of the pair of generalized Ornstein-Uhlenbeck equations

$$dX_t^j = -\alpha_j X_t^j dt + dZ_t^j, \qquad X_0^j \text{ given, independent of } \{(Z_t^1, Z_t^2); t \ge 0\}, \quad j = 1, 2. \tag{8}$$

As in the previous section Z_t^j are two compound Poisson processes with jump sizes ξ_i^j, and jump points $\{T_i\}$. Also, $\alpha_j > 0$ for $j = 1, 2$.It is easy to see that, as $t \to \infty$, the solution of the pair of stochastic differential equations converges to a stationary random vector (X^1, X^2). The distribution of this random vector is given by the following

Proposition 2. *The joint Laplace transform of the stationary random vector* (X^1, X^2), $\Phi(s_1, s_2)$ $:= \mathbb{E}[e^{-s_1X^1 - s_2X^2}]$*, is given by*

$$\Phi(s_1, s_2) = \exp\left(-\lambda \int_0^\infty \left[1 - \phi(s_1e^{-\alpha_1 u}, s_2e^{-\alpha_2 u})\right] du\right) \tag{9}$$

Proof. Suppose that initially $X_0^1 = X_0^2 = 0$. Consider the time interval $[0, t]$ and suppose that there are $N_t := n$ Poisson points in this interval. If U_i, $i = 1, \ldots, n$ are the randomized occurrence times then these are uniformly distributed in $[0, t]$ and independent of each other. Thus $X_t^j = \sum_{i=1}^n \xi_i^j e^{-\alpha_j(t-U_i)}$ Using the above representation consider now the conditional expected value

$$\begin{aligned}
&\mathbb{E}[e^{-s_1X_t^1 - s_2X_t^2} \mid N_t = n, U_1, \ldots, U_n] \\
&\qquad = \mathbb{E}\left[\prod_{i=1}^n e^{-s_1\xi_i^1 e^{-\alpha_1(t-U_i)} - s_2\xi_i^2 e^{-\alpha_2(t-U_i)}} \mid N_t = n, U_1, \ldots, U_n\right] \\
&\qquad = \prod_{i=1}^n \phi(s_1e^{-\alpha_1(t-U_i)}, s_2e^{-\alpha_2(t-U_i)})
\end{aligned}$$

and

$$\mathbb{E}[e^{-s_1X_t^1 - s_2X_t^2} \mid N_t = n] = \left(\frac{1}{t}\int_0^t \phi(s_1e^{-\alpha_1 u}, s_2e^{-\alpha_2 u})du\right)^n. \tag{10}$$

Since $\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$, from the above (9) follows. $\square$

Equation (9) determines the stationary probability that a pulse is counted $\mathbb{P}(X^1 \le u, X^2 \le u)$ provided that the Laplace transform can be inverted numerically. A more explicit expression is possible in the case where (ξ^1, ξ^2) are independent and exponential with rates μ_1 and μ_2 respectively. To further simplify the expressions, we will examine the symmetric system with $\alpha_1 = \alpha_2 = \alpha$. In that case $\phi(s_1, s_2) = \frac{\mu^2}{(\mu+s_1)(\mu+s_2)}$ and substituting this in (9) we obtain the joint Laplace transform

$$\begin{aligned}\Phi(s_1,s_2) &= \exp\left(-\lambda\int_0^\infty\left[1-\frac{\mu_1}{\mu_1+s_1e^{-\alpha u}}\,\frac{\mu_2}{\mu_2+s_2e^{-\alpha u}}\right]du\right)\\ &= \exp\left(-\lambda\int_0^\infty\frac{\mu_1s_2e^{-\alpha u}+\mu_2s_1e^{-\alpha u}+s_1s_2e^{-2\alpha u}}{\left(\mu_1+s_1e^{-\alpha u}\right)\left(\mu_2+s_2e^{-\alpha u}\right)}du\right)\ .\end{aligned}$$

