Discrete Time Risk Processes with After-Effects and Association

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Abstract

We present a simple discrete–time model of a risk process in which primary claims are followed by secondary claims representing after-effects. It is shown that the resulting discrete–time process is *associated*. Estimates for finite and infinite horizon ruin probabilities are then obtained via a diffusion approximation that is based on the classic Functional Central Limit Theorem of Newman and Wright (1981) for sequences of associated random variables.

1 Introduction

We consider a discrete time insurance risk process which models claims with significant after–effects. It is assumed that the occurrence of certain events, besides possibly causing an immediate claim, also generates a stream of secondary claims which occur at subsequent time periods and which are considered after–effects of the original event (or, equivalently, of the original claim).

More specifically, the claim process consists of two types of claims with different statistical characteristics, *primary* and *secondary claims*. It is assumed that primary claims occur independently and in each time period we have a random number of i.i.d. primary claims. A primary claim that occurs at time n may trigger the occurrence of a stream of secondary claims at times $n+1, n+2, \ldots, n+k, \ldots$. These secondary claims are assumed to be *conditionally independent* random variables, given the size of the primary claim that triggers them. Streams of secondary claims triggered by different primary claims are also assumed to be independent. During each period, the increase of the free reserves of the company due to premium income is constant. Our objective is to obtain diffusion approximations for the finite and infinite horizon ruin probability.

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The model proposed can be thought of loosely as a simple, discrete time version of the corresponding continuous time models which are typically used to describe events with after–effects or events whose effect is spread out over a period of time. Such continuous time processes include Poisson shot–noise processes and point processes with secondary points such as the Neyman–Scott and the Bartlett–Lewis models (see e.g. Daley and Vere–Jones, 2003). The proposed risk model, as it will become clear when we give a more detailed description, is not easy to analyze directly since, under the above assumptions, is not in general markovian. However, the total amount of claims in each period turns out to be an *associated* sequence of random variables. Thus we can use the Functional Central Limit Theorem (FCLT) for associated random variables due to Newman and Wright (1981) discussed in the next section in order to obtain a diffusion approximation for the ruin probability in the model we propose.

We first begin with a definition of the concept of association and give a brief overview of the relevant literature.

2 Associated Random Variables and the Central Limit Theorem

Association is a type of positive dependence between random variables.

Definition 1 *A finite collection of random variables $\mathbf{Y} = (Y_1, \ldots, Y_m)$, is said to be* associated *if, for any pair of coordinate-wise nondecreasing real functions h_1, h_2 on $\mathbb{R}^m$ such that $h_j(\mathbf{Y})$, $j = 1, 2$, has finite second moment,*

$$\mathsf{Cov}(h_1(\mathbf{Y}), h_2(\mathbf{Y})) \geq 0.$$

A countable collection of random variables $\{Y_k; k \in \mathbb{N}\}$ is said to be associated if every finite subcollection is associated.

The above definition of association was given in Esary, Proschan, and Walkup (1967). These authors were motivated by considerations in statistics and reliability theory. The same ideas arose also in the context of percolation models and statistical mechanics with the work of Harris (1960) and the seminal paper of Fortuin, Kasteleyn, and Ginibre (1971) which considered covariance inequalities of random variables defined on lattices. These became known as the FKG inequalities. For an overview of association we refer the reader to Szekli (1995).

Note that in the above definition, it suffices to consider functions h_j that are coordinate–wise increasing, *bounded* and *continuous* as one can see by means of a standard approximation argument. We next state three theorems that will be used repeatedly in what follows.

Theorem 2 *(i) If $\mathbf{X}$ and $\mathbf{Y}$ are associated random vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and mutually independent then $(\mathbf{X}, \mathbf{Y})$, considered as a random vector of $\mathbb{R}^{n+m}$, is associated.*

(ii) If $\mathbf{X}$ *is an associated random vector of* $\mathbb{R}^n$ *and* $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ *are increasing (decreasing) then* $(f_1(\mathbf{X}), \ldots, f_m(\mathbf{X}))$ *is associated.*

(This is Theorem 3.10 of Müller and Stoyan, 2002).

Theorem 3 *Suppose that* $\{\mathbf{Y}^{(k)}; k \in \mathbb{N}\}$ *is a sequence of random elements of* $\mathbb{R}^m$ *such that, for each* k, $\mathbf{Y}^{(k)} = (Y_1^{(k)}, \ldots, Y_m^{(k)})$, *is associated and* $\mathbf{Y}^{(k)} \stackrel{d}{\to} \mathbf{Y}$ *as* $k \to \infty$. *Then* $\mathbf{Y}$ *is associated.*

(The proof of this theorem can be found in Esary, Proschan, and Walkup, 1967.)

Theorem 4 *Let* $\{\mathbf{Y}^{(k)}; k \in \mathbb{N}\}$ *be a sequence of independent random elements of* $\mathbb{R}^m$ *such that, for each* k, $\mathbf{Y}^{(k)} := (Y_1^{(k)}, \ldots, Y_m^{(k)})$ *is associated. Then*

(i) $\mathbf{S}_n := \sum_{k=1}^{n} \mathbf{Y}^{(k)}$ *is associated for each* n.

(ii) Suppose that N *is a random variable with values in* $\mathbb{N}$, *independent of the sequence* $\{\mathbf{Y}^{(k)}\}$, *and* $\mathbf{S}_n = \sum_{k=1}^{N} \mathbf{Y}^{(k)}$. *Under the additional assumption that* $Y_i^{(k)} \geq 0$ *w.p. 1 for all* i, k, $\mathbf{S}_n$ *is associated.*

Proof: **(i)** In view of Theorem 2 *(i)* $(\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \ldots, \mathbf{Y}^{(n)})$, considered as a random element of $\mathbb{R}^{mn}$, is associated. The function $\varphi : \mathbb{R}^{mn} \to \mathbb{R}^m$ defined by $\varphi(x_1, x_2, \ldots, x_{mn}) = \left(\sum_{j=0}^{n-1} x_{jm+1}, \sum_{j=0}^{n-1} x_{jm+2}, \ldots, \sum_{j=0}^{n-1} x_{jm+m}\right)$ is increasing with respect to the natural partial orders of its domain and co–domain. Therefore, Theorem 2 *(ii)* implies that $\mathbf{S}_n$ is associated. To establish part **(ii)** suppose $f, g : \mathbb{R}^m \to \mathbb{R}$ are increasing, bounded functions. If we set $\varphi_k := Ef(\mathbf{S}_k)$ and $\gamma_k := Eg(\mathbf{S}_k)$ it holds that $\{\gamma_k\}$, $\{\varphi_k\}$, are nondecreasing sequences, as a result of the non–negativity of the components of $\mathbf{Y}^{(k)}$ and the fact that f and g are increasing functions. Conditioning on N we have

