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ANALYTICITY OF POISSON-DRIVEN STOCHASTIC SYSTEMS

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Abstract

Let ψ be a functional of the sample path of a stochastic system driven by a Poisson process with rate λ. It is shown in a very general setting that the expectation of $\psi, E_\lambda[\psi]$, is an analytic function of λ under certain moment conditions. Instead of following the straightforward approach of proving that derivatives of arbitrary order exist and that the Taylor series converges to the correct value, a novel approach consisting in a change of measure argument in conjunction with *absolute monotonicity* is used. Functionals of non-homogeneous Poisson processes and Wiener processes are also considered and applications to light traffic derivatives are briefly discussed.

POISSON PROCESSES; ANALYTIC FUNCTIONS; ABSOLUTELY MONOTONIC FUNCTIONS; QUEUEING SYSTEMS; LIGHT TRAFFIC DERIVATIVES

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1. Introduction

Quick inspection of transient and stationary expected performance indices for simple Markovian systems (such as the state probabilities for an $M/M/1$ queue) reveals that they are analytic functions of the rates of the Poisson processes that drive them. Specific examples abound but it is not easy to show that, in a certain sense, analyticity is a general property of Poisson-driven systems.

Suppose $\{X_t; 0 \leqq t \leqq t_0\}$ is a stochastic process and $\{N_t; 0 \leqq t \leqq t_0\}$ a Poisson process with rate λ which we assume to vary in some appropriate interval $[a, b)$. A finite-horizon performance index is a real, non-negative functional $\{(X_s, N_s); 0 \leqq s \leqq t_0\} \rightarrow \psi(\{(X_s, N_s); 0 \leqq s \leqq t_0\})$. For instance, X_t could be the workload in an $M/G/1$ queue at time t, N_t the (Poisson) arrival process and $\psi(\{X_s, N_s; 0 \leqq s \leqq t_0\}) = \sup_{s \in [0, t_0]} X_s$. As another example consider a network (open or closed) that includes an exponential server with rate μ. Let Q_t be the queue size at the server at time t. Then the departure counting process can be expressed in terms of a Poisson process N_t with rate μ as $D_t = \int_0^t \mathbf{1}_{(Q_s > 0)}\, dN_s$. Therefore, this would also be a Poisson-driven system and our analyticity conclusions hold as well.

While intuitively plausible, the analyticity of expected performance indices of systems driven by Poisson processes may be hard to establish, particularly in a

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non-Markovian context. A Cameron–Martin–Girsanov change of measure (see for example Liptser and Shiryaev (1977), Brémaud (1981)) can be used to show that $\boldsymbol{E}_\lambda[\psi]$ is a differentiable function of λ under certain regularity conditions. This approach was taken in Reiman and Weiss (1989) and Glynn (1987) for the purpose of developing derivative estimators in simulation. However, even if one imposes enough conditions on ψ to ensure that $\boldsymbol{E}_\lambda[\psi]$ is a C^∞ function in some interval, the problem of showing that the corresponding Taylor series converges to the correct value $\boldsymbol{E}_\lambda[\psi]$ in that interval is in general hard. Our approach circumvents these difficulties by showing that certain related functions are *absolutely monotonic* which, as it turns out, is much easier to establish.

Analyticity for Poisson-driven systems is of particular interest at $\lambda = 0$ since it allows one to apply without qualms the light traffic derivative formulas of Reiman and Simon (1989). While the change of measure approach used here requires that λ be strictly positive, we are able to establish analyticity at $\lambda = 0$ under certain conditions.

2. Absolutely monotonic functions

Absolutely monotonic functions are real, non-negative functions with non-negative derivatives of all orders. They were first studied by Bernstein (1914) who gave the following two *equivalent* definitions.

Definition (D). *A function* $f:\mathbb{R}\to\mathbb{R}^+$ *is* absolutely monotonic (D) *in* $[a, b)$ *iff it has derivatives of all orders that satisfy*

$$f^{(k)}(x) \geqq 0, \qquad x \in (a, b), \qquad k \in \mathbb{N}.$$

Now let $\Delta_h f(x) \equiv f(x+h) - f(x)$ and $\Delta_h^n f(x) \equiv \Delta(\Delta_h^{n-1} f(x))$, $n = 2, 3, \cdots$.