The change of variables $x=e^{-\alpha u}$, $dx=-\alpha e^{-\alpha u}du$, in the integral above gives

$$\begin{aligned}\Phi(s_1,s_2) &= \exp\left(-\lambda\int_0^\infty\frac{\mu_1s_2e^{-\alpha u}+\mu_2s_1e^{-\alpha u}+s_1s_2e^{-2\alpha u}}{\left(\mu_1+s_1e^{-\alpha u}\right)\left(\mu_2+s_2e^{-\alpha u}\right)}du\right)\\ &= \exp\left(\frac{\lambda}{\alpha}\int_1^0\frac{\mu_1s_2+\mu_2s_1+s_1s_2x}{\left(\mu_1+s_1x\right)\left(\mu_2+s_2x\right)}dx\right).\end{aligned}\tag{11}$$

Analysis into partial fractions gives

$$\frac{\mu_1s_2+\mu_2s_1+s_1s_2x}{\left(\mu_1+s_1x\right)\left(\mu_2+s_2x\right)} = \frac{A_1}{\mu_1+s_1x}+\frac{A_2}{\mu_2+s_sx}$$

with

$$A_1 = \frac{s_1^2\mu_2}{s_1\mu_2-\mu_1s_2},\qquad A_2 = -\frac{\mu_1s_2^2}{s_1\mu_2-\mu_1s_2}.$$

Hence, the Laplace transform is

$$\begin{aligned}\Phi(s_1,s_2) &= \exp\left(-\frac{\lambda}{\alpha}A_1\int_0^1\frac{dx}{\mu_1+s_1x}-\frac{\lambda}{\alpha}A_2\int_0^1\frac{dx}{\mu_2+s_2x}\right)\\ &= \exp\left(-\frac{\lambda}{\alpha}\,\frac{\mu_2s_1}{s_1\mu_2-\mu_1s_2}\ln\frac{\mu_1+s_1}{\mu_1}+\frac{\lambda}{\alpha}\,\frac{\mu_1s_2}{s_1\mu_2-\mu_1s_2}\ln\frac{\mu_2+s_2}{\mu_2}\right)\\ &= \left(\frac{\mu_1}{\mu_1+s_1}\right)^{\frac{\lambda}{\alpha}\frac{\mu_2s_1}{s_1\mu_2-\mu_1s_2}}\left(\frac{\mu_2}{\mu_2+s_2}\right)^{-\frac{\lambda}{\alpha}\frac{\mu_1s_2}{s_1\mu_2-\mu_1s_2}}.\end{aligned}\tag{12}$$

It should be noted that the Laplace transforms of the marginal distributions obtained by setting $s_1=0$ or $s_2=0$ are given by $\left(\frac{\mu_i}{\mu_i+s_i}\right)^{\lambda/\alpha}$, $i=1,2$, Thus (12) is the Laplace transform of a bivariate Gamma distribution with positive correlation. It is radically different from standard bivariate Gamma distributions such as the Kibble-Moran distribution [12].

4 Classical Type II Counters and $M/G/\infty$ systems in parallel

Consider now a system consisting of M infinite service systems in parallel. Customers arrive according to a Poisson process with rate λ. The arrival epoch of the ith customer is denoted by T_i and to the ith customer there corresponds a vector of service times $(\xi_i^1,\ldots,\xi_i^M)$.

Upon arrival the customer splits into M parts, each part joining the corresponding infinite server system. The jth component of the ith customer remains in system j for a period of time given by ξ_i^j. Thus the total time in the system for the ith customer is given by $\zeta_i := \max_{1\le j\le M} \xi_i^j$. The sequence of vectors $\{(\xi_i^1, \ldots, \xi_i^M)\}_{i=1,2,\ldots}$ is assumed to be i.i.d. and independent of the Poisson process $\{T_i\}$. The joint distribution of each ξ_i is denoted by $G(z_1, \ldots, z_M) =: \mathbb{P}\left(\xi^1 \le z_1, \ldots, \xi^M \le z_M\right)$.

