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Consistency of Perturbation Analysis for a Queue with Finite Buffer Space and Loss Policy

Y. WARDI,[1] M. A. ZAZANIS,[2] AND M. LUO[3]

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Abstract. The subject of discrete-event dynamical systems has taken on a new direction with the advent of perturbation analysis (PA), an efficient method for estimating the gradients of a steady-state performance measure, by analyzing data obtained from a single-simulation experiment in the time domain. A crucial issue is whether PA gives strongly consistent estimates, namely, whether average time-domain-based gradients converge, over infinite horizon, to the steady-state gradients. In this paper, we investigate this issue for a queue with a finite buffer capacity and a loss policy. The performance measure in question is the average amount of lost customers, as a function of the buffer's capacity, which is assumed to be continuous in our work. It is shown that PA gives strongly consistent estimates. The analysis uses a new technique, based on busy period-dependent inequalities. This technique may have possible extensions to analyses of consistency of PA for more general queueing systems.

Key Words. Perturbation analysis, discrete-event dynamical systems.

1. Introduction

The subject of simulation-based optimization of discrete-event dynamical systems, and especially of queueing networks, has taken on a new direction in the past nine years, with the development of perturbation analysis (see Ref. 1 and the references therein). Consider a discrete-event dynamical system. Let $z_1, z_2, \ldots$ denote the states, let $u_1, u_2, \ldots$ denote the inputs, and let h denote the state transition function. Thus, for $n = 1$,

[1] Assistant Professor, School of Electrical Engineering, Georgia Institute of Technology, Atlanta, Georgia.
[2] Assistant Professor, Department of Industrial Engineering, Northwestern University, Evanston, Illinois.
[3] Graduate Student, School of Electrical Engineering, Georgia Institute of Technology, Atlanta, Georgia.

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$2, \ldots, z_{n+1} = h(z_n, u_n)$. Suppose that $u_1, u_2, \ldots$ are independent, identically distributed random variables. Let (Ω, F, p) be the probability space underlying realizations of the sequences $u_1, u_2, \ldots$. Thus, a sample point $\omega \in \Omega$ gives rise to a realization of an input sequence $u_1, u_2, \ldots$. Consider a given initial state, z_1. The states $z_2, z_3, \ldots$ depend on z_1 and on $\omega \in \Omega$. Now, suppose that the state transition function h depends on a Euclidean variable x and it has the form $h(z, x, u)$, where z is an element in the state space and u is an input. Then, the states $z_2, z_3, \ldots$ also depend on x, by the formula

$$z_{n+1}(x) = h(z_n(x), x, u_n). \tag{1}$$

Notice that $z_n(x)$ also depends on z_1 and on $\omega \in \Omega$. Let g be a real-valued (output) function of z, x, u. Let $f(x)$ be the expected value of $g(z_n(x), x, u_n)$. Under appropriate ergodicity conditions, for every z_1 and for almost every $\omega \in \Omega$,

$$\lim_{N \to \infty} \sum_{n=1}^{N} g(z_n(x), x, u_n)/N = f(x). \tag{2}$$

Suppose that it is desirable to compute an estimate of $df(x)/dx$, by using simulation of the dynamical system. What comes to the mind is to use (2), if (3) below is satisfied for every z_1 and for almost every $\omega \in \Omega$:

$$\lim_{N \to \infty} \sum_{n=1}^{N} \frac{dg}{dx}(z_n(x), x, u_n)/N = \frac{df}{dx}(x). \tag{3}$$

Perturbation analysis (PA) is an efficient technique for evaluating

$$\sum_{n=1}^{N} \frac{dg}{dx}(z_n(x), x, u_n)/N,$$

for a given integer N, in one simulation run (see Ref. 1). An important question is the validity of (3). The question has been answered affirmatively for a number of simple queueing systems: for M/G/1 queues (Ref. 2), GI/G/1 queues (Refs. 3 and 4), M/M/m queues (Ref. 5), and closed Jackson networks (Ref. 6). Sufficient conditions for (3) were given in Ref. 7. If (3) is satisfied, then PA is said to be exact or to give strongly consistent estimates of $df(x)/dx$ (see Ref. 2).

The purpose of this paper is to show that PA gives strongly consistent estimates for a single queue with finite buffer capacity and a policy of discarding customers, or parts thereof, when all the buffer space becomes full. Consider a FIFO queue. Let ζ and s denote its interarrival time and service time random variables, respectively. Suppose that $\bar{\zeta} > \bar{s}$ (overbar denotes expectation). Let x represent the buffer capacity (see Fig. 1). x is expressed in time units; namely, if the buffer is full, then it takes the server x seconds to empty the queue. Let C_n, $n = 1, 2, \ldots$, denote the nth arriving customer. Let $z_n(x)$ denote the waiting time of C_n, namely, the amount of

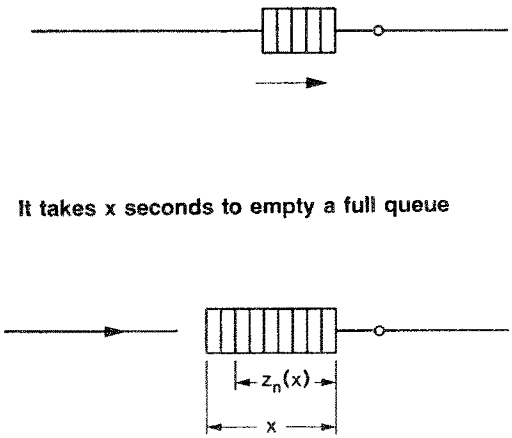

Fig. 1. Queue with policy of discarding customers or parts thereof.

time between the arrival of C_n and the start of its service. Let s_n denote the service time of C_n, if no part of it is being lost. Let t_n denote the time at which C_n arrives. Consider the system at time t_n. If $z_n(x)+s_n \leq x$, then all of C_n fits into the buffer. If $z_n(x)+s_n > x$, then as much as possible of C_n joins the queue, and the part of C_n which does not fit in the queue is discarded.

The queue described above can be used to model a number of systems. For example, in situations where the customer represents the amount of fluid arriving to a reservoir at discrete points in time, the server represents the fluid flow rate out of the reservoir, and any quantity in excess of the reservoir's capacity is directed out of the system. As another example, in packet switching for voice communications, it is generally desirable to limit the packets transmission times; hence, the buffers at the switches could have limited capacities. Typically, a packet arriving to a full queue is discarded. The model described above, allowing for a customer's (packet's) part to be discarded, can act as an approximation (especially, if the packets are transmitted very fast). As a last example, consider a situation where a standard finite buffer queue is used, the service time is a constant (deterministic), and arrivals occur in large, variable-size batches of customers. In such a case, the term customer can be redefined to mean a batch of arrivals, and its service time is proportional to its size. If a batch does not entirely fit in the buffer, a part of it is discarded. Naturally, the discarded amount may be measured in discrete units (corresponding to the nominal customers comprising the batch); but, if the batches are large, and if the service times of nominal customers are small, then our model can provide an adequate approximation. Generally, the amount of buffer can represent a trade-off between the maximum customer delay and the average amount of customer loss.

