Ruin Theory Problems in Simple SDE Models with Large Deviation Asymptotics

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Abstract

We examine hitting probability problems for Ornstein-Uhlenbeck (OU) processes and Geometric Brownian motions (GBM) with respect to exponential boundaries related to problems arising in risk theory and asset and liability models in pension funds. In Section 2 we consider the OU process described by the Stochastic Differential Equation (SDE) $dX_t = \mu X_t dt + \sigma dW_t$ with $X_0 = x_0$ evolving between a lower and an upper deterministic exponential boundary. Both the finite horizon "ruin probability" problem and the corresponding infinite horizon problem is examined in the low noise case, using the Wentzell-Freidlin approach in order to obtain logarithmic asymptotics for the probability of hitting either the lower or the upper boundary. The resulting variational problems are studied in detail. The exponential rate characterizing the ruin probability and the "path to ruin" are obtained by their solution. Logarithmic asymptotics for the meeting probability in a pair of OU processes with different positive drift coefficients, driven by independent Brownian motions is also obtained using Wentzell-Freidlin techniques. The optimal paths followed by the two processes and the meeting time T are determined by solving a variational problem with transversality conditions. In Section 3 a corresponding problem involving a Geometric Brownian motion is considered. Since in this case, an exact, closed form solution is also available and we take advantage of this situation in order to explore numerically the quality of the Large Deviations results obtained using the Wentzell-Freidlin approach.

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1 Introduction

We examine simple linear Stochastic Differential Equations (SDE) describing Ornstein-Uhlenbeck (OU) and Geometric Brownian motion (GBM) processes with positive drift and consider the "ruin problem" of hitting an upper or lower exponential boundary. This problem, is not analytically tractable in closed form for the OU process and we use the Wentzel-Freidlin approach in order to obtain Large Deviations estimates for the ruin probability. More specifically, suppose that the OU process describing the free reserves process $\{X_t\}$ is the solution of the SDE $dX_t = \mu X_t dt + \sigma dW_t$ with $X_0 = x_0$ given, where $\mu > 0$ and $\{W_t\}$ is standard Brownian motion. Suppose further that $V(t) := v_0 e^{\beta t}$ and $U(t) := u_0 e^{\alpha t}$ are two exponential (deterministic) boundary curves, and that initially the free reserves lie between these values, i.e. $0 < v_0 < x_0 < u_0$ and that $0 < \beta < \mu < \alpha$. We examine both the finite horizon and infinite horizon ruin problem, i.e. the probability

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that the OU process remains between the two exponential bounds over a finite or an infinite time horizon. These problems may of course be formulated in terms of a second order PDE with curved (exponential) boundaries in the plane and solved numerically. (An alternative approach, involving a time change argument is also discussed briefly.) The main thrust of the analysis however involves Large Deviations techniques and in particular the Wentzell-Freidlin approach in order to obtain logarithmic asymptotics for the probability of hitting either the lower or the upper boundary. These low-noise asymptotics are valid when the variance σ is small and hence the event of hitting either boundary is rare. The exponential rate characterizing this probability is obtained by solving a variational problem which also gives the "path to ruin". We begin with a careful and detailed analysis of the finite horizon problem of hitting a lower boundary. The infinite horizon problem both for hitting the lower and the upper exponential boundary is treated using the transversality conditions approach of the calculus of variations. In addition, for the OU process with a more general linear drift resulting from the SDE $dX_t = (\mu X_t + r)dt + \sigma dW_t$, the probability of hitting an upper exponential boundary $u_0 e^{\alpha t}$ is examined (with $0 < \mu < \alpha$).

We also consider the problem of two independent OU processes, $\{X_t\}$, $\{Y_t\}$, with initial values $x_0 > y_0$ and average growth rates α and β respectively with $\alpha > \beta$ so that, in the absence of noise, it would hold that $X_t > Y_t$ for all $t > 0$. We examine, again using the Wentzell-Freidlin approach, the probability that the two processes meet. The optimal paths followed by the two processes and the meeting time T is determined by solving a variational problem with trasversality conditions.

In section 4 a corresponding problem involving a Geometric Brownian motion described by the SDE $dX_t = \mu X_t dt + \sigma X_t dW_t$ with $X_0 = x_0$ is examined, together with an upper and a lower exponential boundary. Again the Wentzell-Freidlin theory is used. In this case however, an exact solution is also possible, and therefore we are able to obtain an idea of the accuracy of the logarithmic asymptotics we propose. As expected, when the variance constant σ becomes smaller, the quality of the approximation improves. The case of two correlated Geometric Brownian motions is also discussed. These models are inspired by the Gerber and Shiu (2003) model of assets and liabilities in pension funds.

Such models arise naturally when analyzing systems with compounding assets. Consider the following collective risk model: Claims are i.i.d. random variables $\{Y_i\}$, with distribution F on $\mathbb{R}^+$, and they occur according to an independent Poisson process with points $\{T_n\}$ and rate λ. Denote by $N_t := \sum_{i=1}^{\infty} \mathbf{1}(T_i \leq t)$ the corresponding counting process. Income from premiums comes at a constant rate c and the initial value of the free reserves is x_0. We assume further that free reserves accrue interest at a fixed rate β. If we denote by $Z_t := ct - \sum_{i=1}^{N_t} Y_i$, $t \geq 0$, the process describing net income (i.e. premium income minus liabilities due to claims), then the free reserves process is described by the SDE (Stochastic Differential Equation)

$$dX_t \;=\; \beta X_t dt + dZ_t, \quad X_0 = x_0. \tag{1.1}$$

Along the above lines, Harrison (1977) considered a generalization of the classical model of collective risk theory in which the net income process of a firm, $\{Z_t\}$, has stationary independent increments and finite variance and has paths that have finite variation w.p.1. Then the assets of the firm at time t, X_t, can be represented by a simple path-wise integral with respect to the income process Z as

$$X_t = e^{\beta t} x_0 + \int_0^t e^{\beta(t-s)} dZ_s, \quad t \geq 0, \tag{1.2}$$

where $x_0 > 0$ denotes the initial level of assets and $\beta > 0$ the interest rate. In this setting the Riemann-Stieltjes definition can be used for the integral on the right side of (1.2) which is finite for all $t \geq 0$ and almost every sample path of Z. Thus the process X is defined path-wise in terms of the income process Z_t. A model with Z_t being Brownian motion with drift would be natural as a diffusion approximation of such a model and this leads to the Ornstein-Uhlenbeck model we examine in detail in this paper.