The system can be analyzed by considering a Poisson process, N, on $\mathbb{R}_+^{M+1}$ with mean measure $\lambda dt \times G(dx_1, \ldots, dx_M)$. For $t > 0$ consider the sets

$$\begin{aligned} A_t &:= \{(u, x_1, \ldots, x_M) : 0 \le u \le t,\ x_j \ge 0,\ x_j + u < t,\ j = 1, \ldots, M\}, \\ C_t &:= \{(u, x_1, \ldots, x_M) : 0 \le u \le t,\ x_j \ge 0,\ j = 1, \ldots, M\}. \end{aligned}$$

The number of Poisson points in the set $C_t \setminus A_t$ is equal to the number of customers that are present in the system at time t because these points correspond to customers with at least some components that have not completed service. Thus, $X_t := N(C_t \setminus A_t)$ is the number of customers present in the system at time t, whereas $N(A_t)$ gives the number of departures in the time interval $(0, t]$.

Theorem 1. *Suppose the above system is empty at time* 0*. The probability that the* i*th customer, upon arrival finds the system empty,* $p_i := \mathbb{P}(X_{T_i-} = 0)$ *is given by the expression*

$$p_i = \frac{\lambda^i}{(i-1)!} \int_0^\infty \left(\int_0^t G(u, \ldots, u) du \right)^{i-1} e^{-\lambda t} dt. \tag{13}$$

If $g(z)$ *denotes the probability generating function of the number of customers in a busy period then*

$$g(z) = 1 - \left(\lambda \int_0^\infty e^{-\lambda t + \lambda z \int_0^t G(u,\ldots,u) du} dt \right)^{-1}. \tag{14}$$

Finally

$$\lim_{i\to\infty} p_i = g'(1) = e^{\lambda \int_0^\infty [1 - G(u,\ldots,u)] du}. \tag{15}$$

Proof. The ith customer arrives at time T_i to the system. Due to the Poisson assumption for arrivals T_i has Erlang-i density $f_i(t) = \frac{\lambda^i t^{i-1}}{(i-1)!} e^{-\lambda t}$. Due to the Poisson assumption for the arrivals (together with a *Lack of Anticipation* property which holds for this system - see for instance [23])

$$\mathbb{P}(X_{T_i-} = 0 | T_i = t) = \mathbb{P}(X_t = 0 | N(C_t) = i - 1)$$

Also, by the above definitions and the properties of the Poisson process [8]

$$\mathbb{P}(X_t = 0 | N(C_t) = i-1) = \mathbb{P}(N(A_t) = 0 | N(C_t) = i-1) = \left(\frac{1}{t} \int_0^t G(u, \ldots, u) du \right)^{i-1}.$$

Hence

$$p_i = \int_0^\infty \left(\frac{1}{t} \int_0^t G(u, \ldots, u) du \right)^{i-1} \frac{\lambda^i t^{i-1}}{(i-1)!} e^{-\lambda t} dt$$

which yields (13). Next, let Q_k denote the number of customers in the kth busy period with pgf $g(z) := \mathbb{E}[z_k^Q]$. Clearly, the number of customers in consecutive busy periods are independent and thus, setting $M_1 = 1$, $M_k := M_{k-1} + Q_{k-1}$, $k = 2, 3, \ldots$ we obtain a discrete time renewal process with points corresponding to customers who, upon arrival, find the system empty. Then $\{X_{T_i-} = 0\} = \{i = M_k \text{ for some } k \in \mathbb{N}\}$ and hence

$$\sum_{k=1}^{\infty} z^{M_k} = \sum_{i=1}^{\infty} \mathbf{1}(X_{T_i-} = 0) z^i.$$

Taking expectations in the above and using the fact that $\mathbb{E}[z^{1+Q_1+\cdots+Q_{k-1}}] = zg(z)^{k-1}$ and that $\mathbb{E}\mathbf{1}(X_{T_i-} = 0) = p_i$ we obtain

$$\sum_{k=1}^{\infty} zg(z)^{k-1} = \sum_{i=1}^{\infty} p_i z^i$$

Using equation (13) we obtain

$$\begin{aligned} \frac{z}{1-g(z)} &= \sum_{i=1}^{\infty} z^i \frac{\lambda^i}{(i-1)!} \int_0^{\infty} \left(\int_0^t G(u, \ldots, u) du \right)^{i-1} e^{-\lambda t} dt \\ &= \lambda z \int_0^{\infty} e^{-\lambda t + \lambda z \int_0^t G(u,\ldots,u) du} dt \end{aligned} \tag{16}$$