Let $y_n(x)$ denote the amount of C_n discarded. Figure 2 illustrates the dependence of $y_n(x)$ on $z_n(x)$ and s_n. Functionally,

$$y_n(x) = \max(z_n(x) + s_n - x, 0). \tag{4}$$

If $z_n(x) + s_n - x \leq 0$, then C_n is fully accepted, and it incurs no loss, $y_n(x) = 0$. If $z_n(x) + s_n - x \geq 0$, then C_n is only partially accepted, and it incurs some loss, $y_n(x) = z_n(x) + s_n - x$. Its actual service time, after part of it has been discarded, is $s_n - y_n(x)$. Let $\zeta_n \triangleq t_{n+1} - t_n$. Let $u_n \triangleq (s_n, \zeta_n)$. Then (see Fig. 3),

$$z_{n+1}(x) = \max(z_n(x) + s_n - y_n(x) - \zeta_n, 0). \tag{5}$$

Notice that (5) is a Lindley-like equation. Indeed, with $x = \infty$, $y_n(x) = 0$, and (5) becomes the Lindley equation for the GI/G/1 queue. With (4) and (5), one has the structure of a discrete-event dynamical system driven by random inputs. Events are arrivals of customers, occurring at times t_n, $n = 1, 2, \ldots$. The inputs are $u_n = (s_n, \zeta_n)$. The states are $z_n(x)$. The state transition function h [see (1)] has the form

$$h(z_n(x), x, u_n) = \max(z_n(x) + s_n - \max(z_n(x) + s_n - x, 0) - \zeta_n, 0);$$

see (4), (5). The output function has the form

$$y_n(x) = g(z_n(x), x, u_n) = \max(z_n(x) + s_n - x, 0).$$

Let

$$Y(x) = \text{a.s.} \lim_{N \to \infty} \sum_{n=1}^{N} y_n(x)/N.$$

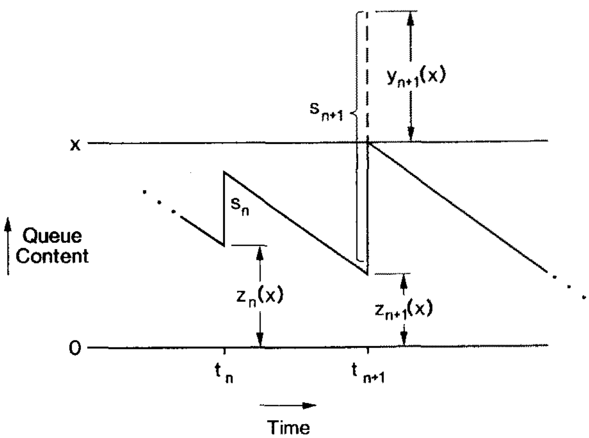

Fig. 2. Time-dependent trajectory of the queue content. Here, $z_n(x) + s_n < x \Rightarrow y_n(x) = 0$, $z_{n+1}(x) + s_{n+1} > x \Rightarrow y_{n+1}(x) = z_{n+1}(x) + s_{n+1} - x$.

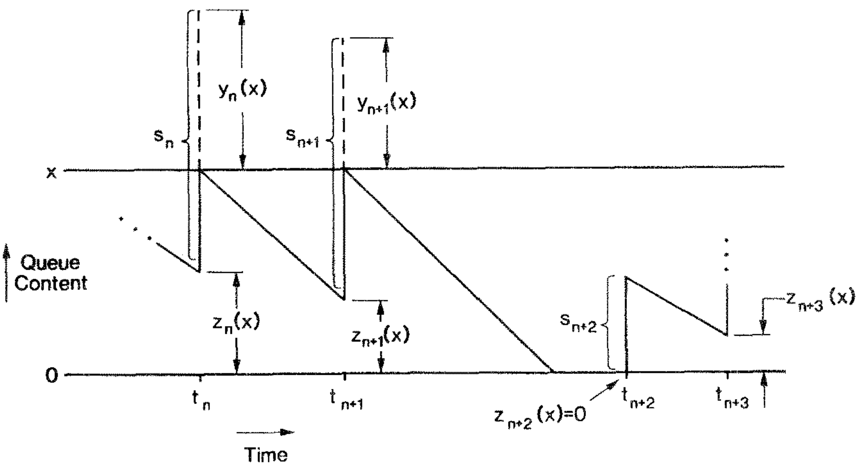

Fig. 3. State transition function. Here,
$y_n(x)=z_n(x)+s_n-x \Rightarrow z_{n+1}(x)=z_n(x)+s_n-y_n(x)-\zeta_n=x-\zeta_n$,
$y_{n+1}(x)=z_{n+1}(x)+s_{n+1}-x,\ x<\zeta_{n+1} \Rightarrow z_{n+2}(x)=0$,
$z_{n+3}(x)=s_{n+2}-\zeta_{n+2}$.

Since $\bar{s}<\bar{\zeta}$, the system has a regenerative structure (see Fig. 4), and $Y(x)$ exists. Moreover, for every $z_1=z_1(x)$, for almost every $\omega\in\Omega$,

$$\lim_{N\to\infty}\sum_{n=1}^{N} y_n(x)/N=Y(x). \tag{6}$$

The following assumption will be made.

Assumption 1.1. (i) The random variable ζ has a bounded density function.

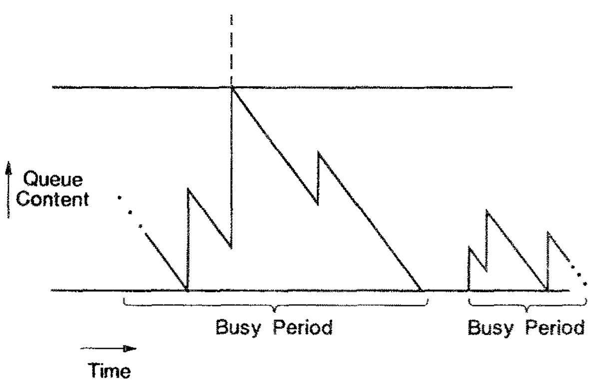

Fig. 4. Regenerative structure.

(ii) The distribution function of the random variable s has at most a finite number of discontinuities. Let $x_1 < x_2 < \cdots < x_J$ be the points at which it is discontinuous. Then, the distribution function of s has a bounded, continuous derivative on $[0, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_J, \infty)$.

(iii) There exists a point $x > x_J$ such that $p(\zeta > x) > 0$.

Remark 1.1. Assumption 1.1 is satisfied under fairly general conditions, for instance, when ζ is supported on the entire half-line R^+, where it has a bounded density function (e.g., exponential distribution), and the distribution function of s is continuously differentiable at all but a finite number of points. It is then clear that (i) and (ii) are satisfied. To see (iii), notice that, for every x, $p(\zeta > x) > 0$. The latter condition on s is satisfied for quite a few distributions, including deterministic, uniform, exponential, and erlangian.