Definition (Δ). *A function* $f:\mathbb{R}\to\mathbb{R}^+$ is absolutely monotonic (Δ) *in* $[a, b)$ *iff*

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh) \geqq 0$$

for all non-negative integers n and for all x and h such that

$$a \leqq x < x+h < \cdots < x+nh < b.$$

Surprisingly, the above alternative definition of absolute monotonicity (which does not assume existence of derivatives at all!) turns out to be equivalent to the first.

Theorem 1 (Bernstein (1914): for a proof see Widder (1946)). *A function f is absolutely monotonic* (D) *in* $[a, b)$ *iff it is absolutely monotonic* (Δ) *there.*

The following theorem guarantees that a function satisfying either definition (Δ) or (D) is analytic. One need not worry about how fast the derivatives grow and whether the Taylor series converges.

Theorem 2 (Bernstein (1914): for a proof see Widder (1946)). *If* $f:\mathbb{R}\to\mathbb{R}^+$ *is absolutely monotonic* (D) *in an interval* $[a, b)$ *then it is analytic there and*

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(x-a)^n,\qquad a\leqq x<b.$$

Remark 1. An immediate consequence of the above theorem is that f can be extended analytically to the left of a since the above power series converges for $|x-a|<|b-a|$.

3. Analyticity of systems driven by Poisson inputs

Consider a stochastic system driven by a Poisson process $\{N_t\}_{t\geqq 0}$ with rate $\lambda\in[a, b)$, $a>0$, and a performance index ψ obtained from the sample path of the system. We will show that, under certain regularity conditions (which in practice can be restrictive), the expected performance measure $f(\lambda)=\boldsymbol{E}_\lambda[\psi]$ is an analytic function of λ in $[a, b)$. Here of course we assume that the performance measure ψ depends on the Poisson stream only through the number and occurrence times of the Poisson events and not directly (as a deterministic function) through λ.

More formally, let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{\boldsymbol{P}})$ be a filtered probability space. Let $(\Omega^N, \mathcal{F}^N, \{\mathcal{F}_t^N\})$ be a filtered space, N_t a point process adapted to $\mathcal{F}_t^N$, and $\boldsymbol{P}_\lambda^N$ $\lambda\in[a, b)$ a family of measures on $(\Omega^N, \mathcal{F}^N)$ such that N_t is a $(\boldsymbol{P}_\lambda, \mathcal{F}_t^N)$-Poisson process with intensity λ for all $\lambda\in[a, b)$. Consider the product space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \boldsymbol{P}_\lambda)$, where $\Omega=\tilde{\Omega}\times\Omega^N$, $\mathcal{F}=\tilde{\mathcal{F}}\times\mathcal{F}^N$, $\mathcal{F}_t=\tilde{\mathcal{F}}_t\times\mathcal{F}_t^N$, and $\boldsymbol{P}_\lambda=\tilde{\boldsymbol{P}}\times\boldsymbol{P}_\lambda^N$. Finally let T be an $\{\mathcal{F}_t\}$-stopping time and $\psi\in\mathcal{F}_T$. We assume that $\psi\geqq 0$ $\boldsymbol{P}_\lambda$-a.s. for all $\lambda\in[a, b)$. This restriction can be easily relaxed by considering separately positive and negative parts as usual. Throughout, $\boldsymbol{E}_\lambda$ denotes expectation with respect to $\boldsymbol{P}_\lambda$ and N_T the number of Poisson arrivals in $[0, T]$. We first establish our result for bounded stopping times.

Theorem 3. *Suppose that* T *is a bounded stopping time, i.e.* $T\leqq t_0$ $\boldsymbol{P}_\lambda$*-a.s. for all* $\lambda\in[a, b)$, $a>0$. *Then, if* $f(\lambda)=\boldsymbol{E}_\lambda[\psi]$ *is finite for all* $\lambda\in[a, b)$ *it is an analytic function of* λ *in the same interval.*

Proof. Let $\mathcal{F}_T=\{A\in\mathcal{F}:A\cap\{T\leqq t\}\in\mathcal{F}_t, t\geqq 0\}$ and denote by $\boldsymbol{P}_{\lambda,T}$ the restriction of $\boldsymbol{P}_\lambda$ on $\mathcal{F}_T$ for $\lambda\in[a, b)$. It is well known that $\boldsymbol{P}_{\lambda,T}$ is an equivalent family of measures on $\mathcal{F}_T$ and in particular that $\boldsymbol{P}_{\lambda,T}\ll\boldsymbol{P}_{a,T}$ with

$$\frac{d\boldsymbol{P}_{\lambda,T}}{d\boldsymbol{P}_{a,T}}=\left(\frac{\lambda}{a}\right)^{N_T}\exp(-T(\lambda-a)),\tag{1}$$