This establishes (14). Finally, from (14) it follows that the renewal process $\{M_k\}$ is aperiodic. We will next show that it is positive recurrent i.e. that $\mathbb{E}[Q_1] = g'(1) < \infty$. Indeed,

$$\begin{aligned} g'(1) &= \lim_{z \uparrow 1} \frac{1-g(z)}{1-z} = \left(\lim_{z \uparrow 1} \lambda(1-z) \int_0^{\infty} e^{-\lambda t(1-z) - \lambda z \int_0^t [1-G(u,\ldots,u)] du} dt \right)^{-1} && (17) \\ &= \left(\lim_{z \uparrow 1} \int_0^{\infty} e^{-x - \lambda z \int_0^{x/(\lambda(1-z))} [1-G(u,\ldots,u)] du} dx \right)^{-1} \quad (\text{with } x = \lambda(1-z)t) && (18) \end{aligned}$$

This last limit is easily evaluated using the monotone convergence theorem and thus we obtain

$$\begin{aligned} g'(1) &= \left(\int_0^{\infty} e^{-x - \lambda \int_0^{\infty} [1-G(u,\ldots,u)] du} dx \right)^{-1} = \left(e^{-\lambda \int_0^{\infty} [1-G(u,\ldots,u)] du} \int_0^{\infty} e^{-x} dx \right)^{-1} \\ &= e^{\lambda \int_0^{\infty} [1-G(u,\ldots,u)] du}. \end{aligned} \tag{19}$$

Hence $g'(1) < \infty$ and therefore the renewal process is positive recurrent. Thus the (Erdös-Feller-Pollard) renewal theorem and (19) show that (15) and positive recurrent. Hence the statement $\lim_{i\to\infty} p_i = g'(1)$ follows from the renewal theorem [1]. Note that a direct computation of the limit from (13) is often harder to carry out. □

For example if the service times are deterministic e.g. $\xi^j = a_j$, $j = 1, \ldots, M$, then $p_i = e^{-\lambda a}$ with $a = \max_{1 \le j \le M} a_j$ and $g(z) = \frac{ze^{-\lambda a}}{1-z(1-e^{-\lambda a})}$ (c.f. [20]).

5 Sensitivity Analysis via Malliavin Calculus on the Poisson Space

A detailed, rigorous account of the Malliavin calculus may be found in [9] and, in what regards functionals of a Poisson processes in particular in [14].

Let $T > 0$ and $\{N_t; t \in [0,T]\}$ a Poisson process with intensity $\lambda > 0$ and jump times $\{T_k\}_{k\geq 1}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. $\mathcal{S}_T$ denotes the set of all smooth Poisson functionals on $[0,T]$ of the form

$$F := f_0 \mathbf{1}(N_T = 0) + \sum_{n=1}^{m} \mathbf{1}(N_T = n) f_n(T_1, \ldots, T_n)$$

where $f_0 \in \mathbb{R}$ and $f_n \in \mathcal{C}^1[0,T]^n$. The functions f_n are assumed to be symmetric in the n variables. $\mathcal{S}_T$ is an algebra of random variables dense in $L^2(\Omega, \mathcal{F}_T, P)$ (where $\mathcal{F}_T$ is the σ–field generated by the process $\{N_t; t \in [0,T]\}$. (For a proof see [14] and [8].)