$$
\begin{aligned}
E[f(\mathbf{S}_n)g(\mathbf{S}_n)] &= \sum_{k=1}^{\infty} P(N=k)\, E[f(S_k)g(S_k)] \geq \sum_{k=1}^{\infty} P(N=k)\, \varphi_k\, \gamma_k \\
&\geq \left(\sum_{k=1}^{\infty} P(N=k)\, \varphi_k\right)\left(\sum_{k=1}^{\infty} P(N=k)\, \gamma_k\right) = Ef(\mathbf{S}_n)\; Eg(\mathbf{S}_n)
\end{aligned}
$$

where the first equality follows from the independence of N and $\{\mathbf{Y}^{(k)}\}$, and the next two inequalities from part **(i)** and the fact that the sequences $\{\gamma_k\}$, $\{\varphi_k\}$, are nondecreasing. This establishes the theorem. ∎

It should be pointed out that without the additional non–negativity assumption part (ii) of the above theorem would not be true.

It can be shown that two associated random variables that are also uncorrelated are necessarily independent (e.g. see Szekli, 1995). This makes intuitively plausible the remarkable Central Limit Theorem for associated random variables due to Newman (1980) and the corresponding invariance principle (or FCLT) of Newman and Wright (1981) which we state below and will use it in the sequel.

Theorem 5 (Newman and Wright) *Assume that* $\{Y_k; k \in \mathbb{N}\}$ *is a nondegenerate, stationary sequence of associated random variables with finite second moment* $EY_1^2 < \infty$. *Let* $S_n = Y_1 + \cdots + Y_n$, *suppose that*

$$\sigma^2 = \mathsf{Var}(Y_1) + 2\sum_{j=2}^{\infty} \mathsf{Cov}(Y_1, Y_j) < \infty \tag{1}$$

holds, and define the sequence of processes $\{Y_n(t); 0 \le t \le T\}_{n=1,2,\ldots}$ *via*

$$Y_n(t) = \frac{S_{[nt]} + (nt - [nt])Y_{[nt]+1} - ntEY_1}{\sigma\sqrt{n}}, \qquad \textit{for } 0 \le t \le T$$

(where, as usual, $[nt]$ *denotes the integer part of* nt*). Then the sequence of processes* $\{Y_n\}$ *converges in distribution to the standard Wiener process* W.

Laws of Large Numbers for associated sequences are given in Newman (1980) and Birkel (1989).

3 The discrete time model with after-effects

3.1 Statistics of the model

We consider a risk process in discrete time with two types of claims, *primary* and *secondary claims*. The latter are assumed to be *after–effects of the former,* occur at a later time than the primary claims that trigger them, and typically have *different statistical characteristics*. In general we will assume that each primary claim triggers a whole sequence of secondary after–effects.

More specifically, during the nth time period N_n primary claims occur, which will be denoted by $\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,N_n}$. The processes we consider will be defined over all integers and thus $n \in \mathbb{Z}$. The jth primary claim which occurs at time n will trigger a stream of secondary claims, $\zeta_1^{n,j}, \zeta_2^{n,j}, \zeta_3^{n,j}, \ldots$ which occur at times $n+1,\ n+2,\ n+3, \ldots$, where $j = 1, 2, \ldots, N_n$. Thus as a result of the N_n primary claims that occur at time n, there will be N_n secondary claim streams that begin at time $n+1$.

To construct our probability model we assume that on the probability space $(\Omega, \mathcal{F}, P)$ a sequence of random variables, $\{N_n; n \in \mathbb{Z}\}$, with values in $\mathbb{N}$, has been defined. The probability space also supports the double array $\{\psi_{n,j};\ n \in \mathbb{Z},\ j \in \mathbb{N}\}$, where $\psi_{n,j} := (\xi_{n,j},\ \zeta_1^{n,j}, \zeta_2^{n,j}, \zeta_3^{n,j}, \ldots)$ denotes a random element of $\mathbb{R}_+^{\infty}$. We will assume that $\{\psi_{nj};\ n \in \mathbb{Z},\ j \in \mathbb{N}\}$ is an independent, identically distributed double array which is also independent of the sequence $\{N_n\}$. Furthermore, the random variables $\zeta_1^{n,j},\ \zeta_2^{n,j}, \zeta_3^{n,j}, \ldots$ are *conditionally independent*, given $\xi_{n,j}$. The corresponding distributions are denoted by

$$\begin{aligned} F(x) &= P(\xi_{n,j} \le x), \\ G_k(y|x) &= P(\zeta_k^{n,j} \le y | \xi_{n,j} = x), \qquad j, k, n \in \mathbb{N}. \end{aligned}$$

Thus the finite–dimensional distributions of the "generic" sequence $(\xi, \zeta_1, \zeta_2, \zeta_3, \ldots)$ are given by

$$P(\xi \le x, \zeta_1 \le y_1, \zeta_2 \le y_2, \ldots, \zeta_k \le y_k) = \int_0^x \prod_{i=1}^k G_i(y_i|u) F(du). \tag{2}$$

We will further assume that

$$G_k(y|x_1) \ge G_k(y|x_2) \;\; \text{for all } y \ge 0 \text{ and } 0 \le x_1 \le x_2, \quad k \in \mathbb{N}. \tag{3}$$

This last stochastic ordering assumption means that the after-effects are more serious for larger primary claims than for smaller ones.

The overall effect of the jth primary claim that occurs at time period n and all its future after–effects is then

$$Z_{n,j} = \xi_{n,j} + \sum_{k=1}^{\infty} \zeta_k^{n,j}. \tag{4}$$

Of particular significance in the sequel will be the fact that, for each $n, j \in \mathbb{N}$, the sequence $(\xi_{n,j},\ \zeta_1^{n,j}, \zeta_2^{n,j}, \zeta_3^{n,j}, \ldots)$ is associated. Dropping the superfluous subscripts and superscripts this is established in the following

Lemma 6 *The collection of random variables* $(\xi, \zeta_1, \zeta_2, \ldots)$ *is associated.*

Proof: It is enough to show that, for all $m \in \mathbb{N}$ and all bounded, increasing functions $f, g : \mathbb{R}^{m+1} \to \mathbb{R}$,

$$E[f(\xi, \zeta_1, \ldots, \zeta_m) g(\xi, \zeta_1, \ldots, \zeta_m)] \ge E[f(\xi, \zeta_1, \ldots, \zeta_m)]\ E[g(\xi, \zeta_1, \ldots, \zeta_m)]. \tag{5}$$