(see for example Brémaud (1981)). Using the Cameron–Martin–Girsanov change of measure given by (1) we have

$$f(\lambda)=\boldsymbol{E}_\lambda[\psi]=\boldsymbol{E}_a\left[\psi\frac{d\boldsymbol{P}_{\lambda,T}}{d\boldsymbol{P}_{a,T}}\right].\tag{2}$$

We can rewrite this expression using (1) as

$$f(\lambda) = \exp(-(\lambda - a)t_0)\boldsymbol{E}_a\left[\psi\left(\frac{\lambda}{a}\right)^{N_T} \exp((\lambda - a)(t_0 - T))\right].$$

Define the function

$$g(\lambda) = \boldsymbol{E}_a\left[\psi\left(\frac{\lambda}{a}\right)^{N_T} \exp((\lambda - a)(t_0 - T))\right].$$

We will show that $g(\lambda)$ is an absolutely monotonic function of λ in $[a, b)$ and therefore analytic there. It is enough to show that g satisfies definition (Δ). Indeed,

$$\Delta_h^n g(\lambda) = \boldsymbol{E}_a\left[\psi \Delta_h^n\left(\frac{\lambda}{a}\right)^{N_T} \exp((\lambda - a)(t_0 - T))\right], \tag{3}$$

since we have assumed that ψ does not depend explicitly on λ, and it is enough to show that $\Delta_h^n g(\lambda) \geqq 0$. For any ω, the function $(\lambda/a)^{N_T(\omega)} \exp((\lambda - a)(t_0 - T(\omega)))$ is absolutely monotonic (D) since it has non-negative derivatives of any order with respect to λ. Therefore, by Theorem 2, it is also absolutely monotonic (Δ) on $[a, b)$, and in view of (3) this immediately implies that $\Delta_h^n g(\lambda) \geqq 0$ and hence that $g(\lambda)$ is absolutely monotonic and analytic on $[a, b)$. The analyticity of $f(\lambda)$ follows immediately from the fact that it is the product of two analytic functions.

Remark 2. Note that in the above theorem it is crucial to assume that a be strictly positive in order to have $\boldsymbol{P}_{\lambda,T} \ll \boldsymbol{P}_{a,T}$.

Corollary 1. *Under the assumptions of Theorem* 3 *and the additional assumption* $b > 2a$, $f(\lambda)$ *is analytic at* $\lambda = 0$ *and the Taylor expansion*

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\lambda^n, \qquad 0 \leqq \lambda < \rho,$$

has radius of convergence ρ *no smaller than* $b - 2a$.

Proof. From Theorem 3, $f(\lambda) = \exp(-(\lambda - a)t_0)g(\lambda)$, where $g(\lambda)$ is absolutely monotonic in $[a, b)$. From Theorem 2, the Taylor series $\sum_{n=0}^{\infty} (1/n!)g^{(n)}(a)(\lambda - a)^n$ converges for $\lambda \in [a, b)$ which implies that it will converge in $|\lambda - a| < b - a$. Therefore, if $b > 2a$, g is analytic at 0 and the radius of convergence of the Taylor expansion there is at least equal to $b - 2a$. The fact that $\exp(-\lambda t_0 - at_0)$ is analytic with infinite radius of convergence there completes the proof.

Remark 3. Analyticity at $\lambda = 0$ is of great interest in applications to queueing systems. Reiman and Simon (1988), (1989) developed a method for computing derivatives of any order in light traffic (i.e. at $\lambda = 0$). Corollary 1 provides some justification for using a Taylor polynomial to extrapolate the performance of such systems for $\lambda > 0$. Specific conditions for doing so, as well as the radius of convergence of such expansions, will be investigated in a future paper.

The above theorem asserts that finite-horizon performance indices of Poisson-driven systems are analytic. Establishing analyticity for the steady-state case is much harder in general, and we will do so only for regenerative systems. The first step towards that is to extend the results of Theorem 3 to stopping times that are finite with probability 1 but not necessarily bounded. In that case stronger assumptions are needed.

We start with a preliminary lemma of independent interest which provides a bound for the tails of N_T. No assumptions are made on the stopping time other than the requirement of having exponentially bounded tails.