Models with compounding assets occur naturally also in the study of pension funds. Gerber and Shiu (2003) have studied such models involving a pair of Geometric Brownian Motion processes with positive drift representing assets and liabilities over time and in this context ruin problems become relevant. With the notable exception of some Geometric Brownian Motion models, analytic solutions in closed form are not possible in general and thus we will study here ruin problems related to these models using Large Deviations techniques. For an overview of the vast subject of ruin probabilities in Risk Theory models see Asmussen and Albrecher (2010). See also Pham (2007) for an overview of large deviation techniques for estimating ruin probabilities.

The use of sample path large deviations for Stochastic Differential Equations with a low noise term was pioneered by Wentzell and Freidlin in the 1970's (see Wentzell and Freidlin, 2012, and the references therein) and Kifer (1974). In the 1980's this problem was studied in particular by Azencott (1980), Priouret (1982) and Azencott (1985). The type of problem we examine in this paper can be is related to the exit probability of a low noise diffusion from an unbounded domain, typically the positive quadrant of $\mathbb{R}^2$. Asymptotics for exit times and hitting probabilities of diffusions as well as applications to the simulation of rare events have been studied in Cottrell, Fort, and Malgouyres (1985), Dupuis and Kushner (1986), Bobrovsky and Zeitouni (1992), Baldi (1995), Simonian (1995), Maier and Stein (1997), and Freidlin, Koralov, and Wentzell (2017).

Exit problems for diffusions with low noise have also been studied extensively using analytic techniques based on singular perturbation theory. Matkowski and Schuss (1977), Williams (1981) and the references therein. See also Ludwig (1975) and Day (1984).

Baldi and Caramellino (2011) extend the Freidlin-Wentzell technique to diffusions on the positive half line with coefficients that are neither bounded nor Lipschitz conditions. Baldi (1995) and Robertson (2010) give applications of the Wentzell-Freidlin Large Deviation techniques to the simulation of diffusion processes. Low noise asymptotics using Wentzell-Freidlin techniques can also be used to study metastability phenomena and *tipping events* in climatology, epidemiology, and ecological systems. For some recent such results see Hill, Zanetell and Gemmer (2022).

Ruin problems related to Gaussian processes may also be analyzed using results from the vast literature on extreme values of Gaussian processes developed in the last four decades, obtaining exact as opposed to logarithmic asymptotic results. We refer the reader to Hüsler (1990), Piterbarg (1996), Hüsler and Piterbarg (1999), Dieker (2005), and the references therein for an overview.

Exact asymptotics have been obtained for various types of ruin problems involing integrated Gaussian processes in Dębicki (2002) and Kobelkov (2005) and also in He and Hu (2007) and Dębicki, Hashorva and Ji (2015) who consider models that include force of interest. Hüsler and Piterbarg (2004) obtain exact aysmptotics for the ruin probability of "physical fractional Brownian motion" which is an integrated fractional Brownian motion with deterministic drift. Hüsler and Piterbarg (2008) obtain the asymptotic distribution of the time to ruin for a class of Gaussian processes with variance function which is regularly varying at infinity and drift $-ct^{\beta}$. More recently multidimensional problems considering simultaneous ruin in Gaussian models have been considered as well. Dębicki, Hashorva, and Wang (2020) and Bisewski, Dębicki and Kriukov (2023) obtain exact asymptotics for simultaneous ruin in multivariate settings.

The domain of applicability of these techniques is of course different from that of the Wentzell-Freidlin large deviation approach. The Wentzell-Freidlin approach used here gives only logarithmic asymptotics. However it is formulated as a variational problem whose solution gives the "path to ruin" and the time to ruin. Thus the solution gives valuable insights, useful in applications. In Section 5.3 we discuss briefly how exact asymptotics may be obtained using techniques from the theory of extremes of Gaussian processes in one of the models examined in this paper.

2 Low noise asymptotics for the Ornstein-Uhlenbeck process

In this section we examine an Ornstein-Uhlenbeck (OU) process with positive infinitesimal drift and consider the probability of hitting an upper or a lower exponential boundary. The problem is approached using the Wentzell-Freidlin theory for obtaining logarithmic asymptotics both for the finite and the infinite horizon problem. An OU process with an additional constant term in the drift is also examined. Interestingly, depending on the value of the constant drift, the variational problem from which the rate function is obtained, may or may not have a unique solution.

We begin by quoting a famous large deviations result for the Brownian motion which will be very important in the sequel.

2.1 Large Deviation Results for the Paths of the Wiener Process

Here we mention some results relating to Large Deviations. We refer the reader to Dembo and Zeitouni (2010) for further background. Recall that a function f is *lower semicontinuous* iff, for every sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$, $\liminf_{n\to\infty} f(x_n) \geq f(x)$. Let $(\mathcal{X}, d)$ a metric space and $\mathcal{B}$ the Borel σ–field generated by the open sets of $\mathcal{X}$. If a function $I : \mathcal{X} \to [0, \infty]$ is lower semicontinuous then the level sets $\Psi_I(y) := \{x \in \mathcal{X} : I(x) \leq y\}$ are *closed subsets of* $\mathcal{X}$. Suppose also that $\{\mu_n\}$ is a family of measures on $(\mathcal{X}, \mathcal{B})$.