Let $\phi(x) := E[f(\xi, \zeta_1, \ldots, \zeta_m)|\xi = x]$, $\gamma(x) := E[g(\xi, \zeta_1, \ldots, \zeta_m)|\xi = x]$ denote the conditional expectations given $\xi = x$. As a result of (3), the implied conditional independence of the ζ_n from (2), and Theorem 3.3.7 of Stoyan and Müller (2002) we have $(\zeta_1, \zeta_2, \ldots, \zeta_m)|_{\xi=x_1} \le_{st} (\zeta_1, \zeta_2, \ldots, \zeta_m)|_{\xi=x_2}$ for $x_1 \le x_2$. Therefore, since f, g are increasing, there exist increasing versions of ϕ and γ. From this (5) follows readily. ∎

3.2 Construction of the stationary version of the claim process and moment computation

Let M be a positive integer. Consider the process $\{Y_n^M;\ n \in \mathbb{Z}\}$ which is defined as $Y_n^M = 0$ for all $n < -M$, $Y_{-M}^M = \sum_{j=1}^{N_{-M}} \xi_{-M,j}$, and, for $n > -M$, by the expression

$$Y_n^M = \sum_{j=1}^{N_n} \xi_{n,j} + \sum_{k=1}^{n+M} \sum_{j=1}^{N_{n-k}} \zeta_k^{n-k,j}. \tag{6}$$

Thus $\{Y_n^M\}$ can be thought of as a claim process starting at time $-M$. In view of the fact that $\zeta_k^{l,j} \ge 0$, a.s. for all values of the indices, it follows that $Y_n^M \uparrow Y_n$ as $M \to \infty$, for all n, w.p.1 with Y_n given by

$$Y_n = \sum_{j=1}^{N_n} \xi_{n,j} + \sum_{k=1}^{\infty} \sum_{j=1}^{N_{n-k}} \zeta_k^{n-k,j}. \tag{7}$$

The first term in the sum above designates the primary claims that occurred at time n while the second gives the total contribution of the after–effects of all primary claims that occurred before time n. As we show in the next proposition, this limit is finite with probability 1 under the finiteness of moments assumptions stated there. We also establish the stationarity of the sequence $\{Y_n; n \in \mathbb{Z}\}$ and we compute the asymptotic variance of the average of the terms of the sequence. From this point on we will be occasionally dropping unnecessary subscripts and superscripts when no confusion can arise.

Proposition 7 *Under the assumption that $EN^2 < \infty$, $E\xi^2 < \infty$, $\sum_{k=1}^{\infty} E\zeta_k < \infty$, and $\sum_{k=1}^{\infty} E\zeta_k^2 < \infty$, the process $\{Y_n; n \in \mathbb{Z}\}$ as given in (7) is finite w.p.1. Furthermore it is stationary with mean*

$$\mu := EY_0 = EN\left(E\xi + \sum_{k=1}^{\infty} E\zeta_k\right) < \infty, \tag{8}$$

and asymptotic variance constant of the sum of the terms of the sequence given by

$$\begin{aligned}\sigma^2 &:= \lim_{n\to\infty} \frac{1}{n}\mathsf{Var}(Y_0 + \cdots + Y_{n-1}) \\ &= \mathsf{Var}(N)\left(E\xi + E\sum_{k=1}^{\infty}\zeta_k\right)^2 + EN\ \mathsf{Var}\left(\xi + \sum_{k=1}^{\infty}\zeta_k\right) < \infty.\end{aligned} \tag{9}$$

Proof. We first establish the a.s. finiteness of Y_n. Note that all terms on the right hand side of (7) are non–negative and write it as

$$\begin{aligned}Y_n &= \sum_{j=1}^{\infty} \xi_{n,j}\mathbf{1}(N_n \geq j) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \zeta_k^{n-k,j}\mathbf{1}(N_{n-k} \geq j) \\ &= \sum_{j=1}^{\infty}\left(\xi_{n,j}\mathbf{1}(N_n \geq j) + \sum_{k=1}^{\infty} \zeta_k^{n-k,j}\mathbf{1}(N_{n-k} \geq j)\right) \\ &= \sum_{j=1}^{\infty} \varphi_{n,j}\end{aligned} \tag{10}$$

with

$$\varphi_{n,j} := \xi_{n,j}\mathbf{1}(N_n \geq j) + \sum_{k=1}^{\infty} \zeta_k^{n-k,j}\mathbf{1}(N_{n-k} \geq j), \qquad n \in \mathbb{Z},\ j \in \mathbb{N}. \tag{11}$$

Since the $\varphi_{n,j}$ are non–negative, we can use the Fubini theorem to evaluate the expectation

$$\begin{aligned}EY_n &= \sum_{j=1}^{\infty} E\left[\xi_{n,j}\mathbf{1}(N_n \geq j) + \sum_{k=1}^{\infty} \zeta_k^{n-k,j}\mathbf{1}(N_{n-k} \geq j)\right] \\ &= \sum_{j=1}^{\infty} E[\xi_{n,j}]P(N_n \geq j) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} E[\zeta_k^{n-k,j}]P(N_{n-k} \geq j) \\ &= E\xi\sum_{j=1}^{\infty} P(N_n \geq j) + \sum_{k=1}^{\infty} E[\zeta_k^{n-k,j}]\sum_{j=1}^{\infty} P(N_{n-k} \geq j) \\ &= E\xi EN + EN\sum_{k=1}^{\infty} E\zeta_k\end{aligned}$$

where, in the above equalities, we have taken into account the independence of N_n and $\xi_{n,j}$ and similarly of N_{n-k} and $\zeta_k^{n-k,j}$. From this expression, dropping the unnecessary superscripts and subscripts and using the fact that the number of claims $\{N_n\}$ are assumed i.i.d., we obtain the finiteness of EY_n, and thus the a.s. finiteness of Y_n, as well as (8).

The stationarity of $\{Y_n; n \in \mathbb{Z}\}$ is also immediate from the expression $Y_n = \sum_{j=1}^{\infty} \varphi_{n,j}$ given in (10). Indeed, for each fixed $j \in \mathbb{N}$, the process $\{\varphi_{n,j}; n \in \mathbb{Z}\}$ is stationary. Also, the family of processes $\{\varphi_{n,j}; n \in \mathbb{Z}\}_{j\in\mathbb{N}}$ is independent because of our stochastic assumptions. This establishes the stationarity of $\{Y_n; n \in \mathbb{Z}\}$.