Lemma 1. *With the above notation assume that* $\boldsymbol{P}(T>u)\leqq A\exp(-su)$ *for some* $A>0$ *and* $s>0$. *Then* $\boldsymbol{P}(N_T/\lambda>u)\leqq B\exp(-u\lambda s/(2\lambda+s))$, *for some* $B>0$.

Proof. Let $x\in(0,1)$. Then,

$$\begin{aligned}\{N_T/\lambda>u\}&=\{N_T/\lambda>u;\,T\leqq ux\}\cup\{N_T/\lambda>u;\,T>ux\}\\&\subseteq\{N_{ux}>\lambda u;\,T\leqq ux\}\cup\{T>ux\}\\&\subseteq\{N_{ux}>\lambda u\}\cup\{T>ux\}\end{aligned}$$

from which follows that

$$\boldsymbol{P}(N_T/\lambda>u)\leqq\boldsymbol{P}(N_{ux}>\lambda u)+\boldsymbol{P}(T>ux).\tag{4}$$

From assumptions of the lemma we have

$$\boldsymbol{P}(T>ux)\leqq A\exp(-sux).\tag{5}$$

On the other hand, since N_{ux} is the number of Poisson arrivals in $[0,ux]$, if we denote by $\lceil\lambda u\rceil$ the smallest integer greater than or equal to λu, we have

$$\boldsymbol{P}(N_{ux}>\lambda u)=\exp(-\lambda ux)\sum_{j=\lceil\lambda u\rceil}^{\infty}\frac{(\lambda ux)^j}{j!}\leqq\exp(-\lambda ux)\frac{(\lambda ux)^{\lceil\lambda u\rceil}}{\lceil\lambda u\rceil!}\sum_{j=0}^{\infty}\left(\frac{\lambda ux}{\lceil\lambda u\rceil}\right)^j.\tag{6}$$

Observe that

$$\sum_{j=0}^{\infty}\left(\frac{\lambda ux}{\lceil\lambda u\rceil}\right)^j\leqq(1-x)^{-1}.$$

From Stirling's formula follows that for some $A_1>0$, $m!\geqq A_1\exp((m+\frac{1}{2})\log m-m)$ for any integer m. Using these bounds together with (6) and the inequality $\lambda u\leqq\lceil\lambda u\rceil$ we get

$$\begin{aligned}\boldsymbol{P}(N_{ux}>\lambda u)&\leqq\frac{A_1}{1-x}\exp\{-\lambda ux+\lceil\lambda u\rceil(\log x+\log\lceil\lambda u\rceil)\\&\quad-(\lceil\lambda u\rceil+\tfrac{1}{2})\log\lceil\lambda u\rceil+\lceil\lambda u\rceil\}\\&\leqq\frac{A_1}{1-x}\exp\{\lambda u(1-x)+\lambda u\log x\}.\end{aligned}\tag{7}$$

From (4), (5), and (7) it follows that

$$(8)\qquad \boldsymbol{P}(N_T/\lambda > u) \leqq A \exp(-usx) + \frac{A_1}{(1-x)} \exp(-\lambda u(x - 1 - \log x)).$$

Since x was arbitrary in $(0, 1)$ we can choose it to be the unique solution x^* in $(0, 1)$ of the equation $sx = \lambda(x - 1 - \log x)$. ($\lambda(x - 1 - \log x)$ is decreasing convex and maps $(0, 1)$ onto $(0, \infty)$.) Notice also that $x^* = \sup_{x \in (0,1)} \min\{sx, \lambda(x - 1 - \log x)\}$. Hence (8) becomes

$$\boldsymbol{P}(N_T/\lambda > u) \leqq B \exp(-usx^*),$$

where B is a positive constant. A more convenient expression for our purposes can be obtained by using the inequality $x^* \leqq (2 + s/\lambda)^{-1}$ which follows from the fact that $x^* = \sup\{x : sx \leqq \lambda(x - 1 - \log x)\}$ and the easily verified inequality $s/(2 + s/\lambda) \leqq \lambda((2 + s\lambda)^{-1} - 1 - \log(2 + s/\lambda)^{-1}$. Therefore

$$\boldsymbol{P}(N_T/\lambda > u) \leqq B \exp(-us\lambda/(2\lambda + s)).$$

We next prove a lemma that will be useful in what follows.