A lower semicontinuous function $I : \mathcal{X} \to [0, \infty]$ is called a *good rate function* if the level sets $\Psi_I(y)$ are *compact* subsets of $\mathcal{X}$. The effective domain of the rate function I is the subset of $\mathcal{X}$, $\mathcal{D}_I := \{x : I(x) < \infty\}$ for which the rate function is finite. As usual, for any $\Gamma \subset \mathcal{X}$, $\bar{\Gamma}$ denotes the *closure* and Γ^o the *interior* of Γ. With the above definitions one may give the following precise statement of the Large Deviation Principle (LDP):

Definition 1. *The family of measures on* $\{\mu_\epsilon\}$ *satisfies an LDP with rate function* I *if for all* $\Gamma \in \mathcal{B}$,

$$-\inf_{x\in\Gamma^o} I(x) \leq \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq -\inf_{x\in\bar{\Gamma}} I(x). \tag{2.1}$$

A fundamental result in sample path Large Deviations theory is the following theorem due to Schilder (1966). Suppose that $\{W(t); t \in [0, 1]\}$ is a Standard Brownian Motion in $\mathbb{R}$ and let $\{W_\epsilon(t); t \in [0, 1]\}$ denote a family of processes, indexed by $\epsilon > 0$ and defined via $W_\epsilon(t) := \sqrt{\epsilon}\, W(t)$. Here $\mathcal{X} = C_0[0, 1]$, is the set of continuous real functions on $[0, 1]$ which vanish at 0, equipped with the norm of uniform convergence.

Theorem 1 (Schilder)**.** *The family of measures* $\{\mu_\epsilon\}$ *induced on* $C_0[0, 1]$ *by the family of processes* $\{W_\epsilon(t); t \in [0, 1]\}$ *satisfies an LDP with good rate function*

$$I = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 f'(s)^2 ds & \text{if } f \in \mathcal{H}^1 \\ +\infty & \text{otherwise} \end{cases}$$

where $\mathcal{H}^1$ *is the* Cameron-Martin space $\{f \in \mathcal{AC}[0, T] : \ f(0) = 0, \ \int_0^1 f'^2(s)ds < \infty\}$ *of absolutely continuous functions with square integrable derivatives.* $\mathcal{AC}[0, T]$ *denotes the set of all real, absolutely continuous functions on* $[0, T]$.

2.2 The Ornstein-Uhlenbeck SDE and the time to exit from a deterministic boundary

Consider the Ornstein-Uhlenbeck Stochastic Differential Equation (SDE)

$$dX_t = \mu X_t dt + \sigma dW_t, \qquad X_0 = x_0 \tag{2.2}$$

where $\mu > 0$. Note that the expectation $\mathbb{E}X_t = x_0 e^{\mu t}$ increases exponentially with time and consider also the deterministic exponential function

$$V(t) = v_0 e^{\beta t} \quad \text{where} \quad 0 \le \beta < \mu \quad \text{and} \quad 0 < v_0 < x_0. \tag{2.3}$$

Let

$$p_l(x_0, T) \;=\; \mathbb{P}(X_t > V(t);\; 0 \le t \le T) \tag{2.4}$$

denote the probability that the process $\{X_t\}$ stays above the exponential boundary $V(t)$. Here $1 - p_l(x_0, T)$ may be thought of as a type of *ruin probability*. We are interested in evaluating $p_l(x_0, T)$ and the limiting probability $p_l(x_0) := \mathbb{P}(X_t > V(t);\; 0 \le t)$. Due to the Markovian property of $\{X_t\}$, the "non-ruin probability" defined in (2.4) satisfies the PDE

$$\frac{1}{2}\sigma^2 f_{xx} + \mu x f_x + f_t \;=\; 0, \quad \text{in } D := \{(x,t) : 0 < t < T,\, x > v_0 e^{\beta t}\} \tag{2.5}$$

with boundary conditions $f(v_0 e^{\beta t}, t) = 0$ for $t \in [0, T]$ and $f(x, T) = 1$ for $x > v_0 e^{\beta T}$.

We will not attempt to obtain an expression for the solution of (2.5) due to the difficulties that arise as a result of the shape of the domain D though one may of course obtain numerical results for the ruin probability based on the above formulation. We will instead use Wentzell-Freidlin "low noise asymptotics" in order to obtain a large deviations estimate for the probability that X_t crosses the path of $V(t)$ for some $t \in [0, T]$ (Freidlin and Wentzell, 2012).

Similarly, if $\alpha > \mu$, $u_0 > x_0 > 0$, and $U(t) = u_0 e^{\alpha t}$ we will consider the probability $p_u(x_0, T) = \mathbb{P}(X_t < U(t);\; 0 \le t \le T)$ of remaining below the upper boundary U, as well as the corresponding infinite horizon probability $p_u(x_0) := \mathbb{P}(X_t < U(t);\; 0 \le t)$.

2.3 The Wentzell-Freidlin framework - Finite horizon problem

Wentzell-Freidlin theory generalizes the ideas in Schilder's theorem to the paths of Stochastic Differential Equations. To express the problem discussed in the previous section in the Wentzell-Freidlin framework we consider the family of processes $\{X_t^\epsilon\}$

$$dX_t^\epsilon \;=\; \mu X_t^\epsilon dt + \sqrt{\epsilon}\,\sigma\, dW_t, \quad X_0^\epsilon = x_0 \tag{2.6}$$

together with the deterministic process

$$\dot{x}(t) \;=\; \mu x(t), \quad x(0) = x_0.$$

Denote by $C[0, T]$ the set of continuous functions on $[0, T]$, and by $C_{x_0}[0, T]$ the set of all continuous functions $f : [0, T] \to \mathbb{R}$ with $f(0) = x_0$. Consider the transformation $F : C[0, T] \to C_{x_0}[0, T]$ defined by

$$f \;=\; F(g) \quad \text{with} \quad f(t) \;:=\; x_0 + \int_0^t \mu f(s) ds \;+\; \sigma g(t), \quad t \in [0, T]. \tag{2.7}$$