For technicalities involved with the proof of Lemma 2.2, below, we will prove the strong consistency of PA only for points x satisfying $x > x_J$. This restriction may be satisfied by the following argument: Loss of a part of a customer is generally an undesired phenomenon. Therefore, it is natural to have the value of x in such a way that only customers which arrive to a heavily loaded queue could incur loss. In this case, x should be outside of the support of s, preventing the possibility that an arriving customer, which finds the queue empty, could incur loss. In aiming at a more general analysis, we allow x to be within the support of s (which would be the case if s were supported on all of R^+), but we insist that x be greater than any jump point of the distribution function of s, i.e., $x > x_J$. Thus, we will consider x belonging to a closed interval Γ, whose left point is greater than x_J. Let $\Gamma = [\tilde{x}, \bar{x}]$, for some points $\tilde{x} > x_J$ and $\bar{x} > \tilde{x}$. We will also assume that $p(\zeta > \bar{x}) > 0$. Notice that the latter assumption is trivially satisfied whenever ζ is supported on all of R^+ (e.g., ζ is exponentially distributed); in general, the existence of $\bar{x}$ satisfying $p(\zeta > \bar{x}) > 0$ follows from Assumption 1.1(iii). Assumption 1.1 and the conditions on Γ are satisfied in the following situations: ζ is supported on R^+, where it has a bounded density function (e.g., exponentially distributed), and either one of the following four conditions is satisfied:

(i) s has a bounded density function (namely, its distribution function has a bounded derivative) and $\Gamma = [\tilde{x}, \bar{x}]$ is any closed interval in R^+;

(ii) s has a deterministic distribution, $s = x_1$ w.p.1 for some $x_1 > 0$, $\tilde{x} > x_1$, and $\bar{x} > \tilde{x}$;

(iii) s is uniformly distributed on an interval $[x_1, x_2]$, $\tilde{x} > x_2$ and $\bar{x} > \tilde{x}$;

(iv) s has a discrete distribution; it can have values only from a finite set, $x_1, \ldots, x_J$, $\tilde{x} > x_J$, and $\bar{x} > \tilde{x}$.

It can thus be seen that Assumption 1.1 and the conditions on Γ are quite general. Consider $x \in \Gamma$. By Assumption 1.1, (4), and (5), for every $z_1 = z_1(x)$, for almost every $\omega \in \Omega$, $dy_n(x)/dx$ exists for all $n = 1, 2, \ldots$.

Let prime denote derivative with respect to x. The following proposition will be established.

Proposition 1.1. The function $x \to Y(x)$ is continuously differentiable on Γ.

Theorem 1.1. Perturbation analysis gives strongly consistent estimates of $Y'(x)$, $x \in \Gamma$; namely, for every $z_1 = z_1(x)$ and for almost every $\omega \in \Omega$,

$$\lim_{N\to\infty} \sum_{n=1}^{N} y'_n(x)/N = Y'(x). \tag{7}$$

The analysis is carried out in Section 2. Section 3 contains a numerical example. Section 4 is a conclusion. The appendix contains proofs not provided in the body of the paper.

2. Analysis

In this section, Proposition 1.1 and Theorem 1.1 will be proved. The structure of the proofs is the following: Consider an $x \in \Gamma$. Recall that a.s. $y'_n(x)$ exists for every $n = 1, 2, \ldots$. In order to avoid a situation where $y'_n(x)$ does not exist (which occurs with probability 0), we will define a sequence of random functions $\eta_n(x)$, $n = 1, 2, \ldots$, such that $\eta_n(x)$ always exists and $\eta_n(x) = y'_n(x)$ whenever the latter exists. We will show that $\eta_n(x)$ can have the values 0 or -1, specifically: $\eta_n(x) = -1$, if the nth customer arriving to the queue is the first customer in its busy period which incurs loss; and $\eta_n(x) = 0$, otherwise. Thus, we get an expression for $y'_n(x)$, reflecting the fact that only the first customer in a busy period which incurs loss absorbs any additional infinitesimal buffer space (this point will be proved in Lemma 2.1). Next, it will be shown that there exists a function $G(x)$, defined on Γ, such that, for every $x \in \Gamma$,

$$\text{w.p.1} \sum_{n=1}^{N} \eta_n(x)/N \to -G(x), \qquad \text{as } N \to \infty;$$

see Eq. (9). Continuity of $G(x)$ will then be proved (Lemma 2.2). Finally, using the inequalities in (10) and (16), finite-difference arguments will establish that $Y'(x)$ exists and that $Y'(x) = -G(x)$, from which the proofs of the proposition and theorem will follow directly. With the expression of $y'_n(x)$ obtained above, Algorithm 2.1 (below) computes a PA estimate of

$Y'(x)$, $\bar{Y}'(x)$. It is computed along a simulation run of the system, keeping track of whether an arriving customer is the first one in its busy period to incur a loss.

Algorithm 2.1.

Data. $x \in \Gamma$, an integer N.

Step 0. Set sum $= 0$; set $j = 1$.

Step 1. A customer just arrived. If the customer is the first one in its busy period which incurs loss, set sum $=$ sum $- 1$.

Step 2. If $j < N$, set $j = j + 1$, and go to Step 1. Otherwise, go to Step 3.

Step 3. Stop the simulation-run; set $\bar{Y}'(x) = \text{sum}/N$, and exit.

We now address the proofs of Theorem 1.1 and Proposition 1.1. Let C_n denote the nth customer arriving to the queue. Consider $x \in \Gamma$. Let $\bar{\Omega} \subset \Omega$ denote the event that, for every $n = 1, 2, \ldots,$

$$z_n(x) + s_n - x \neq 0 \quad \text{and} \quad z_n(x) + s_n - y_n(x) - \zeta_n \neq 0;$$

see (4) and (5). By Assumption 1.1, $p(\bar{\Omega}) = 1$. Moreover, for every $\omega \in \bar{\Omega}$, $y'_n(x)$ and $z'_n(x)$ exist, $n = 1, 2, \ldots$. For every $n = 1, 2, \ldots$, let

$$k(n, x) \triangleq \max(k \leq n \mid z_k(x) = 0);$$

if $z_k(x) > 0$, for every $k \leq n$, set $k(n, x) = 1$. In other words, $k(n, x) = k$ such that C_k is the first customer in the busy period containing C_n. Let

$$K(n, x) \triangleq \min(k \geq n \mid z_{k+1}(x) = 0).$$

In other words, $K(n, x) = k$ such that C_k is the last customer in the busy period containing C_n. Define the random functions $\eta_n(x)$, $n = 1, 2, \ldots$, as follows: if $y_n(x) > 0$ and either $n = k(n, x)$ or $y_m(x) = 0$ for every $m = k(n, x), \ldots, n - 1$, then set $\eta_n(x) = -1$. Otherwise, set $\eta_n(x) = 0$. Notice that $\eta_n(x) = -1$ if and only if C_n is the first customer in its busy period which incurs loss. $\eta_n(x)$ depends on ω and z_1 (z_1 is assumed fixed). It is illustrated in Fig. 5.

Lemma 2.1. w.p.1, $\eta_n(x) = y'_n(x)$, for every $n = 1, 2, \ldots$.