Lemma 2. *Assume that* $\boldsymbol{E}_\lambda[\psi^4] < \infty$ *and that* $\boldsymbol{E}_\lambda[\exp(sT)] < \infty$ *for some* $s \in (0, \lambda)$. *Then for any* ε *such that* $0 < \varepsilon < s/6$, $\boldsymbol{E}_\lambda[\psi(1 + \varepsilon/\lambda)^{N_T} \exp(\varepsilon T)] < \infty$.

Proof. Using twice the Cauchy–Schwarz inequality together with the elementary inequality $(1 + \varepsilon/\lambda)^{N_T} \leqq \exp(\varepsilon N_T/\lambda)$, we obtain

$$(9)\qquad \boldsymbol{E}_\lambda\left[\psi\left(\frac{\lambda + \varepsilon}{\lambda}\right)^{N_T} \exp(\varepsilon T)\right] \leqq \boldsymbol{E}_\lambda^{\frac{1}{4}}[\psi^4]\boldsymbol{E}_\lambda^{\frac{1}{4}}[\exp(4\varepsilon T)]\boldsymbol{E}_\lambda^{\frac{1}{2}}[\exp(2\varepsilon N_T/\lambda)].$$

In view of the assumptions of the lemma, the first two terms in the right-hand side of (9) are finite for $\varepsilon < s/6$ and it is enough to show that $\boldsymbol{E}_\lambda[\exp(2\varepsilon N_T/\lambda)]$ is finite. In fact, it is enough to show that $\boldsymbol{P}_\lambda(N_T/\lambda > u) \leqq B \exp(-us/3)$ for some $B > 0$. But our assumption that $\boldsymbol{E}_\lambda[\exp(sT)] < \infty$ implies that $\boldsymbol{P}(T > u) < A \exp(-us)$ for some $A > 0$, and from Lemma 1 it follows that

$$\boldsymbol{P}_\lambda(N_T/\lambda > u) \leqq B \exp\left(-u \frac{s}{2 + s/3\lambda}\right) \leqq B \exp(-us/3),$$

which implies that $\boldsymbol{E}_\lambda[\exp(2\varepsilon N_T/\lambda)] < \infty$, in view of the fact that $\varepsilon < s/6$.

We now are ready to state our main result.

Theorem 4. *Assume that* T *is an* $\mathcal{F}_t$*-stopping time that is finite* P_λ*-a.s. for* $\lambda \in [a, c)$, $a > 0$, *and suppose that for some* $s \in (0, a)$, $\boldsymbol{E}_a[\psi^4]$ *and* $\boldsymbol{E}_a[\exp(sT)]$ *are finite. Then* $f(\lambda)$ *is analytic on* $[a, b)$, *where* $b \leqq \min(c, a + s/6)$.

Proof. Since $T < \infty$ P_λ-a.s. and P_a-a.s., $P_{\lambda,T} \ll P_{a,T}$ with

$$\frac{dP_{\lambda,T}}{dP_{a,T}} = \left(\frac{\lambda}{a}\right)^{N_T} \exp(-(\lambda - a)T).$$

Use the Cameron–Girsanov–Martin change of measure as in the proof of Theorem 3 to write

$$f(\lambda) = \boldsymbol{E}_\lambda[\psi] = \boldsymbol{E}_a\left[\psi\left(\frac{\lambda}{a}\right)^{N_T} \exp\left(-(\lambda - a)T\right)\right].$$

Consider now the function

$$g_1(\lambda) = \boldsymbol{E}_a\left[\psi\left(\frac{\lambda}{a}\right)^{N_T} \exp\left((\lambda - a)T\right)\right].$$

This is finite in $[a, b)$ in view of our assumptions and Lemma 2. We show that $g_1(\lambda)$ is an absolutely monotonic function of λ in $[a, b)$ and therefore analytic there. It is enough to show that g_1 satisfies definition (Δ), i.e. that $\Delta_h^n g_1(\lambda) \geqq 0$. Indeed,

$$\Delta_h^n g_1(\lambda) = \boldsymbol{E}_a\left[\psi \Delta_h^n\left(\frac{\lambda}{a}\right)^{N_T} \exp\left((\lambda - a)T\right)\right] \tag{10}$$

since we have assumed that ψ does not depend explicitly on λ, and it is enough to show that $\Delta_h^n(\lambda/a)^{N_T} \exp((\lambda - a)T) \geqq 0$. For any ω, the function $(\lambda/a)^{N_T(\omega)} \exp((\lambda - a)T(\omega))$ is absolutely monotonic (D) since it has non-negative derivatives of any order with respect to λ. Therefore, by Theorem 1 it is also absolutely monotonic (Δ) on $[a, b)$, and, in view of (10) this immediately implies that $\Delta_h^n g_1(\lambda) \geqq 0$, and hence that $g_1(\lambda)$ is absolutely monotonic and analytic on $[a, b)$.