In view of the stationarity of $\{Y_n; n \in \mathbb{Z}\}$, to obtain (9) we start with the fact that

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \mathsf{Var}(Y_0 + \cdots + Y_{n-1}) = \mathsf{Var}(Y_0) + 2\sum_{j=1}^{\infty} \mathsf{Cov}(Y_0, Y_j) \tag{12}$$

provided that the infinite series converges. In our case we have

$$\begin{aligned}
\mathsf{Var}(Y_0) &= \mathsf{Var}\left(\sum_{j=1}^{N_0} \xi_{0,j}\right) + \mathsf{Var}\left(\sum_{k=1}^{\infty}\sum_{j=1}^{N_{-k}} \zeta_k^{-k,j}\right) \\
&= EN_0 \mathsf{Var}(\xi_{0,1}) + \mathsf{Var}(N_0)\,(E\xi_{0,1})^2 + EN_0 \sum_{k=1}^{\infty} \mathsf{Var}(\zeta_k^{-k,1}) + \mathsf{Var}(N_0)\left(E\sum_{k=1}^{\infty}\zeta_k^{-k,1}\right)^2 \\
&= EN\left(\mathsf{Var}(\xi) + \sum_{k=1}^{\infty}\mathsf{Var}(\zeta_k)\right) + \mathsf{Var}(N)\left(E\xi + \sum_{k=1}^{\infty} E\zeta_k\right)^2 \qquad (13)
\end{aligned}$$

where in the last equality we have made use of the i.i.d. assumptions and dropped superfluous subscripts and superscripts. A straightforward computation leads to the following expression for the covariance

$$\mathsf{Cov}(Y_0, Y_j) = EN\left(\mathsf{Cov}(\xi, \zeta_j) + \sum_{k=1}^{\infty}\mathsf{Cov}(\zeta_k, \zeta_{k+j})\right). \tag{14}$$

Substituting (13) and (14) into (12), after straightforward manipulations we obtain

$$\begin{aligned}
\sigma^2 &= EN\mathsf{Var}(\xi) + \mathsf{Var}(N)\left(E\xi + \sum_{k=1}^{\infty} E\zeta_k\right)^2 + 2EN\mathsf{Cov}\left(\xi, \sum_{k=1}^{\infty}\zeta_k\right) + EN\mathsf{Var}\left(\sum_{k=1}^{\infty}\zeta_k\right) \\
&= \mathsf{Var}(N)\left(E\xi + E\sum_{k=1}^{\infty}\zeta_k\right)^2 + EN\mathsf{Var}\left(\xi + \sum_{k=1}^{\infty}\zeta_k\right).
\end{aligned}$$

This conclude the proof. ∎

3.3 Association of the process with after–effects

We are now ready to prove our main result upon which hinges the diffusion approximation of the next section, namely the fact that the process with after–effects introduced in 3.1 is associated.

Proposition 8 *The claims $\{Y_n; n \in \mathbb{Z}\}$ form an associated sequence of random variables.*

Proof. We will show that $(Y_1, \ldots, Y_m)$ is associated for each fixed $m \in \mathbb{N}$. This, together with the stationarity of the sequence, establishes the proposition. To this effect, for each $M \in \mathbb{N}$ consider the finite collection of random variables $(Y_1^M, \ldots, Y_m^M)$ where Y_n^M is given by (6). If we set $\mathbf{Y}_m := (Y_1, \ldots, Y_m)$ and $\mathbf{Y}_m^M := (Y_1^M, \ldots, Y_m^M)$ then $\mathbf{Y}_m^M \to \mathbf{Y}_m$ w.p.1 based on the results of the previous section. This in turn implies convergence in distribution, i.e. $\mathbf{Y}_m^M \xrightarrow{d} \mathbf{Y}_m$. Thus, in view of Theorem 3, it is enough to show that $\mathbf{Y}_m^M$ is associated.

$\mathbf{Y}_m$ admits the following decomposition:

$$\mathbf{Y}_m = \sum_{j=1}^{N_m} \mathbf{V}_j^m + \sum_{j=1}^{N_{m-1}} \mathbf{V}_j^{m-1} + \cdots + \sum_{j=1}^{N_1} \mathbf{V}_j^1 + \sum_{j=1}^{N_0} \mathbf{V}_j^0 + \sum_{j=1}^{N_{-1}} \mathbf{V}_j^{-1} \cdots + \sum_{j=1}^{N_{-M}} \mathbf{V}_j^{-M} + \cdots \quad (15)$$

where

$$\mathbf{V}_j^m = (0, 0, \ldots, 0, \xi_{m,j}), \qquad \mathbf{V}_j^{m-1} = \left(0, 0, \ldots, \xi_{m-1,j}, \zeta_{m-1,j}^1\right), \ldots,$$

$$\mathbf{V}_j^0 = \left(\zeta_{0,j}^1, \zeta_{0,j}^2, \ldots, \zeta_{0,j}^{m-1}, \zeta_{0,j}^m\right), \qquad \mathbf{V}_j^{-1} = \left(\zeta_{-1,j}^2, \zeta_{-1,j}^3, \ldots, \zeta_{-1,j}^m, \zeta_{-1,j}^{m+1}\right), \ldots,$$

$$\mathbf{V}_j^{-M} = \left(\zeta_{-M,j}^{M+1}, \zeta_{-M,j}^{M+2}, \ldots, \zeta_{-M,j}^{M+m-1}, \zeta_{-M,j}^{M+m}\right), \ldots.$$

Here $\mathbf{V}_j^k$ represents the contribution to $\mathbf{Y}_m$ of the jth claim event occurring at time $k = m, m-1, \ldots, 0, -1, \ldots$ either directly by means of the primary claim $\xi_{k,j}$ that occurs at time k, or indirectly, by means of the stream of secondary claims. If we denote by $\mathbf{W}_k := \sum_{j=1}^{N_k} \mathbf{V}_j^k$, $k = m, m-1, \ldots, 1, 0, -1, -2, -3, \ldots$, the overall contribution of claim events that occur at epoch k to the claim vector vector $\mathbf{Y}_m$ we have

$$\mathbf{Y}_m = \mathbf{W}_m + \mathbf{W}_{m-1} + \cdots + \mathbf{W}_0 + \mathbf{W}_{-1} + \mathbf{W}_{-2} + \cdots.$$

The corresponding claim vector with history truncated at time $-M$ is then

$$\mathbf{Y}_m^M = \mathbf{W}_m + \mathbf{W}_{m-1} + \cdots + \mathbf{W}_0 + \mathbf{W}_{-1} + \cdots + \mathbf{W}_{-M+1} + \mathbf{W}_{-M}.$$