Proof. Recall that $p(\bar{\Omega}) = 1$. Therefore, it suffices to show that, for every $\omega \in \bar{\Omega}$ and $n = 1, 2, \ldots$, $y'_n(x) = \eta_n(x)$. Consider a busy period, consisting of customers $C_k, \ldots, C_K$. By (5), for all $n = k, \ldots, K - 1$,

$$z_{n+1}(x) = z_n(x) + s_n - y_n(x) - \zeta_n.$$

Hence,

$$z'_{n+1}(x) = z'_n(x) - y'_n(x). \tag{8}$$

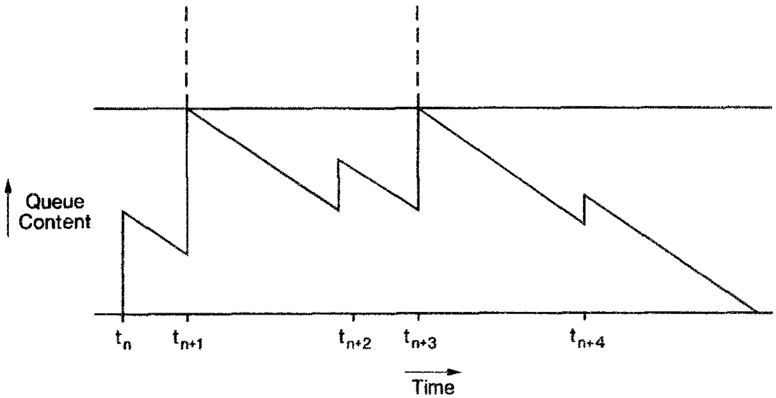

Fig. 5. The function $\eta_n(x)$. Here, $y_{n+1}(x) > 0$, $y_{n+3}(x) > 0$. But $\eta_{n+1}(x) = -1$, $\eta_{n+3}(x) = 0$. Also, $\eta_n(x) = \eta_{n+2}(x) = \eta_{n+4}(x) = 0$.

Consider first the case where, for every $n = k, \ldots, K$, $y_n(x) = 0$. Since $\omega \in \bar{\Omega}$ and by (4), $y'_n(x) = 0$. By the definition of $\eta_n(\cdot)$, $\eta_n(x) = 0$.

Next, suppose that $y_n(x) > 0$ for some $n = k, \ldots, K$. Let

$$m = \min(n = k, \ldots, K \,|\, y_n(x) > 0).$$

Then, $\eta_m(x) = -1$ and, for every $\bar{m} = k, \ldots, K$, $\bar{m} \neq m$, $\eta_{\bar{m}}(x) = 0$. By (4), for every $\bar{m} = k, \ldots, K$, if $y_{\bar{m}}(x) = 0$, then $y'_{\bar{m}}(x) = 0$; and if $y_{\bar{m}}(x) > 0$, then $y_{\bar{m}}(x) = z_{\bar{m}}(x) + s_{\bar{m}} - x$, hence $y'_{\bar{m}}(x) = z'_{\bar{m}}(x) - 1$. Now, $z'_k(x) = 0$, since $z_k(x) = 0$ and $\omega \in \bar{\Omega}$. For every $\bar{m} = k, \ldots, m-1$, $y'_{\bar{m}}(x) = 0$; hence, by (8), $z'_{\bar{m}+1}(x) = 0$. Hence, $z'_m(x) = 0$. Therefore and since $y_m(x) > 0$,

$$y'_m(x) = z'_m(x) - 1 = 0 - 1 = -1.$$

But $\eta_m(x) = -1$, hence $y'_m(x) = \eta_m(x)$. We have shown that, for every $\bar{m} = k, \ldots, m$, $y'_{\bar{m}}(x) = \eta_{\bar{m}}(x)$. It remains to show the latter equality for $\bar{m} = m+1, \ldots, K$.

We will now show by induction that, for every $\bar{m} = m+1, \ldots, K$, $y'_{\bar{m}}(x) = 0$ and $z'_{\bar{m}}(x) = 1$. Consider first the case where $\bar{m} = m+1$. By (8), $z'_{m+1}(x) = 0 - (-1) = 1$. If $y_{m+1}(x) = 0$, then $y'_{m+1}(x) = 0$. If $y_{m+1}(x) > 0$, then $y'_{m+1}(x) = z'_{m+1}(x) - 1 = 0$. In any case, $y'_{m+1}(x) = 0$, and as we have seen, $z'_{m+1}(x) = 1$. Next, we turn to the inductive argument. Suppose that, for some $\bar{m} = m+1, \ldots, K-1$, $z'_{\bar{m}}(x) = 1$ and $y'_{\bar{m}}(x) = 0$. By (8), $z'_{\bar{m}+1}(x) = 1$. If $y_{\bar{m}+1}(x) = 0$, then $y'_{\bar{m}+1}(x) = 0$. If $y_{\bar{m}+1}(x) > 0$, then $y'_{\bar{m}+1}(x) = z'_{\bar{m}+1}(x) - 1 = 0$. In any event, $y'_{\bar{m}+1}(x) = 0$. This completes the inductive argument. Therefore, for every $\bar{m} = m+1, \ldots, K$, $y'_{\bar{m}}(x) = 0$. But $\eta_{\bar{m}}(x) = 0$; hence, the proof of the lemma is complete. □

Let $G_1(x)$ denote the probability that at least one customer incurs loss in a busy period. Let $G_2(x)$ denote the average number of customers in a busy period. Let $G(x)=G_1(x)/G_2(x)$. By the definition of $\eta_n(x)$, for every $\omega\in\Omega$, a.s.,

$$\lim_{N\to\infty}\sum_{n=1}^{N}\eta_n(x)/N=-G(x). \tag{9}$$

Lemma 2.2. $G(\cdot)$ is continuous on Γ.

Proof. See the appendix. □

Proof of Theorem 1.1. By (9) and Lemma 2.1, it suffices to show that $Y'(x)$ exists and that $Y'(x)=-G(x)$. The main arguments in the proof are the following: Let $x\in\Gamma$ and $\delta>0$ be given, such that $x+\delta\in\Gamma$. For an $\omega\in\Omega$, consider a busy period of the queue, where x is the buffer capacity (ω may belong to $\bar{\Omega}^C$; in this case, a busy period ends when the next one begins; we consider them as two distinct busy periods). Suppose that the busy period starts with customer C_k and ends with customer C_K, for some integers k and K. We will show that

$$\delta\sum_{n=k}^{K}\eta_n(x)\le\sum_{n=k}^{K}(y_n(x+\delta)-y_n(x)). \tag{10}$$

By (6) and (9), a.s.,

$$\lim_{N\to\infty}\sum_{n=1}^{N}y_n(x+\delta)/N=Y(x+\delta), \tag{11}$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}y_n(x)/N=Y(x), \tag{12}$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}\eta_n(x)/N=-G(x). \tag{13}$$