Now let $g_2(\lambda) = f(\lambda) + g_1(\lambda)$. Then

$$\begin{aligned} g_2(\lambda) &= \boldsymbol{E}_a\left\{\psi\left(\frac{\lambda}{a}\right)^{N_T} \left[\exp\left(-(\lambda - a)T\right) + \exp\left((\lambda - a)T\right)\right]\right\} \\ &= 2\boldsymbol{E}_a\left[\psi\left(\frac{\lambda}{a}\right)^{N_T} \cosh\,(\lambda - a)T\right]. \end{aligned}$$

Again, one sees that $(\lambda/a)^{N_T(\omega)} \cosh\left[(\lambda - a)T(\omega)\right]$ is an absolutely monotonic function of λ since all derivatives with respect to λ are non-negative. From the above and Theorem 2 it follows that $g_2(\lambda)$ is analytic on $[a, b)$. Since $f(\lambda) = g_1(\lambda) - g_2(\lambda)$ is the difference of two analytic functions on $[a, b)$, it is analytic itself there.

Remark 4. In the above proof we take advantage of the two alternative definitions (D) and (Δ) for absolute monotonicity in order to avoid proving that the derivatives

$$\frac{d^n}{d\lambda^n} \boldsymbol{E}_a\left[\psi\left(\frac{\lambda}{a}\right)^{N_T} \exp\left(-(\lambda - a)T\right)\right], \qquad n = 1, 2, \cdots$$

exist and that the corresponding Taylor series converges to $f(\lambda)$.

Remark 5. The extension to ψ that are not necessarily non-negative follows easily by decomposing ψ into positive and negative parts.

Corollary 2. *Suppose that for all* $\lambda \in [a, b)$ *with* $a > 0$, $\boldsymbol{E}_\lambda[\psi^4] < \infty$ *and there exists* $s > 0$ (*possibly depending on* λ) *such that* $\boldsymbol{E}_\lambda[\exp(sT)] < \infty$. *Then* $f(\lambda)$ *is analytic on* $[a, b)$.

Proof. Fix $\delta \in (0, b-a)$. From Theorem 4, for any rational point r_i in the interval $[a, b)$ there exists $s(r_i) > 0$ such that $f(\lambda)$ is analytic in the open interval $(r_i, r_i + \varepsilon(r_i))$ with $\varepsilon(r_i) < \min(b, r_i + s(r_i)/6)$. This collection of open intervals constitutes an open cover for the closed interval $[a+\delta, b-\delta]$ and by the Heine–Borel theorem there exists a finite subcover of open intervals, $(r_j, r_j + \varepsilon(r_j))$, $j = 1, 2, \cdots, m$. A straightforward analytic continuation procedure shows that f is analytic on $[a+\delta, b-\delta]$. Since δ was arbitrary, f is analytic on (a, b). Using Theorem 4 once more at $\lambda = a$ concludes the proof.

4. Steady-state results in a regenerative framework

Consider now in the above framework a stochastic process $\{X_t; t \geqq 0\}$ and a renewal process $\{S_n; n = 0, 1, 2 \cdots\}$, both adapted to $\mathscr{F}_t$, and suppose that X_t is regenerative with respect to $\{S_n\}$. Let $S_0 = 0$, $S_1 = T$, and $\psi = \int_0^T \phi(X_s)\, ds$ with $\phi: \mathbb{R} \to \mathbb{R}^+$. Suppose the family of distributions $\boldsymbol{P}_\lambda(T \leqq x)$ is spread-out, $0 < \boldsymbol{E}_\lambda[T] < \infty$, and $\boldsymbol{E}_\lambda \int_0^T \phi(X_s)\, ds < \infty$ for all $\lambda \in [a, b)$. It is well known that under these assumptions $X_t \xrightarrow{D} X_\infty$ and $\boldsymbol{E}_\lambda[\phi(X_\infty)] = \boldsymbol{E}_\lambda[\psi]/\boldsymbol{E}_\lambda[T]$.