Suppose that F is a square integrable functional of the Poisson process $\{N_t, t \in [0,T]\}$, depending on a real parameter α in such a way that $\partial_\alpha F$ exists with probability 1 as an element of $L^2(\Omega, \mathcal{F}_T, P)$ and h is a bounded, measurable function. Thus $J(\alpha) := \mathbb{E}[h(F)]$ is a function of α. We are interested in obtaining a Monte-Carlo estimator for the sensitivity $J(\alpha)$ which does not rely on finite difference estimates since these are characterized by particularly poor statistical properties, namely high variance, bias, and convergence rates lower than the standard $N^{1/2}$ one expects in Monte-Carlo estimation. In order to achieve this goal we will use an integration-by-parts approach based on the Malliavin calculus for Poisson functionals which has been pioneered by Privault [14], [15]. This is made possible by obtaining a weight W_α such that

$$\frac{d}{d\alpha}\mathbb{E}[h(F)] \;=\; \mathbb{E}[W_\alpha h(F)]. \tag{20}$$

Thus if F_i, $W_{\alpha,i}$ are i.i.d. copies of these random variables, an efficient estimator for $J'(\alpha)$ is

$$\widehat{J'}(\alpha) := \frac{1}{N}\sum_{i=1}^{N} W_{\alpha,i} h(F_i)\ .$$

Denote by $\mathcal{C}_0[0,T]$ and $\mathcal{C}_0^1[0,T]$ the space of all continuous functions respectively continuously differentiable functions with $w : [0,T] \to \mathbb{R}$ with $w(0) = w(T) = 0$.

Definition 1. *For $w \in \mathcal{C}_0[0,T]$ denote by D_w the Malliavin directional derivative in the direction w for random variables in $\mathcal{S}_T$ which is defined by*

$$D_w F \;=\; -\sum_{n=1}^{m} \mathbf{1}(N_T = n) \sum_{i=1}^{n} w(T_i)\partial_i f_n(T_1, \ldots, T_n).$$

In the special case where the random variable F is given in terms of a stochastic integral with respect to the Poisson process, $F = \int_0^T g(t)dN_t$, where $g \in L^2[0,T]$ it holds that ([15])

$$
\begin{aligned}
D_w F &= D_w \sum_{n=1}^{\infty} \mathbf{1}(N_T = n) \sum_{i=1}^{n} g(T_i) = -\sum_{n=1}^{\infty} \mathbf{1}(N_T = n) \sum_{i=1}^{n} w(T_i) g'(T_i) \\
&= -\int_0^T w(t) g'(t) dN_t. \qquad (21)
\end{aligned}
$$

When $w \in C_0^1[0,T]$ the operator D_w is closable and admits a closable adjoint D_w^* such that

$$
\mathbb{E}[G D_w F] = \mathbb{E}[F D_w^* G], \qquad F, G \in \mathcal{S}_T. \qquad (22)
$$

For all $F \in \mathrm{Dom}(D_w) \cap L^4(\Omega)$, it holds that $F \in \mathrm{Dom}(D^*)$ and

$$
D_w^* F = F \int_0^T w'(t) dN_t - D_w F. \qquad (23)
$$

Theorem 2. *For any measurable function h for which $h(F)$ is square integrable, (20) holds with*

$$
W_\alpha = \frac{\partial_\alpha F}{D_w F} \left(\int_0^T w'(t) dN_t + \frac{D_w D_w F}{D_w F} \right) - \frac{D_w \partial_\alpha F}{D_w F}. \qquad (24)
$$

Proof. We will first establish (20) for infinitely differentiable functions with bounded derivatives of all orders, $h \in C_b^\infty$. Then

$$
\partial_\alpha \mathbb{E}[h(F)] = \mathbb{E}\left[h'(F) \partial_\alpha F\right] = \mathbb{E}\left[\frac{\partial_\alpha F}{D_w F} D_w h(F)\right], \qquad (25)
$$

the last equation following from the fact that $D_w h(F) = h'(F) D_w F$. However, from (22),

$$
\mathbb{E}\left[\frac{\partial_\alpha F}{D_w F} D_w h(F)\right] = \mathbb{E}\left[h(F)\, D^* \left(\frac{\partial_\alpha F}{D_w F}\right)\right]. \qquad (26)
$$