Fix an ω where (11)-(13) are satisfied. Take the limits in (11)-(13) over a sequence of integers $N(m)$, $m=1,2,\ldots$, such that $C_{N(m)}$ ends the mth busy period of the queue, with x being the buffer capacity (ω was fixed above). By (10), for all m,

$$\delta\sum_{n=1}^{N(m)}\eta_n(x)/N(m)\le\sum_{n=1}^{N(m)}(y_n(x+\delta)-y_n(x))/N(m). \tag{14}$$

Hence and by (11)-(13),

$$(Y(x+\delta)-Y(x))/\delta\ge-G(x).$$

By taking $\delta \to 0$ ($\delta > 0$),

$$\underline{\lim}(Y(x+\delta) - Y(x))/\delta \geq -G(x). \tag{15}$$

Next, consider a busy period of the queue, whre $x+\delta$ is the buffer capacity. Suppose that it starts with C_k and ends with C_K [k and K are different from those in the discussion leading to (10)]. We will show that

$$\delta \sum_{n=k}^{K} \eta_n(x+\delta) \geq \sum_{n=k}^{K} (y_n(x+\delta) - y_n(x)). \tag{16}$$

Therefore,

$$(Y(x+\delta) - Y(x))/\delta \leq -G(x+\delta).$$

By taking $\delta \to 0$ ($\delta > 0$) and by Lemma 2.2,

$$\overline{\lim}(Y(x+\delta) - Y(x))/\delta \leq -G(x). \tag{17}$$

By (15) and (17),

$$\lim_{\delta \searrow 0}(Y(x+\delta) - Y(x))/\delta = -G(x). \tag{18}$$

The limit where $\delta < 0$ can be shown in a similar way.

It remains to establish (10) and (16).

Proof of (10). The busy period in question is with x being the buffer capacity. It starts with C_k and ends with C_K. Therefore, and by (5), $z_k(x) = 0$, and for $n = k, \ldots, K-1$,

$$z_{n+1}(x) = z_n(x) + s_n - y_n(x) - \zeta_n > 0.$$

$z_k(x+\delta)$ may be positive, but $z_n(\cdot)$ is monotone nondecreasing in x; hence, for every $n = k, \ldots, K-1$, $z_{n+1}(x+\delta) > 0$. By (5),

$$z_{n+1}(x+\delta) = z_n(x+\delta) + s_n - y_n(x+\delta) - \zeta_n > 0.$$

In showing (10), we analyze two cases.

Case I. $k = K$. If $y_k(x) = 0$, then $y_k(x+\delta) = 0$ [since $y_k(\cdot)$ is monotone nonincreasing], and $\eta_n(x) = 0$. Hence, (10) is satisfied. If $y_k(x) > 0$, then by (4),

$$y_k(x) = s_k - x, \qquad \eta_k(x) = -1,$$

and by (4),

$$y_k(x+\delta) \geq z_k(x+\delta) + s_k - (x+\delta).$$

Hence,

$$y_k(x+\delta) - y_k(x) \geq z_k(x+\delta) - \delta \geq -\delta,$$

implying (10).

Case II. $k < K$. We will show that, for every $n = k, \ldots, K-1$,

$$z_{n+1}(x) - z_{n+1}(x+\delta) \leq \sum_{m=k}^{n} (y_m(x+\delta) - y_m(x)). \tag{19}$$

The proof of (19) is by induction. First, consider the case where $n = k$. By (5), the fact that $K > k$, and the fact that $z_k(x) = 0$,

$$\begin{aligned} &z_{k+1}(x) - z_{k+1}(x+\delta) \\ &= s_k - y_k(x) - \zeta_k - (z_k(x+\delta) + s_k - y_k(x+\delta) - \zeta_k) \\ &= y_k(x+\delta) - y_k(x) - z_k(x+\delta) \leq y_k(x+\delta) - y_k(x). \end{aligned}$$

Hence, (19) is satisfied with $n = k$.

Next, suppose that (19) is satisfied for some $n = k, \ldots, K-2$. We now show that (19) is satisfied for $n+1$. We have

$$\begin{aligned} &z_{n+2}(x) - z_{n+2}(x+\delta) \\ &= z_{n+1}(x) + s_{n+1} - y_{n+1}(x) - \zeta_{n+1} \\ &- (z_{n+1}(x+\delta) + s_{n+1} - y_{n+1}(x+\delta) - \zeta_{n+1}) \quad \text{[by (5)]} \\ &= z_{n+1}(x) - z_{n+1}(x+\delta) + y_{n+1}(x+\delta) - y_{n+1}(x) \\ &\leq \sum_{m=k}^{n+1} (y_m(x+\delta) - y_m(x)) \quad \text{[by (19) and the inductive argument]}. \end{aligned} \tag{20}$$

Hence, (19) is satisfied for $n+1$. This establishes (19) for $n = k, \ldots, K-1$.

We now use (19) to show (10). In the proof of Lemma 2.1, we saw that $z_n'(x)$ can be 0 or 1. Hence,

$$z_k(x+\delta) - z_k(x) \leq \delta.$$

This fact will be used below. We consider two cases.

Case II(a). $\sum_{n=k}^{K} \eta_n(x) = 0$. In this case, for all $n = k, \ldots, K$, $y_n(x) = 0$. Hence, $y_n(x+\delta) = 0$, since $y_n(x)$ is monotone nonincreasing. Since $\eta_n(x) \leq 0$, this implies (10).

Case II(b). $\sum_{n=k}^{K} \eta_n(x) < 0$. By the definition of $\eta_n(\cdot)$,

$$\sum_{n=k}^{K} \eta_n(x) = -1.$$

It suffices to show that

$$\sum_{n=k}^{K} (y_n(x+\delta) - y_n(x)) \geq -\delta. \tag{21}$$

By (19), with $n = K - 1$,

$$\begin{aligned}\sum_{n=k}^{K} (y_n(x+\delta) - y_n(x)) \\ &= \sum_{n=k}^{K-1} (y_n(x+\delta) - y_n(x)) + y_K(x+\delta) - y_K(x) \\ &\geq z_K(x) - z_K(x+\delta) + y_K(x+\delta) - y_K(x). \qquad (22)\end{aligned}$$

We consider two cases.

Case II(b1). $y_K(x) = 0$. Then, $y_K(x+\delta) = 0$. But $z_K(x) - z_K(x+\delta) \geq -\delta$, since $z'_K(\cdot)$ can be 0 or 1. Hence, and by (22), (21) is satisfied.

Case II(b2). $y_K(x) > 0$. Then, by (4),

$$y_K(x) = z_K(x) + s_K - x$$

and

$$y_K(x+\delta) \geq z_K(x+\delta) + s_K - (x+\delta).$$

Therefore,

$$z_K(x) - z_K(x+\delta) + y_K(x+\delta) - y_K(x) \geq x - s_K + s_K - (x+\delta) = -\delta.$$

Hence, by (22), (21) is satisfied.

This completes the proof of (21), hence of (10).

The proof of (16) is similar to that of (10), hence it is relegated to the appendix.