If, in addition to the above assumptions, for all $\lambda \in [a, b)$ and for some $s > 0$ (possibly depending on λ) $\boldsymbol{E}_\lambda[\exp(sT)] < \infty$ and $\boldsymbol{E}_\lambda[\psi^4] < \infty$, then $f(\lambda) = \boldsymbol{E}_\lambda \phi(X_\infty)$ is analytic in $[a, b)$. Of particular interest is the case $\phi(X_s) = \mathbf{1}_{(X_s \in B)}$ for some Borel set B. Then it is easily seen that $\boldsymbol{E}_\lambda[e^{sT}] < \infty$ for all $\lambda \in [a, b)$ is a sufficient condition for the analyticity of $\boldsymbol{P}_\lambda[X_\infty \in B)$ on $[a, b)$.

5. Non-homogeneous Poisson processes

We use the same framework as in Section 3, except for the fact that here $\{N_s; 0 \leqq s \leqq t\}$ is a non-homogeneous Poisson process with rate $\lambda(s, \theta)$. We assume that for all s in $[0, t]$ $\lambda(s, \theta)$ is an absolutely monotonic, strictly positive function of θ. This includes multiplicative (i.e. $\lambda(s, \theta) = \theta\lambda(s)$ with $\lambda(s) > 0$ for $s \in [0, t]$) and additive (i.e. $\lambda(s, \theta) = \theta + \lambda(s)$) parameters. Assume that $\theta \in [a, b)$ and that $f(\theta) = \boldsymbol{E}_\theta[\psi] < \infty$ in $[a, b)$. We conclude again that $f(\theta)$ is analytic on $[a, b)$. The argument is identical to the proof of Theorem 3:

$$(11) \quad f(\theta) = \boldsymbol{E}_a\left[\psi \frac{dP_{\theta,t}}{dP_{a,t}}\right] = \boldsymbol{E}_a\left[\psi \prod_{i=1}^{N_t} \frac{\lambda(T_i, \theta)}{\lambda(T_i, a)}\right] \exp\left(-\int_0^t [\lambda(s, \theta) - \lambda(s, a)]\, ds\right)$$

where N_t is the total number of Poisson points in $[0, t]$ and T_i, $i = 1, 2, \cdots, N_t$ the epochs of their occurrence. If $N_t = 0$ then the product in (11) is defined to equal 1.

For any realization, $\prod_{i=1}^{N_t} (\lambda(T_i, \theta)/\lambda(T_i, a))$ is an absolutely monotonic function of θ since it is the product of absolutely monotonic functions (Widder (1946)). The analyticity of $f(\theta)$ follows from the arguments used in Theorem 3.

6. Functionals of the Wiener process

The above approach can be used to establish the analyticity of functionals of the Wiener process. In this section we examine the simplest case. Let $\mathcal{W} = \{W_s; 0 \leqq s \leqq t\}$ be a Wiener process on $(\Omega, \mathcal{F}, \boldsymbol{P}_0)$ with drift 0 and variance σ^2, and $\mathcal{W} \to \psi(\mathcal{W})$ a real, non-negative functional on the paths $\mathcal{W}$. Assume that $\boldsymbol{P}_0(W_0 = 0) = 1$ and define a family of equivalent probability measures $\boldsymbol{P}_\mu$, $\mu \in [a, b)$, on $\mathcal{F}$ by

$$\frac{d\boldsymbol{P}_\mu}{d\boldsymbol{P}_0} = \exp(W_t\mu/\sigma^2 - \tfrac{1}{2}(\mu/\sigma)^2 t). \tag{12}$$

It is well known (see for example Liptser and Shiryaev (1977), Wong and Hajek (1986)) that under $\boldsymbol{P}_\mu$, $\mathcal{W}$ is a Wiener process with drift μ and variance σ^2. We assume that ψ depends on μ only through the sample path $\mathcal{W}$ and not explicitly as a function of μ. An argument similar to the proof of Theorem 3 can now be used to show that $f(\mu) = \boldsymbol{E}_\mu[\psi]$ is analytic on $[a, b)$, where $\boldsymbol{E}_\mu$ denotes expectation with respect to $\boldsymbol{P}_\mu$.