Clearly the random vectors $\mathbf{W}_k$, $k = m, m-1, \ldots, 0, -1, -2, \ldots$, are independent in view of our assumptions. Furthermore, each one of these random vectors is associated as can be seen from Lemma (6) and theorem (4), part *(ii)*. Therefore, from part *(i)* of theorem (4) if follows that $\mathbf{Y}_m^M$ is associated and hence the proof is complete. ∎

3.4 A model with Bernoulli after–effects

Here we give a concrete example of such a discrete process with after–effects. For the sake of simplicity of exposition we will assume that only a single primary claim occurs during each time period. (By allowing an atom at zero for the distribution of these primary claims we can also allow periods with no primary claims at all.) The extension to the situation where a random number of primary claims occurs in each time period, as before, is straightforward. Let ξ_n

denote the primary claim that occurs in the nth time period. We will assume that $\{\xi_n; n \in \mathbb{Z}\}$ is a sequence of i.i.d. non–negative random variables with $E\xi^2 < \infty$. The after–effects of each claim are Bernoulli in the following sense. Suppose that $\{\eta_k^n; k \in \mathbb{N}, n \in \mathbb{Z}\}$ is a double array of independent Bernoulli random variables with values in $\{0, 1\}$ that are also independent of $\{\xi_n\}$. We assume that, for all $n \in \mathbb{Z}$, $P(\eta_k^n = 1) = p_k$, $k = 1, 2, \ldots$. We further assume that

$$\sum_{k=1}^{\infty} p_k < \infty. \tag{16}$$

Also let $\{v_k\}$, $k = 1, 2, \ldots$, denote a sequence of measurable, nonnegative functions $v_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\sup_k E[v_k(\xi)^2] = L < \infty. \tag{17}$$

Then the stream of after–effects of the primary claim ξ_n is the sequence of secondary claims $\zeta_k^n := \eta_k^n v_k(\xi_n)$, $k = 1, 2, 3 \ldots$, which occur at times $n+1, n+2, n+3, \ldots$. Thus, in the stationary model, the total amount due to both types of claims that occur at time n, which includes the present primary claim and the after–effects of all past primary claims, is equal to

$$Y_n = \xi_n + \sum_{k=1}^{\infty} \eta_k^{n-k} v_k(\xi_{n-k}). \tag{18}$$

The overall effect of the nth primary claim together with all its future after–effects is

$$Z_n = \xi_n + \sum_{k=1}^{\infty} \eta_k^n v_k(\xi_n). \tag{19}$$

As a result of (16) and of the first Borel–Cantelli lemma, with probability 1, only finitely many of the $\{\eta_k^n; k = 1, 2, 3, \ldots\}$ are equal to 1 and hence the right hand side of (19) is a.s. finite. Similarly, only finitely many of $\{\eta_k^{n-k}; k = 1, 2, 3, \ldots\}$ are equal to 1 a.s. and hence Y_n is also finite w.p.1.

We have

$$EZ = E\xi + \sum_{k=1}^{\infty} p_k E[v_k(\xi)]$$

which, as a consequence of assumptions (17) and (16), together with the fact that $0 \le Ev_k(\xi) \le \left(E[v_k(\xi)^2]\right)^{1/2} \le L^{1/2}$, can be seen to be finite. Conditioning first on ξ, we also obtain

$$\begin{aligned}\mathsf{Var}(Z) \quad = \quad & \mathsf{Var}(\xi) + 2\sum_{k=1}^{\infty} p_k \mathsf{Cov}(\xi, v_k(\xi)) + \sum_{k=1}^{\infty} p_k \mathsf{Var}(v_k(\xi)) \\ & + \sum_{k=1}^{\infty} p_k(1-p_k)(Ev_k(\xi))^2 + \sum_{k=1}^{\infty} \sum_{\substack{l=1 \\ l \neq k}}^{\infty} p_k p_l \mathsf{Cov}(v_k(\xi), v_l(\xi)).\end{aligned}$$

It is easy to see that the convergence of the series $\sum_{k=1}^{\infty} p_k$ implies that of $\sum_{k=1}^{\infty} p_k^2$, of $\sum_{k=1}^{\infty} p_k(1-p_k)$, and of $\sum_{k=1}^{\infty} \sum_{\substack{l=1 \\ l \neq k}}^{\infty} p_k p_l$. Hence, using a simple argument based on the

Cauchy–Schwarz inequality, on the finiteness of $E\xi^2$, and on assumption (17), we see that $\mathsf{Var}(Z) < \infty$.

A special case of particular interest is $p_k = \alpha^k$ where $\alpha \in (0,1)$ and $v_k(x) = x$ for all $x \in \mathbb{R}^+$. We then have $EZ = E\xi/(1-\alpha)$ and

$$\mathsf{Var}(Z) = \frac{1+2\alpha-\alpha^2}{(1-\alpha)^2(1+\alpha)} + (E\xi)^2 \frac{\alpha}{1-\alpha^2}.$$

4 The Risk Model and the Diffusion approximation

Consider now the following discrete time free reserves process that corresponds to the stationary claims process $\{Y_n; n \in \mathbb{N}\}$ defined in the previous section,

$$R_n = u + cn - \sum_{m=1}^{n} Y_m, \tag{20}$$

where u is the initial risk reserve and c is the gross risk premium per unit time. The finite horizon ruin probability of the process $\{R_n; n \in \mathbb{N}\}$ for some fixed $n_0 \in \mathbb{N}$ is defined as

$$\Psi(u, n_0) = P\left(\inf_{n=1,2,\ldots,n_0} R_n < 0\right) \tag{21}$$

and the corresponding infinite horizon ruin probability is

$$\Psi(u) = P\left(\inf_{n=1,2,\ldots} R_n < 0\right). \tag{22}$$

We will use the Newman-Wright FCLT for associated sequences in order to obtain a diffusion approximation for the discrete time risk model (20). Suppose that we are given the stationary claims process $\{Y_n\}$ defined in (7), a sequence of premiums $\{c^\nu\}$, and a sequence of initial capitals $\{u^\nu\}$, $\nu = 1, 2, \ldots$. We then define a sequence of continuous time free reserves processes $\{R^\nu(t); t \in [0, \infty)\}$ via

$$R^\nu(t) := \frac{1}{\sqrt{\nu}} \left(u^\nu + \nu t c^\nu - \sum_{j=1}^{[\nu t]} Y_j - (\nu t - [\nu t]) Y_{[\nu t]+1} \right), \qquad t \geq 0. \tag{23}$$

The following then holds.