This completes the proof of Theorem 1.1. □

Proof of Proposition 1.1. The proof follows directly from Theorem 1.1, Lemma 2.2, and the fact that $Y'(x) = -G(x)$. □

3. Numerical Tests

Results of numerical experiments for computing estimates of $Y'(x)$ are reported in this section. We assume on a deterministic service time of $s = 1.0$, that the arrival process is Poisson distributed with rates $\lambda = 0.8$ and $\lambda = 0.95$ (these values of λ correspond to moderate and high traffic intensities), and

Table 1. Results for $\lambda = 0.8$.

N	1000	3000	5000	7000	9000
PA Estimates	−0.147	−0.154	−0.147	−0.148	−0.151
FD Estimates ($\delta = 0.1$)	+0.201	−0.208	−0.113	−0.162	−0.154
FD Estimate ($\delta = 0.01$)	+2.850	+0.286	+0.332	+0.032	−1.073

that the buffer capacity is $x = 1.6$. Two estimates are computed: PA estimates, namely,

$$\sum_{n=1}^{N} y_n'(x)/N; \tag{23}$$

and FD estimates (finite-difference estimates), namely,

$$\left[\sum_{n=1}^{N} y_n(x+\delta)/N - \sum_{n=1}^{N} y_n(x)/N\right] \Big/ \delta, \tag{24}$$

for $\delta = 0.1$ and $\delta = 0.01$, and by using the same seed. The results are shown in Tables 1 and 2, for various values of N. For $\lambda = 0.8$ (Table 1), it can be seen that PA estimates yield approximations with error under 10%, for as few as 1000 iterates. On the other hand, finite-difference estimates with $\delta = 0.1$ require at least 7000 iterates for similar precision, and with $\delta = 0.01$, over 9000 iterates. It is not surprising to see that PA estimates are better than finite-difference estimates. It is interesting to note that finite-difference estimates are worse for $\delta = 0.01$ than for $\delta = 0.1$; this is likely caused by the fact that the variance of finite-difference estimates increases as δ is reduced. For $\lambda = 0.95$ (Table 2), the results indicate that PA estimates require 3000 iterates to achieve an error under 10%, but FD estimates require over 9000 iterates for similar precision, with both $\delta = 0.1$ and $\delta = 0.01$.

Finally, in order to verify that the PA and FD estimates do indeed converge to similar limits, we ran the simulation for 100,000 customers. The results are shown in Tables 3 and 4. The estimates of $Y(x)$ were done by straightforward simulations, and were then used to evaluate the FD estimates. Thus, the FD estimates at $x = 1.6$, with $\delta = 0.1$ and $\delta = 0.01$, were

Table 2. Results for $\lambda = 0.95$.

N	1000	3000	5000	7000	9000
PA Estimates	−0.128	−0.144	−0.144	−0.141	−0.142
FD Estimates ($\delta = 0.1$)	−0.323	−0.190	−0.174	−0.156	−0.213
FD Estimate ($\delta = 0.01$)	−1.700	−0.402	−0.011	−0.062	−1.198

Table 3. Results for $\lambda = 0.8$, $N = 100{,}000$.

x	1.60	1.70	1.61
$Y(x)$	0.1929872	0.1785946	0.1914803
$Y'(x)$	−0.1517100	−0.1368400	−0.1502500

done by $[Y(1.7)-Y(1.6)]/0.1$ and $[Y(1.61)-Y(1.6)]/0.01$, respectively. The estimates of $Y'(x)$ were done by using PA.

The following can be seen: For $\lambda = 0.8$ and $x = 1.6$, the finite difference estimate of $Y'(x)$ with $\delta = 0.1$ is $[Y(1.7)-Y(1.6)]/0.1 = -0.143926$. Notice that it is between the PA estimates at $x = 1.6$ (−0.1517100) and at $x = 1.7$ (−0.136840). When $\delta = 0.01$, the results are more precise: The FD estimate is $[Y(1.61)-Y(1.6)]/0.01 = -0.150690$, which is between the PA estimates at $x = 1.6$ (−0.1517100) and at $x = 1.61$ (−0.1502500).

When $\lambda = 0.95$, similar precision can be seen. When $\delta = 0.1$, the finite-difference estimate of $Y'(x)$ is $[Y(1.7)-Y(1.6)]/0.1 = -0.134365$, between the PA estimates at $x = 1.6$ (−0.1413100) and at $x = 1.7$ (−0.1279500). When $\delta = 0.01$, the FD estimate is $[Y(1.61)-Y(1.6)]/0.01 = -0.140410$, between the PA estimates at $x = 1.6$ (−0.1413100) and at $x = 1.61$ (−0.1398800).

In summary, it can be seen that the PA algorithm gives consistent estimates of $Y'(x)$. Moreover, Tables 1 and 2 indicate that PA estimates converge faster than FD estimates. FD estimates converge faster when $\delta = 0.1$ than when $\delta = 0.01$, but to a less precise estimate. This should not be surprising, since precision and variance of the FD estimator are generally smaller for larger values of δ.

4. Conclusions

Consistency of PA for a single queue with finite buffer space and loss policy has been established. Numerical tests for estimating the derivative of the average quantity of customer loss as a function of the buffer capacity were conducted. PA and finite-difference estimates were compared. The

Table 4. Results for $\lambda = 0.95$, $N = 100{,}000$.

x	1.60	1.70	1.61
$Y(x)$	0.2427748	0.2293383	0.2413707
$Y'(x)$	−0.1413100	−0.1279500	−0.1398800

tests show conclusively that PA estimates are far more accurate than finite-difference estimates, for comparable numbers of simulation events. The proof of consistency of PA estimates uses a novel technique, based on busy period-based inequalities [Eqs. (10) and (16)]. This technique may be generalized to prove the consistency of PA estimates for more general queueing network configurations.

5. Appendix

In this appendix, we prove Lemma 2.2 and Eq. (16).

Proof of Lemma 2.2. It suffices to prove that $G_1(x)$ and $G_2(x)$ are continuous on Γ. We start with G_2.

Proof of Continuity of G_2. Consider $x \in \Gamma$ and $\delta > 0$, such that $x+\delta \in \Gamma$. We can suppose that $z_1(x) = 0$, and consider the first busy period. Let

$$M(x) = \min(m = 1, 2, \ldots \mid z_{m+1}(x) = 0);$$

and let

$$M(x+\delta) = \min(m = 1, 2, \ldots \mid z_{m+1}(x+\delta) = 0).$$

In other words, $C_{M(x)}$ [$C_{M(x+\delta)}$, resp.] is the last customer in the first busy period of the queue, where x [$x+\delta$, resp.] is the buffer capacity. Then for every $\omega \in \Omega$,

$$M(x+\delta) \geq M(x).$$

Notice that

$$G_2(x) = E(M(x)), \qquad G_2(x+\delta) = E(M(x+\delta)).$$

Moreover,

$$G_2(x+\delta) - G_2(x) = E(M(x+\delta) - M(x)).$$

Notice that

$$G_2(x+\delta) - G_2(x)$$
$$= E(M(x+\delta) - M(x) \mid M(x+\delta) > M(x))p(M(x+\delta) > M(x)). \tag{25}$$