Let f_i, denote the solution of (2.7) when the driving function is $g_i \in C[0, T]$, $i = 1, 2$. We may then establish the continuity of the map F by means of a Gronwall argument which shows that

$$\|f_1 - f_2\| \;\le\; \sigma\, e^{\mu T} \|g_1 - g_2\|$$

where $\|f\| := \sup\{|f(t)| : t \in [0,T]\}$ denotes the sup norm. Theorem 5.6.7 of Dembo and Zeitouni (2010) applies and therefore the solution of (2.6) satisfies a Large Deviation Principle with good rate function

$$I(f,T) := \begin{cases} \frac{1}{2}\int_0^T \left(f'(t) - \mu f(t)\right)^2 \sigma^{-2} dt & \text{if } f \in \mathcal{H}^1_{x_0}[0,T] \\ +\infty & \text{otherwise} \end{cases} \tag{2.8}$$

where $\mathcal{H}^1_{x_0}[0,T] := \{f : [0,T] \to \mathbb{R},\ f(t) = x_0 + \int_0^t \phi(s)ds\ ,\ t \in [0,T], \phi \in L^2[0,T]\}$ is the Cameron-Martin space of absolutely continuous functions with square integrable derivatives and initial value $f(0) = x_0$.

We can now state our first result.

Theorem 2. *In the above framework, if the lower boundary curve is $V(t) = v_0 e^{\beta t}$,*

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\min_{t \in [0,T]} X^\epsilon_t - V(t) \leq 0\right) = -I_V(T). \tag{2.9}$$

The rate function $I_V(T)$ is given by

$$I_V(T) = \frac{\mu}{\sigma^2} \frac{\left(v_0 e^{\beta(T \wedge t^o_V)} - x_0 e^{\mu(T \wedge t^o_V)}\right)^2}{e^{2\mu(T \wedge t^o_V)} - 1} \tag{2.10}$$

where t^o_V is the unique positive solution of the equation

$$\phi_V(t) := \left(1 - \frac{\beta}{\mu}\right) e^{(\mu+\beta)t} + \frac{\beta}{\mu} e^{(\beta-\mu)t} = \frac{x_0}{v_0}. \tag{2.11}$$

Similarly, for the upper boundary curve $U(t) = u_0 e^{\alpha t}$,

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\max_{t \in [0,T]} X^\epsilon_t - U(t) \geq 0\right) = -I_U(T) \tag{2.12}$$

with

$$I_U(T) = \frac{\mu}{\sigma^2} \frac{\left(u_0 e^{\alpha(T \wedge t^o_U)} - x_0 e^{\mu(T \wedge t^o_U)}\right)^2}{e^{2\mu(T \wedge t^o_U)} - 1} \tag{2.13}$$

where t^o_U is the unique positive solution of the equation

$$\phi_U(t) := \frac{\alpha}{\mu} e^{(\alpha-\mu)t} - \left(\frac{\alpha}{\mu} - 1\right) e^{(\mu+\alpha)t} = \frac{x_0}{u_0}. \tag{2.14}$$

Proof. The proof is long and will be divided into three parts for clarity of exposition.

Part 1. We begin by fixing $t > 0$ and considering paths that start at x_0 at time 0 and end at $V(t) = v_0 e^{\beta t}$ at time t. Consider the set

$$\mathcal{H}^1_{x_0,V(t)} := \left\{h : [0,t] \to \mathbb{R} : h(s) = x_0 + \int_0^s \phi(u)du\ ,\ s \in [0,t],\ h(t) = V(t), \phi \in L^2[0,t]\right\}.$$

Then, for $\eta > 0$,

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\sup_{0 \leq s \leq t} |X^\epsilon_s - h(s)| < \eta\right) = -J_*(t). \tag{2.15}$$

where $J_*(t)$ is the solution of the variational problem

$$J_*(t) := \inf \left\{ J(x;t) : x \in \mathcal{H}^1_{x_0,V(t)} \right\} \tag{2.16}$$

with

$$J(x;t) = \int_0^t F(x,x',s)ds, \qquad \text{and} \quad F(x,x',s) = \frac{1}{2\sigma^2}\left(x' - \mu x\right)^2. \tag{2.17}$$

$J(x;t)$ gives the rate function for a path $x(\cdot)$ that starts at x_0 and meets the lower boundary at the point $(t, v_0 e^{\beta t})$ i.e. satisfies the boundary conditions

$$x(0) = x_0, \qquad x(t) = v_0 e^{\beta t}. \tag{2.18}$$

The infimum in (2.16) is taken over all absolutely continuous functions on $[0,t]$ with derivative in L^2. The function $x \in \mathcal{H}^1_{x_0,V(t)}[0,t]$ that minimizes the integral defining the rate function is the solution of the Euler-Lagrange equation (e.g. see Brechtken-Manderscheid, 1994)

$$F_x - \frac{d}{ds}F_{x'} = 0 \tag{2.19}$$

and the boundary conditions (2.18). With the given form of F in (2.17) the Euler-Lagrange equation becomes

$$x''(s) = \mu^2 x(s) \tag{2.20}$$

which has the general solution

$$x(s) = c_1 e^{\mu s} + c_2 e^{-\mu s}. \tag{2.21}$$

The values of c_1, c_2 for which x satisfies the boundary conditions are given by

$$c_1 = \frac{v_0 e^{\beta t} - x_0 e^{-\mu t}}{e^{\mu t} - e^{-\mu t}}, \qquad c_2 = \frac{x_0 e^{\mu t} - v_0 e^{\beta t}}{e^{\mu t} - e^{-\mu t}}. \tag{2.22}$$