Proposition 9 *For the sequence of risk processes defined in (23) we assume that the mean μ of $\{Y_n\}$ given in (8) and the variance σ^2 given in (9) are finite. We further assume that $\lim_{\nu\to\infty} \nu^{-1/2} u^\nu = u$ and $\lim_{\nu\to\infty} \nu^{1/2} (c^\nu - \mu) = \gamma > 0$ for some strictly positive u and γ. Then*

$$R^\nu(t) \xrightarrow{\mathrm{d}} u + \gamma t + \sigma W(t), \qquad t \in [0, \infty) \tag{24}$$

where W is standard brownian motion.

Proof. Rewrite (23) as

$$R^\nu(t) = \frac{1}{\sqrt{\nu}} u^\nu + \sqrt{\nu}\,(c^\nu - \mu)\,t - \frac{1}{\sqrt{\nu}} \left(\sum_{j=1}^{[\nu t]} Y_j + (\nu t - [\nu t]) Y_{[\nu t]+1} - \mu \nu t \right).$$

The first term on the right hand side of the above equation converges to u and the second to γt by assumption. The third term converges weakly in $\mathbb{C}[0,T]$ to $\{\sigma W(t); t \in [0,T]\}$ for any $T > 0$ by a direct application of theorem (5) and proposition (8). The extension to $\mathbb{C}[0,\infty)$ follows readily by Theorem 5 of Whitt (1970). ∎

The above result can next be applied, in the same fashion as in Iglehart (1969), Grandell (1977), and Grandell (1991), to approximate the surplus process (20) by

$$\hat{R}(t) = u + (c-\mu)t + \sigma W(t). \tag{25}$$

The finite time ruin probability $\Psi(u,t)$, of the surplus process (20) can then be approximated by

$$\hat{\Psi}(u,t) = 1 - \Phi\left(\tfrac{u+\rho\mu t}{\sigma\sqrt{t}}\right) + \Phi\left(\tfrac{-u+\rho\mu t}{\sigma\sqrt{t}}\right) e^{-R_D u}, \tag{26}$$

where

$$R_D = \frac{2\rho\mu}{\sigma^2},$$

and $\rho = c/\mu - 1$. The corresponding infinite horizon ruin probability $\Psi(u)$ of the surplus process (20) is approximated by

$$\hat{\Psi}(u) = e^{-R_D u}. \tag{27}$$

5 Experimental results

Here we present numerical results that compare the actual finite horizon ruin probabilities, estimated by means of Monte–Carlo experiments, with the corresponding diffusion approximations for two models with after-effects.

5.1 Model 1

In this model primary claims are i.i.d. random variables that are gamma distributed with mean 2 (shape parameter 2, scale parameter 1). The number of primary claims in each time period is i.i.d. Poisson with mean 5. A primary claim $\xi_{n,j}$ at time n triggers a sequence $\{\zeta_k^{n,j}; k = 1,2,3,\ldots\}$ of secondary claims that occur at times $n+1, n+2, \ldots$ with $\zeta_k^{n,j} := \xi_{n,j} a^k$ where $0 < a < 1$. In figures 1 and 2 ten simulated sample paths of $R(t)$ and $\hat{R}(t)$ are compared. In these plots the time horizon is $t = 1000$, the initial capital is $u = 100$, $a = 0.2$, and $\rho = 0.04$.

In tables 1 and 2, the corresponding finite time ruin probabilities Ψ and $\hat{\Psi}$ are presented for time horizon $t = 2000$ and $10,000$, respectively. The relative error $\varepsilon\%$ is defined by $\varepsilon\% = 100(\hat{\Psi} - \Psi)/\Psi$.

u	$\Psi \pm 1.96\widehat{\sigma}/\sqrt{N}$	$\hat{\Psi}(u,t)$	$\varepsilon\%$
100	$(4.89 \pm 0.05)10^{-1}$	$4.93 \cdot 10^{-1}$	0.76
200	$(2.42 \pm 0.04)10^{-1}$	$2.28 \cdot 10^{-1}$	-5.73
300	$(1.10 \pm 0.03)10^{-1}$	$0.98 \cdot 10^{-1}$	-11.04
400	$(4.92 \pm 0.21)10^{-2}$	$3.84 \cdot 10^{-2}$	-21.85
500	$(1.95 \pm 0.14)10^{-2}$	$1.38 \cdot 10^{-2}$	-29.38
600	$(7.70 \pm 0.86)10^{-3}$	$4.48 \cdot 10^{-3}$	-41.82
700	$(2.88 \pm 0.52)10^{-3}$	$1.31 \cdot 10^{-3}$	-54.34
800	$(1.10 \pm 0.32)10^{-3}$	$3.46 \cdot 10^{-4}$	-68.56
900	$(2.75 \pm 1.62)10^{-4}$	$8.17 \cdot 10^{-5}$	-70.29
1000	$(7.50 \pm 8.49)10^{-5}$	$1.73 \cdot 10^{-5}$	-76.99

Table 1: Comparison of the finite-time ruin probabilities Ψ and $\hat{\Psi}$, $t = 2000$.

u	$\Psi \pm 1.96\widehat{\sigma}/\sqrt{N}$	$\hat{\Psi}(u,t)$	$\varepsilon\%$
100	$(5.39 \pm 0.05)10^{-1}$	$5.43 \cdot 10^{-1}$	8.80
200	$(3.03 \pm 0.05)10^{-1}$	$2.94 \cdot 10^{-1}$	-2.70
300	$(1.70 \pm 0.04)10^{-1}$	$1.59 \cdot 10^{-1}$	-6.58
400	$(9.33 \pm 0.28)10^{-2}$	$8.54 \cdot 10^{-2}$	-8.39
500	$(5.37 \pm 0.22)10^{-2}$	$4.57 \cdot 10^{-2}$	-14.95
600	$(3.06 \pm 0.17)10^{-2}$	$2.42 \cdot 10^{-2}$	-20.75
700	$(1.73 \pm 0.13)10^{-2}$	$1.28 \cdot 10^{-2}$	-26.05
800	$(9.60 \pm 0.96)10^{-3}$	$6.65 \cdot 10^{-3}$	-30.71
900	$(5.30 \pm 0.71)10^{-3}$	$3.43 \cdot 10^{-3}$	-35.25
1000	$(2.83 \pm 0.52)10^{-3}$	$1.75 \cdot 10^{-3}$	-38.06
1100	$(1.68 \pm 0.40)10^{-3}$	$8.80 \cdot 10^{-4}$	-47.44
1200	$(8.50 \pm 2.86)10^{-4}$	$4.37 \cdot 10^{-4}$	-48.61
1300	$(6.50 \pm 2.50)10^{-4}$	$2.13 \cdot 10^{-4}$	-67.16
1400	$(2.75 \pm 1.62)10^{-4}$	$1.03 \cdot 10^{-4}$	-62.66
1500	$(7.50 \pm 8.49)10^{-5}$	$4.86 \cdot 10^{-5}$	-35.23

Table 2: Comparison of the finite-time ruin probabilities Ψ and $\hat{\Psi}$, $t = 10,000$.