Let $\Omega_1 \subset \Omega$ denote the event that $M(x+\delta) > M(x)$. If $\omega \in \Omega_1$, then $C_{M(x)+1}$ belongs to the first busy period with $x+\delta$ being the buffer capacity. Hence, by (5),

$$z_{M(x)+1}(x+\delta) = z_{M(x)}(x+\delta) + s_{M(x)} - y_{M(x)}(x+\delta) - \zeta_{M(x)} > 0.$$

Since $z_{M(x)+1}(x) = 0$,

$$z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x) - \zeta_{M(x)} \le 0.$$

We saw in the proof of Lemma 2.1 that, for every $n = 1, 2, \ldots, z'_n(x)$ can be 0 or 1, and $y'_n(x)$ can be 0 or -1. Therefore,

$$z_{M(x)}(x+\delta) - z_{M(x)}(x) \le \delta,$$

$$y_{M(x)}(x+\delta) - y_{M(x)}(x) \ge -\delta.$$

Hence, if $\omega \in \Omega_1$, since $z_{M(x)+1}(x+\delta) > 0$,

$$\begin{aligned}\zeta_{M(x)} &< z_{M(x)}(x+\delta) + s_{M(x)} - y_{M(x)}(x+\delta) \\ &\le z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x) + 2\delta;\end{aligned} \tag{26}$$

and since $z_{M(x)+1}(x) = 0$,

$$\zeta_{M(x)} \ge z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x). \tag{27}$$

Let $\Omega_2 \subset \Omega$ denote the event that (26) and (27) are satisfied. Then, we have seen that $\Omega_1 \subset \Omega_2$. By Assumption 1.1, the density function of ζ is bounded by some constant K_1. By conditioning on $M(x)$, and by (26) and (27),

$$\begin{aligned}p(\Omega_1) \le p(\Omega_2) &\le p(\zeta_{M(x)} < z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x) \\ &\qquad + 2\delta \mid \zeta_{M(x)} \ge z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x)) \\ &\le 2\delta K_1 / p(\zeta_{M(x)} \ge z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x)).\end{aligned} \tag{28}$$

But

$$z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x) \le x \le \bar{x},$$

where $\bar{x}$ is the right-hand point of Γ. Hence,

$$p(\zeta_{M(x)} \ge z_{M(x)}(x) + s_{M(x)} - y_{M(x)}(x)) \ge p(\zeta_{M(x)} \ge \bar{x}) \triangleq K_2.$$

By Assumption 1.1 and the conditions imposed on Γ, $K_2 > 0$. Therefore, and by (28),

$$p(\Omega_1) \le 2\delta K_1 / K_2. \tag{29}$$

Next, if $M(x+\delta)>M(x)$, then $M(x+\delta)-M(x)$ is the number of customers in the first busy period with $x+\delta$ being the buffer capacity, beyond the $M(x)$th customer. Since $z_{M(x)+1}(x)=0$,

$$z_{M(x)+1}(x+\delta)=z_{M(x)+1}(x+\delta)-z_{M(x)+1}(x)\leq\delta.$$

Therefore, and since $E(M(\bar{x}|z_1(x)=\delta)<\infty$ (by the assumption that $p(\zeta>\bar{x})>0$),

$$\begin{aligned} E(M(x+\delta)-M(x)\,|\,M(x+\delta)>M(x)) &\leq E(M(x+\delta)\,|\,z_1(x)=\delta) \\ &\leq E(M(\infty)\,|\,z_1(x)=\delta) \\ &\triangleq K_3 \\ &<\infty. \end{aligned} \tag{30}$$

By (25), (30), and (29),

$$0\leq G_2(x+\delta)-G_2(x)\leq 2\delta K_1K_3/K_2.$$

This shows that $G_2(\cdot)$ is continuous on Γ.

Proof of Continuity of G_1. Consider $x\in\Gamma$ and $\delta>0$, such that $x+\delta\in\Gamma$. Let $z_1(x)=z_1(x+\delta)=0$. Let $\Omega(x)$ $[\Omega(x+\delta)$, resp.$]\subset\Omega$ denote the event that

$$\sum_{n=1}^{M(x)} y_n(x)>0 \quad \left[\sum_{n=1}^{M(x+\delta)} y_n(x+\delta)>0, \text{resp.}\right].$$

Then,

$$G_1(x)=p(\Omega(x)), \qquad G_1(x+\delta)=p(\Omega(x+\delta)).$$

Let $\Omega_1\subset\Omega$ denote the event that $M(x+\delta)>M(x)$. We saw by (29) that

$$p(\Omega_1)\leq 2\delta K_1/K_2.$$

Now, if $\omega\in\Omega(x+\delta)\cap\Omega(x)^c$, then for some $n=1,\ldots,M(x+\delta)$, $y_n(x+\delta)>0$, and for all $n=1,\ldots,M(x)$, $y_n(x)=0$. Since $y_n(\cdot)$ is monotone nonincreasing, $\Omega(x+\delta)\cap\Omega(x)^c\subset\Omega_1$. Hence,

$$p(\Omega(x+\delta)\cap\Omega(x)^c)\leq 2\delta K_1/K_2. \tag{31}$$

Now,

$$\begin{aligned} &|G_1(x)-G_1(x+\delta)| \\ &=|p(\Omega(x))-p(\Omega(x+\delta))| \\ &\leq p(\Omega(x)\cap\Omega(x+\delta)^c)+p(\Omega(x+\delta)\cap\Omega(x)^c) \\ &\leq p(\Omega(x)\cap\Omega(x+\delta)^c)+2\delta K_1/K_2 \\ &\leq p(\Omega(x)\cap\Omega(x+\delta)^c\cap\Omega_1^c)+p(\Omega_1)+2\delta K_1/K_2 \\ &\leq p(\Omega(x)\cap\Omega(x+\delta)^c\cap\Omega_1^c)+4\delta K_1/K_2. \end{aligned} \tag{32}$$

Recall that Ω_1^c is the event that $M(x)=M(x+\delta)$. For every $m=1,2,\ldots,$ let $H_m\subset\Omega$ denote the event that $M(x)\geq m$ and $0=y_m(x+\delta)<y_m(x)$. Then,

$$\Omega(x)\cap\Omega(x+\delta)^c\cap\Omega_1^c\subset\bigcup_{m=1}^{\infty} H_m. \tag{33}$$

We now find an upper bound on $p(H_m)$. Let L be an upper bound on the derivative of the distribution function of s, whenever it exists (see Assumption 1.1). By Assumption 1.1 and the fact that $x\in\Gamma$,

$$p(H_1)=p(x<s_1\leq x+\delta)\leq L\delta.$$

Consider $m=2,3,\ldots,$. Then,

$$p(H_m)=p(H_m\,|\,M(x)\geq m)p(M(x)\geq m).$$

Now,

$$p(H_m\,|\,M(x)\geq m)$$
$$=p(y_m(x)>0,\, y_m(x+\delta)=0\,|\,z_2(x)>0,\ldots,z_m(x)>0). \tag{34}$$