Expressing the adjoint of D_w in terms of Poisson stochastic integrals using (23) we obtain

$$
D^* \left(\frac{\partial_\alpha F}{D_w F}\right) = \frac{\partial_\alpha F}{D_w F} \int_0^T w'(t) dN_t - D_w \left(\frac{\partial_\alpha F}{D_w F}\right). \qquad (27)
$$

Equations (25), (26), (27) establish (20) with W given by (24) for smooth functions h. To establish the result for general functions h we need to use an approximation procedure considering a sequence h_n is C_b^∞ converging pointwise to h following ([15]). □

Consider now the shot noise counter discussed in section 2 with pulses of fixed size (say equal to 1) and let

$$
F = \int_0^T e^{-\alpha(T-t)} dN_t \qquad (28)
$$

denote the "saturation level" of the counter at time T. It is easy to see that $F \in L^2(\Omega)$. The parameter of interest with respect to which we want to estimate sensitivities is the decay rate α. Clearly

$$\partial_\alpha F \;=\; \int_0^T (T-t)e^{-\alpha(T-t)} dN_t \quad \text{w.p. 1.} \tag{29}$$

Also let w be an appropriate perturbation function vanishing at 0 and T. Then, using (21), we obtain

$$D_w F \;=\; -\int_0^T w(t)\alpha e^{-\alpha(T-t)} dN_t. \tag{30}$$

We also have

$$D_w D_w F \;=\; \int_0^T w(t)\alpha \left(w'(t) + \alpha w(t)\right) e^{-\alpha(T-t)} dN_t, \tag{31}$$

$$D_w \partial_\alpha F \;=\; \int_0^T w(t)(\alpha(T-t)-1)e^{-\alpha(T-t)} dN_t. \tag{32}$$

Then the following holds

Proposition 3. *The sensitivity of the probability that F exceeds the threshold u with respect to the decay rate α can be estimated using the estimator arising from*

$$\frac{d}{d\alpha}\mathbb{P}(F > u) \;=\; \mathbb{E}[W_\alpha \mathbf{1}(F > u)] \tag{33}$$

where the weight in the above equation is given by

$$\begin{aligned} W_\alpha \;=\; & -\frac{\int_0^T (T-t)e^{-\alpha(T-t)} dN_t}{\int_0^T w(t)\alpha e^{-\alpha(T-t)} dN_t} \left(\int_0^T w'(t) dN_t \;-\; \frac{\int_0^T w(t)\left(w'(t) + \alpha w(t)\right)\alpha e^{-\alpha(T-t)} dN_t}{\int_0^T w(t)\alpha e^{-\alpha(T-t)} dN_t} \right) \\ & -\frac{\int_0^T w(t)\left(\alpha(T-t)-1\right)\alpha e^{-\alpha(T-t)} dN_t}{\int_0^T w(t)\alpha e^{-\alpha(T-t)} dN_t}. \end{aligned} \tag{34}$$

Remark: Appropriate perturbation functions are for instance $w(t) = t(T-t)$, or $w(t) = \sin(\pi t/T)$.

6 Numerical Results

We simulate the shot noise counter with pulses of fixed size and consider the performance criterion $P(F > u)$ where F is defined in (28). We simulate the process with $\alpha = 0.5$, time horizon $T = 100$, Poisson pulses of unit rate $\lambda = 1$. We perform $N = 10^4$ independent replications and besides $P(F > u)$ we also estimate the gradient $\frac{\partial}{\partial\alpha}P(F > u)$ using the estimator $\frac{1}{N}\sum_{i=1}^N W_{\alpha,i}\mathbf{1}(F_i > u)$. The weight W_α in the above estimator is given by (34).