Thus (2.21) with the constants c_1, c_2 given by (2.22) gives the optimal path

$$x(s) \;=\; \frac{v_0 e^{\beta t}\left(e^{\mu s} - e^{-\mu s}\right) + x_0\left(e^{\mu(t-s)} - e^{-\mu(t-s)}\right)}{e^{\mu t} - e^{-\mu t}} \;=\; \frac{v_0 e^{\beta t}\sinh(\mu s) + x_0 \sinh(\mu(t-s))}{\sinh(\mu t)}. \tag{2.23}$$

From (2.21) $x'(s) - \mu x(s) = -2\mu c_2 e^{-\mu s}$ and, taking into account (2.17),

$$J_*(t) \;=\; \frac{4\mu^2 c_2^2}{2\sigma^2}\int_0^t e^{-2\mu s} ds \;=\; \frac{\mu c_2^2}{\sigma^2}\left(1 - e^{-2\mu t}\right).$$

Using the expression for c_2 we have

$$J_*(t) \;=\; \frac{\mu}{\sigma^2}\frac{\left(v_0 e^{\beta t} - x_0 e^{\mu t}\right)^2}{e^{2\mu t} - 1}. \tag{2.24}$$

There remains to show that there is no path $x(s)$ with piecewise continuous derivative which achieves a smaller value of the criterion, i.e. that the optimal solution does not have corners. To this end we consider the Erdmann corner conditions (see Brechtken-Manderscheid,1994, p. 33). The first condition requires that $F_{x'}$ evaluated at the critical path be a continuous function of s. Since $F_{x'} = \frac{1}{\sigma^2}\left(x' - \mu x\right)$ and $x(s)$ is necessarily continuous, the first Erdmann condition implies the continuity of $x'(s)$ as well. Therefore, by virtue of the first Erdmann condition alone we may conclude that the optimal solution cannot have discontinuities in its derivative. For the sake of completeness we mention that the second Erdmann condition requires that $F - x'F_{x'}$

evaluated at the critical path $x(s)$ be also a continuous function of s. Since $F - x'F_{x'} = -\frac{1}{2\sigma^2}\left((x')^2 - \mu^2 x^2\right)$ and because of the continuity of $x(s)$, this second condition by itself would allow the existence of corners at which the first derivative changes sign. Such corners are of course precluded by the first condition.

Finally, we claim that the solution we have found corresponds to a global minimum. To see this we appeal to Theorem 3.16 of Brechtken-Manderscheid (1994), p. 45, according to which it suffices to show that $F(x, x') := \frac{1}{2\sigma^2}(x' - \mu x)^2$ (abusing slightly the notation) is a convex function on $\mathbb{R}^2$. Indeed, we can show that, for any $(x_0, x_0') \in \mathbb{R}^2$,

$$F(x, x') \geq F(x_0, x_0') + F_x(x_0, x_0')\,(x - x_0) + F_{x'}(x_0, x_0')\,(x' - x_0')$$

or

$$\frac{1}{2}\left(x' - \mu x\right)^2 \geq \frac{1}{2}\left(x_0' - \mu x_0\right)^2 - \mu\left(x_0' - \mu x_0\right)\,(x - x_0) + \left(x_0' - \mu x_0\right)\,(x' - x_0').$$

This last inequality can be seen to be equivalent to

$$\left(x' - \mu x\right)^2 + \left(x_0' - \mu x_0\right)^2 \; - \; 2\left(x' - \mu x\right)\left(x_0' - \mu x_0\right) \geq 0$$

which is clearly true and thus the convexity of F and therefore the global optimality of x is established. Note however that this optimal solution was obtained without taking into account the *path inequality constraint*

$$x(s) \geq V(s) \quad \text{for all } s \in [0, t]. \tag{2.25}$$

Therefore the optimal path (2.23) may violate this inequality.

Part 2. In the first part we obtained the *fixed time optimal solution* under the boundary conditions (2.18). In this part however we will solve the optimization problem with finite time horizon $[0, T]$,

$$I(T) := \inf\{J(x, t) : 0 \leq t \leq T, x \in \mathcal{H}^1_{x_0, V(t)}, \text{ i.e. } x \text{ satisfies the conditions (2.18) }\} \tag{2.26}$$

still ignoring the *path inequality constraint* (2.25).

Clearly $I(T) = \inf_{t\in[0,T]} J_*(t)$. From (2.24) we see that $J_*(t)$ is a continuously differentiable function for $t > 0$. We will establish that it is strictly convex on $[0, T]$. Indeed

$$J_*'(t) \;=\; \frac{2v_0\mu^2 e^{\mu t}\left(x_0 e^{\mu t} - v_0 e^{\beta t}\right)}{\sigma^2 (e^{2\mu t} - 1)^2}\; \left[\left(1 - \frac{\beta}{\mu}\right) e^{(\beta+\mu)t} + \frac{\beta}{\mu} e^{(\beta-\mu)t} - \frac{x_0}{v_0}\right]. \tag{2.27}$$