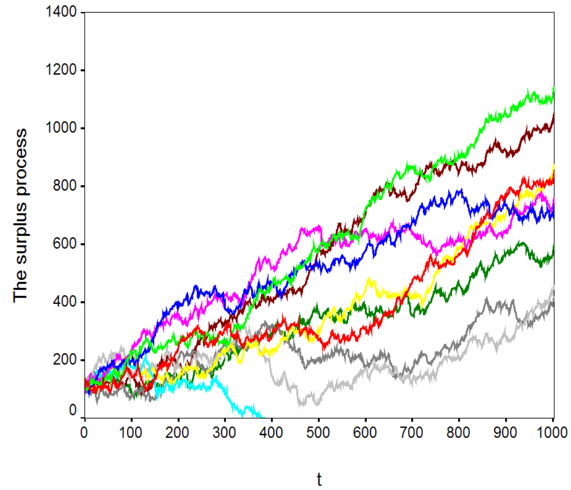

Figure 1: Simulations of $R(t)$, model 1.

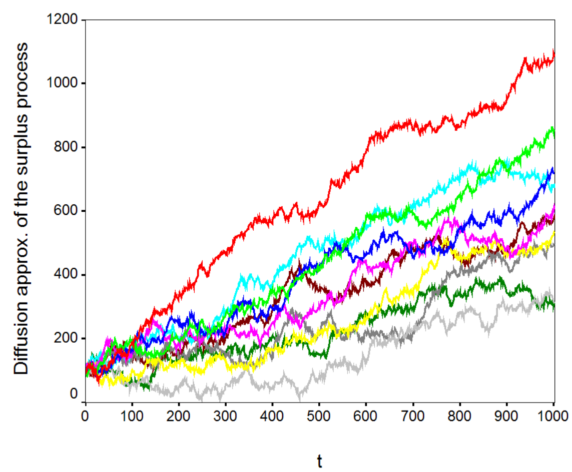

Figure 2: Simulations of $\hat{R}(t)$, model 1.

5.2 Model 2

The second model is essentially in the framework of section 3.4. Primary claims are again i.i.d. random variables gamma distributed with shape parameter 2 and scale parameter 1 and the number of primary claims in each time period are i.i.d. Poisson distributed with mean 5. A primary claim $\xi_{n,j}$ at time n triggers a sequence $\{\zeta_k^{n,j}; k = 1, 2, 3, \ldots\}$ of secondary claims that occur at times $n+1, n+2, \ldots,$ with $\zeta_k^{n,j} := \xi_{n,j}\eta_k^{n,j}$. Here $\{\eta_k^{n,j}; n \in \mathbb{Z}; k, j \in \mathbb{N}\}$ is a triple array of independent Bernoulli random variables such that

$$\eta_k^{n,j} = \begin{cases} 1, & \text{with probability } p_k = a^k \\ 0 & \text{with probability } 1 - p_k \end{cases},$$

where $0 < a < 1$. Thus, the total contribution of the secondary claims triggered by a primary claim ξ is $Z = \xi \sum_{k=1}^{\infty} \eta_k$ which is finite with probability 1 by virtue of the Borel–Cantelli lemma.

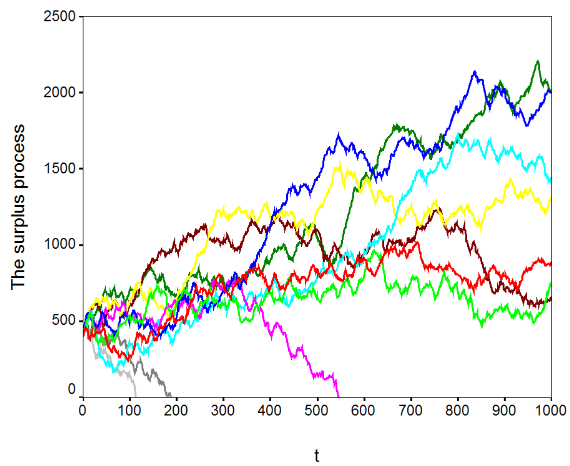

Figure 3: Simulations of $R(t)$, model 2.

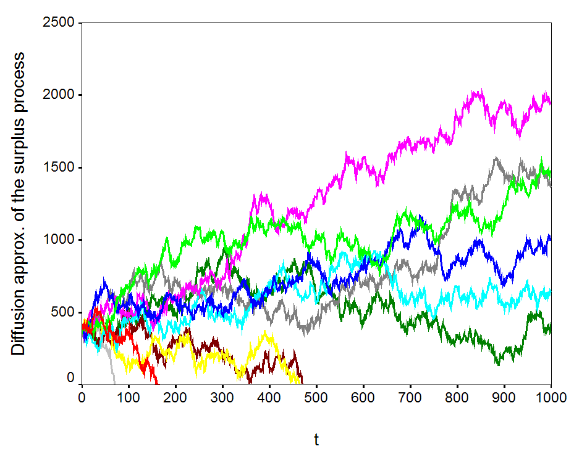

Figure 4: Simulations of $\hat{R}(t)$, model 2.

In figures 3 and 4 ten simulated sample paths of $R(t)$ and $\hat{R}(t)$ are compared. The time horizon is $t = 1000$, the initial capital is $u = 400$, $\rho = 0.01$, and $a = 0.7$.

u	$\Psi \pm 1.96\widehat{\sigma}/\sqrt{N}$	$\hat{\Psi}(u,t)$	$\varepsilon\%$
100	$(8.47 \pm 0.04)10^{-1}$	$8.70 \cdot 10^{-1}$	2.83
200	$(7.49 \pm 0.04)10^{-1}$	$7.54 \cdot 10^{-1}$	0.62
500	$(4.95 \pm 0.05)10^{-1}$	$4.75 \cdot 10^{-1}$	-3.93
1000	$(2.28 \pm 0.04)10^{-1}$	$1.96 \cdot 10^{-1}$	-13.98
2000	$(3.29 \pm 0.17)10^{-2}$	$2.05 \cdot 10^{-2}$	-37.65
3000	$(1.50 \pm 0.38)10^{-3}$	$1.03 \cdot 10^{-3}$	-31.32
4000	(0 ± 0)	$2.37 \cdot 10^{-5}$	$---$
5000	(0 ± 0)	$2.54 \cdot 10^{-7}$	$---$