The first of the above estimators is obtained using the weight function $w(t) = t(T-t)$ and the second using $w(t) = \sin\frac{\pi t}{T}$.

u	0.4	0.6	0.8	1.0	1.2	1.4	1.6
Der. Est.	0.07372	-0.55850	-0.67975	-1.08215	-1.15916	-1.42970	-1.59342
st. dev.	0.11849	0.11732	0.11360	0.11011	0.10300	0.09863	0.09141
Der. Est 2	0.17883	-0.58503	-0.68806	-1.01166	-1.30899	-1.19217	-1.40720
st. dev.	0.14142	0.13549	0.13362	0.12981	0.12593	0.11559	0.11131
$E\mathbf{1}(F>u)$	0.97500	0.94320	0.90370	0.83850	0.77150	0.69750	0.60970
st. dev.	0.00156	0.00231	0.00295	0.00368	0.00420	0.00459	0.00488
u	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Der. Est.	-1.50717	-1.58173	-1.50320	-1.46432	-1.2361	-1.19499	-1.05961
st. dev.	0.08703	0.08036	0.07507	0.06808	0.06261	0.05702	0.05154
Der. Est 2	-1.53923	-1.51252	-1.58921	-1.39094	-1.23613	-1.0214	-1.03679
st. dev	0.10453	0.09399	0.09016	0.08111	0.07506	0.0654	0.05835
$E\mathbf{1}(F>u)$	0.53620	0.4488	0.3760	0.31200	0.25060	0.20420	0.16010
st. def.	0.00499	0.0050	0.0048	0.00463	0.00433	0.00403	0.00367
u	3.2	3.4	3.6	3.8	4.0	4.2	4.4
Der. Est.	-0.75207	-0.72924	-0.56096	-0.41407	-0.31755	-0.24445	-0.20168
st. dev.	0.04472	0.04102	0.03766	0.03106	0.02722	0.02353	0.02149
Der. Est. 2	-0.83723	-0.63393	-0.5857	-0.44039	-0.4077	-0.30665	-0.17447
st. dev.	0.05417	0.04479	0.0420	0.03682	0.0336	0.02960	0.02217
$E\mathbf{1}(F>u)$	0.11600	0.09140	0.0689	0.05140	0.03470	0.02750	0.01900
st. dev.	0.00320	0.00288	0.0025	0.00221	0.00183	0.00164	0.00137

Table 1: Derivative Estimate 1 refers to the weight $w(t) = t(T-t)$. Derivative Estimate 2 refers to the weight $w(t) = \sin(\pi t/T)$.

In Figures 1 and 2 we show simulation results for a shot-noise counter with exponential pulses. In this case $F = \sum_{i=1}^{N(T)} \xi_i e^{-\alpha(T-t_i)}$. Here $\{t_i\}$ is again a Poisson process with rate λ and $\{xi_i\}$ a sequence of i.i.d. exponentially distributed random variables with rate μ, independent of the Poisson process. The estimator weight can, in this case be written as

$$W_\alpha := \frac{\partial_\alpha F}{D_w F}\left(\sum_{i=1}^{N(T)} \xi_i w'(t_i) + \frac{D^2 w F}{D_w F}\right) - \frac{D_w \partial_\alpha F}{D_w F}$$

where

$$\partial_\alpha F = -\sum_{i=1}^{N(T)} \xi_i (T - t_i) e^{-\alpha(T-t_i)}, \qquad D_w F = -\sum_{i=1}^{N(T)} \xi_i \alpha w(t_i) e^{-\alpha(T-t_i)},$$

$$D_w \partial_\alpha F = \sum_{i=1}^{N(T)} \xi_i (\alpha(T - t_i) - 1) e^{-\alpha(T-t_i)}, \quad D_w^2 F = \sum_{i=1}^{N(T)} w(t_i) \xi_i \alpha \left[w'(t_i) + \alpha w(t_i)\right] e^{-\alpha(T-t_i)}.$$

Simulation experiments with $\lambda = \mu = 1$, $\alpha = 0.5$, and $T = 100$ are conducted to estimate the derivative $\partial_\alpha P(F > u)$ for various values of u. In Figure 1 results using the weight function $w(t) = t(T - t)$ are shown whereas Figure 2 shows the corresponding results for the weight function $w(t) = \sin(\pi t/T)$. In both cases the dots correspond to the exact value of the derivative computed numerically whereas the red and green lines respectively the estimated derivative values using the Malliavin calculus based estimator.