Given the definition of ϕ_V in (2.11) we note that the quantity inside the brackets above is $\phi_V(t) - \frac{x_0}{v_0}$ Since $0 < \beta < \mu$ and $0 < v_0 < x_0$, $x_0 e^{\mu t} - v_0 e^{\beta t} > 0$ for all $t \geq 0$ and thus the sign of $J_*'(t)$ is that of $\phi_V(t) - \frac{x_0}{v_0}$. Note that $\phi_V'(t) = \frac{\mu-\beta}{\mu} e^{(\beta+\mu)t}\left[\mu + \beta(1 - e^{-2\mu t})\right] > 0$ for all $t \geq 0$ and thus ϕ_V is strictly increasing. Also, given the definition of ϕ_V we have $\lim_{t\to\infty} \phi_V(t) = +\infty$ and $\phi_V(0) = 1$. Further, $\frac{x_0}{v_0} > 1$ hence there exists a unique $t_V^o > 0$ such that

$$\phi_V(t_V^o) \;=\; \frac{x_0}{v_0} > 1. \tag{2.28}$$

In view of the expression (2.27), $J_*'(t) < 0$ for $0 \leq t < t_V^o$, $J_*(t_V^o) = 0$, and $J_*'(t) > 0$ for $t > t_V^o$. Thus t_V^o, the unique solution of (2.11), is a point of global minimum for J_*. Then

$$I(T) \;=\; \inf_{t\in[0,T]} J_*(t) \;=\; \begin{cases} J_*(T) & \text{if} \quad T \leq t_V^o \\ J_*(t^o) & \text{if} \quad T > t_V^o \end{cases}. \tag{2.29}$$

Figure 1 illustrates the behavior of the function $J_*(t)$ and that of $I(t)$.

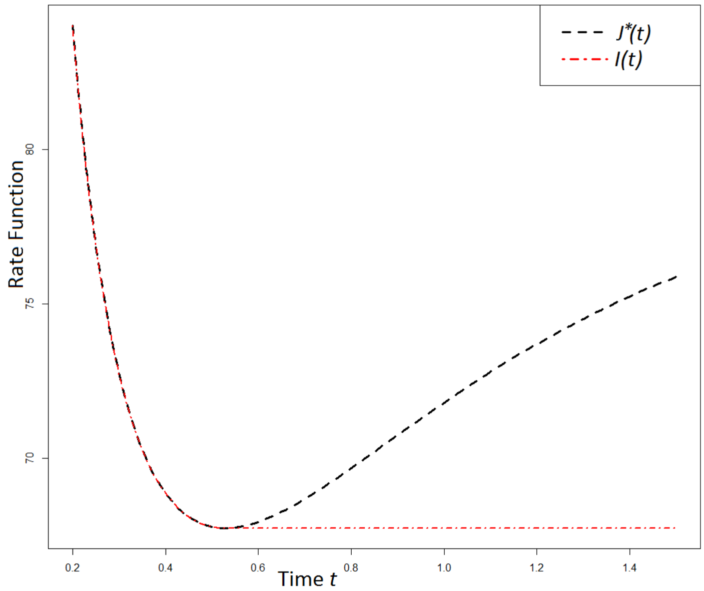

Figure 1: The dotted black line denotes the function $J_*(t)$. The dotted red line denotes the rate function $I(t)$. Here $\mu = 2.5$, $\beta = 1.0$, $x_0 = 4$, $v_0 = 1$ and $t_V^o \approxeq 0.529$.

Part 3. We complete the proof by showing that the optimal rate given by (2.29) remains valid even after taking into account the additional path inequality constraint (2.25). Define

$$J_{**}(t) := \inf\left\{J(x;t) : x \in \mathcal{H}^1_{x_0,V(t)}, x(s) \geq V(s) \text{ for } s \in [0,t].\right\} \tag{2.30}$$

Consider the optimal path $x(s)$ of Part 1 given in (2.22) , (2.21), (2.23), for all $s \geq 0$. Note that $c_2 > 0$ (since $\mu > \beta$ and $x_0 > v_0$). The sign of c_1 depends on t: $c_1 > 0 \Leftrightarrow v_0 e^{(\mu+\beta)t} - x_0 > 0$ and this is equivalent to

$$t > t_1 := \frac{1}{\mu+\beta} \log \frac{x_0}{v_0}. \tag{2.31}$$

We also point out that

$$t_1 < t_V^o. \tag{2.32}$$

This follows by the fact that ϕ_V is a strictly increasing function and

$$\phi_V(t_1) = \left(1 - \frac{\beta}{\mu}\right) e^{(\beta+\mu)t_1} + e^{-2\mu t}\frac{\beta}{\mu} e^{(\beta+\mu)t_1} = \frac{x_0}{v_0}\left(1 - \frac{\beta}{\mu}(1 - e^{-2\mu t})\right) < \frac{x_0}{v_0} = \phi_V(t_V^o).$$

We distinguish three cases according to the relationship between t and t_1.

Case 1: $t < t_1$. This implies that $c_1 < 0$. Because $x(0) > V(0) = v_0$, $x'(s) = \mu c_1 e^{\mu s} - \mu c_2 e^{-\mu s} < 0$ for all $s \geq 0$, and $\lim_{s\to\infty} x(s) = -\infty$, t is the unique intersection point of the paths $x(\cdot)$ and $V(\cdot)$ and the inequality constraint (2.25) is satisfied.

Case 2: $t = t_1$. Then, from (2.22) $c_1 = 0$ and $c_2 = x_0$ and hence $x(s) = x_0 e^{-\mu s}$. Again, the paths $x(\cdot)$ and $V(\cdot)$ intersect only once, at t, $x(s) > V(s)$ for $s \in [0,t)$, and the path inequality constraint is satisfied.