Table 3: Comparison of the finite-time ruin probabilities Ψ and $\hat{\Psi}$, $t = 10,000$, $a = 0.5$.

u	$\Psi \pm 1.96\widehat{\sigma}/\sqrt{N}$	$\hat{\Psi}(u,t)$	$\varepsilon\%$
100	$(8.60 \pm 0.03)10^{-1}$	$9.18 \cdot 10^{-1}$	6.84
200	$(8.00 \pm 0.04)10^{-1}$	$8.42 \cdot 10^{-1}$	5.30
500	$(6.37 \pm 0.05)10^{-1}$	$6.42 \cdot 10^{-1}$	-7.92
1000	$(4.09 \pm 0.05)10^{-1}$	$3.92 \cdot 10^{-1}$	-4.06
2000	$(1.41 \pm 0.03)10^{-1}$	$1.24 \cdot 10^{-1}$	-12.25
3000	$(4.32 \pm 0.20)10^{-2}$	$3.06 \cdot 10^{-2}$	-29.21
4000	$(9.80 \pm 0.97)10^{-3}$	$5.77 \cdot 10^{-3}$	-41.10
5000	$(1.20 \pm 0.34)10^{-3}$	$8.20 \cdot 10^{-3}$	-31.70
6000	$(4.50 \pm 2.08)10^{-4}$	$8.68 \cdot 10^{-5}$	-80.71
7000	(0 ± 0)	$6.83 \cdot 10^{-6}$	$---$

Table 4: Comparison of the finite-time ruin probabilities Ψ and $\hat{\Psi}$, $t = 10,000$, $a = 0.7$.

In tables 3, 4, 5, and 6 the corresponding finite time ruin probabilities Ψ and $\hat{\Psi}$ are presented for time horizon $t = 10,000$, $a = 0.5$, 0.7, 0.8, and 0.9 respectively, $\rho = 0.003$.

Each entry in the six tables above involved 40,000 independent simulation runs. As expected, when the initial capital u is so large that the ruin probability is of the order of 10^{-2} or smaller, the quality of the diffusion approximation is not very satisfactory. At the same time, the computational effort necessary in order to estimate the ruin probability using simulation becomes prohibitively large. We should point out that for a number of runs with large values of u in tables 3, 4, 5, and 6 (all referring to model 2) the results of the Monte Carlo estimation for the ruin probability give 0 because no sample paths leading to ruin were observed.

In closing we should point out that, from a practical point of view, the simple diffusion approximation is a relatively " blunt instrument" which does not provide a very accurate approximation of the ruin probability, but instead a qualitative estimate. After all, as the CLT itself, it is only based on the first two moments of the risk process and does not take into account its tail behavior. Nontheless, such approximations have their uses in the analysis of risk and insurance processes. For a detailed account we refer the reader to [2], [7], and [8].

u	$\Psi \pm 1.96\widehat{\sigma}/\sqrt{N}$	$\hat{\Psi}(u,t)$	$\varepsilon\%$
100	$(8.27 \pm 0.04)10^{-1}$	$9.44 \cdot 10^{-1}$	14.11
200	$(7.88 \pm 0.04)10^{-1}$	$8.90 \cdot 10^{-1}$	12.94
500	$(6.70 \pm 0.05)10^{-1}$	$7.42 \cdot 10^{-1}$	10.84
1000	$(5.04 \pm 0.05)10^{-1}$	$5.40 \cdot 10^{-1}$	7.05
2000	$(2.67 \pm 0.04)10^{-1}$	$2.65 \cdot 10^{-1}$	-4.64
5000	$(1.96 \pm 0.14)10^{-2}$	$1.63 \cdot 10^{-2}$	-16.86
6000	$(7.73 \pm 0.86)10^{-3}$	$5.02 \cdot 10^{-3}$	-34.99
7000	$(1.58 \pm 0.39)10^{-3}$	$1.36 \cdot 10^{-3}$	-13.63
8000	$(4.00 \pm 1.96)10^{-4}$	$3.22 \cdot 10^{-4}$	-19.39
9000	$(1.75 \pm 1.30)10^{-4}$	$6.67 \cdot 10^{-5}$	-61.87
10000	(0 ± 0)	$1.20 \cdot 10^{-5}$	$---$

Table 5: Comparison of the finite-time ruin probabilities Ψ and $\hat{\Psi}$, $t = 10,000$, $a = 0.8$.

u	$\Psi \pm 1.96\widehat{\sigma}/\sqrt{N}$	$\hat{\Psi}(u,t)$	$\varepsilon\%$
100	$(7.25 \pm 0.04)10^{-1}$	$9.71 \cdot 10^{-1}$	33.94
200	$(7.06 \pm 0.04)10^{-1}$	$9.42 \cdot 10^{-1}$	33.42
500	$(6.44 \pm 0.05)10^{-1}$	$8.60 \cdot 10^{-1}$	33.61
1000	$(5.58 \pm 0.05)10^{-1}$	$7.36 \cdot 10^{-1}$	31.85
2000	$(4.05 \pm 0.05)10^{-1}$	$5.29 \cdot 10^{-1}$	30.67
5000	$(1.35 \pm 0.03)10^{-1}$	$1.70 \cdot 10^{-1}$	25.74
10000	$(1.32 \pm 0.11)10^{-2}$	$1.42 \cdot 10^{-2}$	7.55
11000	$(7.38 \pm 0.84)10^{-3}$	$7.85 \cdot 10^{-3}$	6.48
12000	$(3.90 \pm 0.61)10^{-3}$	$4.20 \cdot 10^{-3}$	7.71
13000	$(1.88 \pm 0.42)10^{-3}$	$2.17 \cdot 10^{-3}$	15.83
14000	$(9.00 \pm 2.94)10^{-4}$	$1.08 \cdot 10^{-3}$	20.56
15000	$(5.25 \pm 2.24)10^{-4}$	$5.24 \cdot 10^{-4}$	-0.29
16000	$(1.75 \pm 1.30)10^{-4}$	$2.44 \cdot 10^{-4}$	39.38
17000	$(2.00 \pm 1.39)10^{-4}$	$1.10 \cdot 10^{-4}$	-45.17
18000	(0 ± 0)	$4.76 \cdot 10^{-5}$	$---$
19000	(0 ± 0)	$1.99 \cdot 10^{-5}$	$---$
20000	(0 ± 0)	$8.02 \cdot 10^{-6}$	$---$

Table 6: Comparison of the finite-time ruin probabilities Ψ and $\hat{\Psi}$, $t = 10,000$, $a = 0.9$.

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