But

$$y_m(x+\delta)\geq y_m(x)-\delta.$$

Hence, by (34) and the fact that $z_m(x)\leq z_m(x+\delta)$,

$$p(H_m\,|\,M(x)\geq m)\leq p(x<z_m(x)+s_m\leq x+\delta\,|\,z_2(x)>0,\ldots,z_m(x)>0). \tag{35}$$

Let $x_1,\ldots,x_J$ be the real-valued, positive points where the distribution function of s is discontinuous. If

$$z_m(x)\notin\bigcup_{i=1}^{J}[x-x_i,\, x+\delta-x_i],$$

then

$$p(x<z_m(x)+s_m\leq x+\delta\,|\,z_m(x))\leq L\delta.$$

Now, by (5),

$$z_m(x)=z_{m-1}(x)+s_{m-1}-y_{m-1}(x)-\zeta_{m-1};$$

and since ζ has a bounded density function, there exists a constant $K_4>0$ such that

$$p(z_m(x)\in\bigcup_{i=1}^{J}[x-x_i,\, x+\delta-x_i]z_m(x)>0)\leq K_4\delta.$$

Hence, and by (35),

$$p(H_m\,|\,M(x)\geq m)\leq\delta(L+K_4). \tag{36}$$

By (32), (33), and (36), we have

$$\begin{aligned} &|G_1(x) - G_1(x+\delta)| \\ &\le \sum_{m=1}^{\infty} p(H_m) + 4\delta K_1/K_2 \\ &\le \sum_{m=1}^{\infty} p(H_m | M(x) \ge m) p(M(x) \ge m) + 4\delta K_1/K_2 \\ &\le \delta(L + K_4) \sum_{m=1}^{\infty} p(M(x) \ge m) + 4\delta K_1/K_2 \\ &= \delta(L + K_4) E(M(x)) + 4\delta K_1/K_2 \\ &\le \delta(L + K_4) E(M(\bar{x})) + 4\delta K_1/K_2. \end{aligned} \tag{37}$$

Hence, G_1 is continuous on Γ.

This completes the proof of the lemma. □

Proof of Equation (16). The busy period in question is with $x+\delta$ being the buffer capacity. Therefore, $z_k(x+\delta)=0$, and for every $n = k, \ldots, K-1$, $z_{n+1}(x+\delta) > 0$. Also, $z_k(x) = 0$, and by (5), for every $n = k, \ldots, K-1$,

$$z_{n+1}(x) \ge z_n(x) + s_n - y_n(x) - \zeta_n, \tag{38}$$

$$z_{n+1}(x+\delta) = z_n(x+\delta) + s_n - y_n(x+\delta) - \zeta_n. \tag{39}$$

We analyze two cases.

Case I. $k = K$. If $y_k(x+\delta) = 0$, then $\eta_k(x+\delta) = 0$. Since $y_k(x) \ge 0$, (16) is satisfied. If $y_k(x+\delta) > 0$, then $y_k(x) > 0$ and $\eta_k(x+\delta) = -1$. Therefore, and since $z_k(x) = z_k(x+\delta) = 0$,

$$\begin{aligned} y_k(x+\delta) - y_k(x) &= z_k(x+\delta) + s_k - (x+\delta) - (z_k(x) + s_k - x) \\ &= -\delta = \delta\eta_k(x+\delta). \end{aligned}$$

Hence, (16) is satisfied.

Case II. $k < K$. We will show that, for every $n = k, \ldots, K-1$,

$$z_{n+1}(x) - z_{n+1}(x+\delta) \ge \sum_{m=k}^{n} (y_m(x+\delta) - y_m(x)). \tag{40}$$

The proof of (40) is by induction. First, consider the case where $n = k$. By (38) and (39), and since $z_k(x) = z_k(x+\delta) = 0$,

$$\begin{aligned} &z_{k+1}(x) - z_{k+1}(x+\delta) \\ &\ge z_k(x) + s_k - y_k(x) - \zeta_k - (z_k(x+\delta) + s_k - y_k(x+\delta) - \zeta_k) \\ &= y_k(x+\delta) - y_k(x). \end{aligned}$$

Hence, (40) is satisfied with $n = k$. Next, suppose that (40) is satisfied for some $n = k, \ldots, K-2$. We now show that (40) is satisfied for $n+1$. We have

$$\begin{aligned}
&z_{n+2}(x) - z_{n+2}(x+\delta) \\
&\geq z_{n+1}(x) + s_{n+1} - y_{n+1}(x) - \zeta_{n+1} \\
&\quad - (z_{n+1}(x+\delta) + s_{n+1} - y_{n+1}(x+\delta) - \zeta_{n+1}) \\
&= z_{n+1}(x) - z_{n+1}(x+\delta) + y_{n+1}(x+\delta) - y_{n+1}(x) \quad \text{[by (38) and (39)]} \\
&\geq \sum_{m=k}^{n} (y_m(x+\delta) - y_m(x)) + y_{n+1}(x+\delta) - y_{n+1}(x) \\
&\qquad \text{[by (40) and the inductive argument]} \\
&= \sum_{m=k}^{n+1} (y_m(x+\delta) - y_m(x)). \qquad (41)
\end{aligned}$$

This shows (40) for $n+1$. Hence, (40) is satisfied for $n = k, \ldots, K-1$.

We now turn to establish (16). We consider two cases.

Case II(a). $\sum_{n=k}^{K} \eta_n(x+\delta) = 0$. Then, for every $n = k, \ldots, K$, $y_n(x+\delta) = 0$. Since $y_n(x) \geq 0$, (16) is satisfied.

Case II(b). $\sum_{n=k}^{K} \eta_n(x+\delta) = -1$. It suffices to show that

$$\sum_{n=k}^{K} (y_n(x+\delta) - y_n(x)) \leq -\delta. \qquad (42)$$

Since

$$\sum_{n=k}^{K} \eta_n(x+\delta) = -1,$$

there exists $n = k, \ldots, K$, such that $y_n(x+\delta) > 0$. Let

$$l \triangleq \max(n = k, \ldots, K \mid y_n(x+\delta) > 0).$$

Then,

$$y_l(x+\delta) = z_l(x+\delta) + s_l - (x+\delta) > 0.$$

Therefore, and by the fact that $y_l(\cdot)$ is monotone nonincreasing,

$$y_l(x) = z_l(x) + s_l - x > 0.$$

Therefore, and by (40) with $n = l-1$,

$$\begin{aligned}
&\sum_{m=k}^{l} (y_m(x+\delta) - y_m(x)) \\
&\leq y_l(x+\delta) - y_l(x) + z_l(x) - z_l(x+\delta) \\
&= z_l(x+\delta) + s_l - (x+\delta) - (z_l(x) + s_l - x) + z_l(x) - z_l(x+\delta) \\
&= -\delta. \qquad (43)
\end{aligned}$$

Since for all $n = l+1, \ldots, K$, $y_n(x+\delta) = 0$ [by the definition of l] and $y_n(x) \geq 0$, (42) follows. Hence, (16) follows. □

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