Case 3: $t > t_1$. Here both $c_1 > 0$ and $c_2 > 0$ and thus $x(s) > 0$ for all $s > 0$. Therefore, as a result of (2.20), $x''(s) > 0$ and the function x is strictly convex for all $s \geq 0$. In this case, as is shown in the Appendix, the paths $x(\cdot)$ and $V(\cdot)$ intersect at precisely two points, one of which is of course t while the other will be

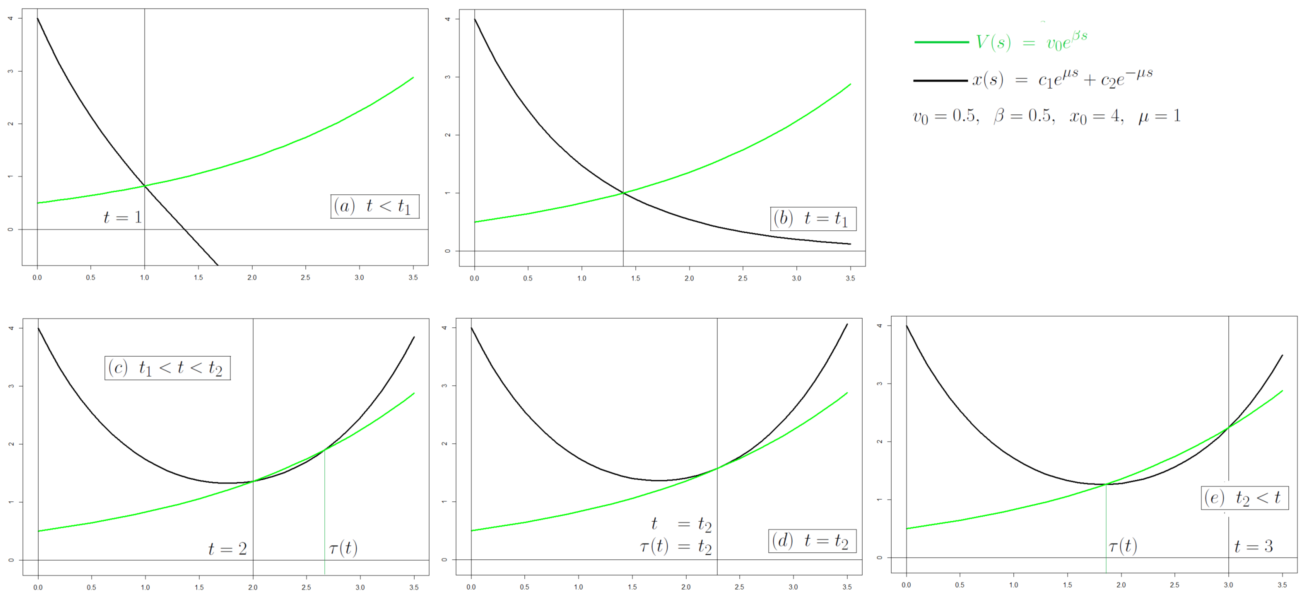

Figure 2: The three cases. In this example $\mu = 1$, $\beta = 0.5$, $x_0 = 4$, $v_0 = 0.5$. These give $t_1 \approx 1.39$ and $t_2 \approx 2.29$. (a) shows the behavior when $t = 1 < t_1$. This is case 1 and $x(\cdot)$ decreases monotonically. The inequality constraints are satisfied. In (b) $t = t_1$. This is case 2 and again the inequality conditions are satisfied. The remaining tree plots illustrate case 3. In (c) $t = 2 < t_2$ and in (d) $t = t_2 \approx 2.29$. Since in these two cases $t \le \tau(t)$ the inequality constraints are satisfied. In (e) however, where $t > \tau(t) = t_2$ the inequality constraint (2.25) is not satisfied.

denoted by $\tau(t)$. Figure 2 illustrates this for specific values of the parameters μ, β, x_0, v_0. For the values of the parameters in Figure 2 $t_1 = \frac{2}{3}\log 8 \approx 1.39$. In this figure, in the upper left, the path $x(s)$ is decreasing and eventually becomes negative. There is a single intersection between the curves $x(s)$ and $V(s)$. On the other hand in the graph lower, in the middle, ($t = 2$) and in the right ($t = 3$) the path $x(s)$ is strictly convex, as is $V(s)$, and thus the two curves intersect in two points. For $t = 2$ the path $x(s)$ satisfies (5.4) and therefore (5.1) and the path inequality constraint (2.25) while for $t = 3$ it does not.

The key remark is the following: If $x'(t) < V'(t)$ then the path $x(\cdot)$ intersects $V(\cdot)$ from above at t, then again from below at $\tau(t) > t$. If, conversely, $x'(t) > V'(t)$ then $x(\cdot)$ intersects $V(\cdot)$ from below at t. Since $x(0) > V(0)$ this necessarily implies that there was an earlier crossing from above at $\tau(t) < t$. (The case $x'(t) = V'(t)$ corresponds to $t = \tau(t)$. The path $x(\cdot)$ is tangent to $V(\cdot)$ at t and $x(s) > V(s)$ for all $s \neq t$.)

The situation in Case 3 is examined in more detailed in Section 5.1 of the Appendix where it is established that there exists a time t_2 such that $t_1 < t_2$ and the relationship between t and t_2 determines whether the path $x(\cdot)$ satisfies the inequality constraints (2.25) or not. Specifically

> If $t_1 < t < t_2$ then $x(\cdot)$ intersects $V(\cdot)$ from above at t and hence it satisfies the inequality constraint $x(s) > V(s)$ for $s \in [0, t)$. It crosses $V(t)$ once again at $\tau(t) > t$, this time from below.
>
> If $t = t_2$ then $x(t)$ is tangent to $V(t)$ at t. It satisfies the inequality constraint $x(s) > V(s)$ for $s \in [0, t)$ (and in fact even beyond t though this is of no interest for our purposes).
>
> If $t > t_2$ then $x(t)$ crosses $V(t)$ from below. This means that there was a first crossing from above at $\tau(t) < t$. As a result $x(s) < V(s)$ when $s \in (\tau(t), t]$ and the inequality constraint (2.25) is not satisfied in this case.