Theorem 5. *Assume that $\boldsymbol{E}_\mu[\psi] < \infty$ for all $\mu \in [a, b)$. Then $f(\mu)$ is analytic on $[a, b)$.*

Proof. Let $-\mathcal{W} = \{-W_s; 0 \leqq s \leqq t\}$ and define the functionals ϕ_1 and ϕ_2 by

$$\phi_1(\mathcal{W}) = \tfrac{1}{2}[\psi(\mathcal{W}) + \psi(-\mathcal{W})], \qquad \phi_2(\mathcal{W}) = \tfrac{1}{2}[\psi(\mathcal{W}) - \psi(-\mathcal{W})].$$

Define the functions $f_i(\mu) = \boldsymbol{E}_\mu[\phi_i]$, $i = 1, 2$, which are finite on $[a, b]$. We have

$$f_i(\mu) = \boldsymbol{E}_0[\phi_i(\mathcal{W}) \exp(W_t\mu/\sigma^2)] \exp(-\tfrac{1}{2}(\mu/\sigma)^2 t) \qquad i = 1, 2. \tag{13}$$

Since $\phi_1(\mathcal{W}) = \phi_1(-\mathcal{W})$, we also have

$$\begin{aligned} f_1(\mu) = \boldsymbol{E}_\mu[\phi_1(-\mathcal{W})] &= \boldsymbol{E}_0[\phi_1(-\mathcal{W}) \exp(-W_t\mu/\sigma^2)] \exp(-\tfrac{1}{2}(\mu/\sigma)^2 t) \\ &= \boldsymbol{E}_0[\phi_1(\mathcal{W}) \exp(-W_t\mu/\sigma^2)] \exp(-\tfrac{1}{2}(\mu/\sigma)^2 t). \end{aligned} \tag{14}$$

Similarly, from $\phi_2(\mathcal{W}) = -\phi_2(\mathcal{W})$ it follows that

$$f_2(\mu) = -\boldsymbol{E}_0[\phi_2(\mathcal{W}) \exp(-W_t\mu/\sigma^2)] \exp(-\tfrac{1}{2}(\mu/\sigma)^2 t). \tag{15}$$

From (13) and (14) it follows that

$$f_1(\mu) = \boldsymbol{E}_0[\phi_1(\mathcal{W}) \cosh(W_t\mu/\sigma^2)] \exp(-\tfrac{1}{2}(\mu/\sigma)^2 t). \tag{16}$$

Since $\cosh(W_t\mu/\sigma^2)$ is an absolutely monotonic function of μ for all ω, using the same argument as in the proof of Theorem 3 we conclude that $\boldsymbol{E}_0[\phi_1(\mathcal{W}) \cosh(W_t\mu/\sigma^2)]$ is absolutely monotonic and therefore analytic in $[a, b)$.

From (16) then follows that $f_1(\mu)$ is analytic in the same interval. Similarly from (13) and (15) we obtain

$$f_2(\mu) = \boldsymbol{E}_0[\phi_2(\boldsymbol{W}) \sinh(W_t\mu/\sigma^2)] \exp(-\tfrac{1}{2}(\mu/\sigma)^2 t), \tag{17}$$

which is also seen to be analytic because $\sinh(W_t\mu/\sigma^2)$ is an absolutely monotonic function of μ for all ω. Since $f(\mu) = f_1(\mu) + f_2(\mu)$, this concludes the proof.

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References

Bernstein, S. (1914) Sur la définition et les propriétés des fonctions analytiques d'une variable réelle. *Math. Ann.* **75,** 449–468.

Brémaud, P. (1981) *Point Processes and Queues.* Springer-Verlag, New York.

Glynn, P. W. (1987) Likelihood ratio gradient estimation: an overview. *Proc. Winter Simulation Conf.,* ed. A. Thesen, H. Grant and W.D. Kelton, pp. 366–374.

Lipster, E. and Shiryaev, B. (1977) *Statistics of Stochastic Processes,* Vol. I. Springer-Verlag, New York.

Reiman, M. and Simon, B. (1988) Light traffic limits of sojourn time distributions in Markovian queueing networks. *Commun. Statist.—Stoch. Models* **4,** 191–233.

Reiman, M. and Simon, B. (1989) Open queueing systems in light traffic. *Math. Operat. Res.* **14,** 26–59.

Reiman, M. and Weiss, A. (1989) Simulation via likelihood ratios. *Operat. Res.* **37,** 830–846.

Widder, D. V. (1946) *The Laplace Transform.* Princeton University Press.

Wong, E. and Hajek, B. (1986) *Stochastic Processes in Engineering Systems.* Springer-Verlag, New York.