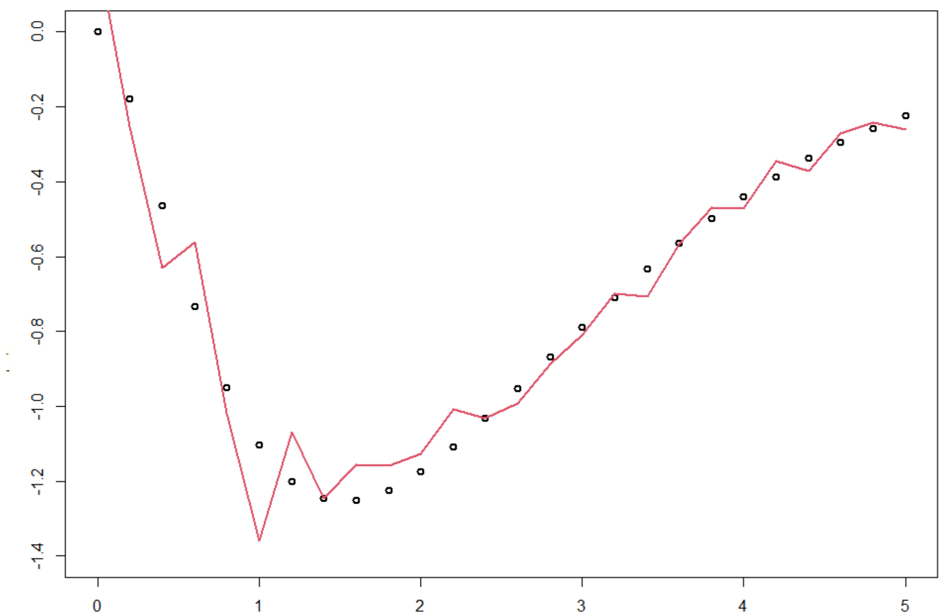

Figure 1: The first estimator $w(t) = t(T - t)$.

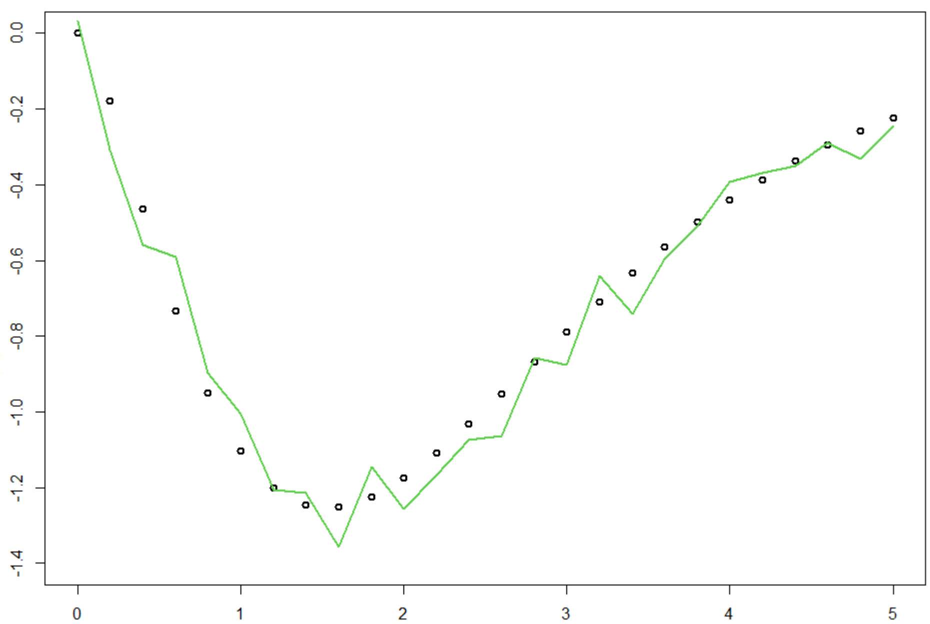

Figure 2: The second estimator $w(t) = \sin(\pi t/T)$.

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