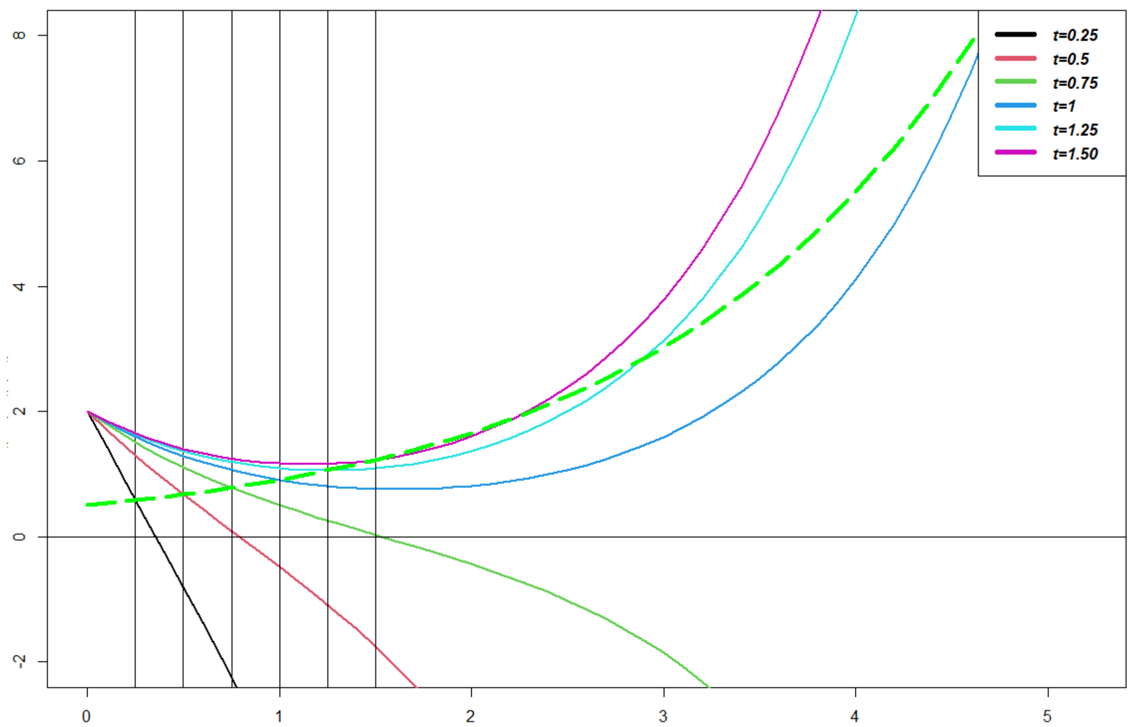

Figure 3: Here $\mu = 1$, $\beta = 0.6$, $x_0 = 2$, $v_0 = 0.5$. Thus $t_1 = 0.8664$. The hitting times range from $t = 0.25$ to $t = 1.50$. Note that, for $t = 0.25, 0.50$, and 0.75 the path $x(s)$ eventually becomes negative, after hitting $V(s)$, the thick green line. When the hitting times are greater than t_1, (i.e. $t = 1,\ 1.25,$ and 1.5) the path $x(s)$ is a convex function and has two intersection points with the dotted green line.

Figure 3 illustrates these cases. For $t = 0.25,\ 0.5,$ and 0.75 (black, red, and green paths) the paths eventually become negative and intersect the dotted green line (i.e. the function $V(\cdot)$) once. In the rest of the cases the paths remain positive and intersect the dotted green line twice.

Thus the optimal path of Part 1 also satisfies the constraint (2.25) iff $t \le t_2$. In that case the path given by (2.23) minimizes the functional $J(x,t)$ in (2.17) under the boundary conditions (2.18) and the path inequality constraints (2.25). Then

$$J_{**}(t) = J_*(t) \quad \text{when} \quad t < t_2. \tag{2.33}$$

If $t > t_2$ then t is *the second point of intersection of* $x(s)$ *with* $V(s)$ and (2.25) is not satisfied. This means that the path $x(s)$ is not feasible under the additional constraint $x(s) > V(s)$ and therefore that the minimum value $J_*(t)$ obtained without taking into account the inequality constraint is smaller than $J_{**}(t)$. Thus we have

$$\begin{array}{rcl} J_{**}(t) &=& J_*(t) \ \text{ if } t \le t_2 \\ J_{**}(t) &\ge& J_*(t) \ \text{ if } t > t_2 \end{array} \tag{2.34}$$

Then, the rate function in (2.9), defined as

$$I_V(T) \ := \ \inf \left\{ J(x;t) : x \in \mathcal{H}^1_{x_0,V(t)}, x(s) \ge V(s) \text{ for } 0 < s < t, \ \ 0 < t \le T. \right\} \tag{2.35}$$

can be obtained as

$$I_V(T) \ := \ \min_{t \in (0,T]} J_{**}(t). \tag{2.36}$$

If $T \le t_2$ then $J_{**}(t) = J_*(t)$ and hence $I_V(T) = \min_{t\in(0,T]} J_*(t) = J_*(t_V^o \wedge T)$ due to the fact that J_* is strictly decreasing in $(0, t_V^o)$ and strictly increasing in (t_V^o, ∞).

Therefore we conclude that $I_V(T)$ is also given by (2.24). This concludes the proof of the first part of Theorem 2. The proof of the second part, pertaining to the upper boundary curve, is similar and will be omitted. □

2.4 The infinite horizon problem – lower and upper boundary curves

We now turn to the infinite horizon problem of obtaining a large deviations estimate for the probability $\mathbb{P}(\inf_{t\ge0} X_t - v_0 e^{\beta t} \le 0)$ and $\mathbb{P}(\inf_{t\ge0} X_t - u_0 e^{\alpha t} \ge 0)$ in the same context as that of the previous section. It is of course possible, after having solved the corresponding finite horizon problem as we saw in the previous section, to minimize over the horizon T. Instead of this, we will use here the standard *transversality conditions* approach of the Calculus of Variations in order to tackle in one step the infinite horizon problem by considering variational problems with variable end-points.

Theorem 3. *Suppose $\{X_t^\epsilon\}$, $\epsilon > 0$, is the family of diffusions described by the solution of the SDE (2.6). Suppose also that the upper bounding curve $U(t) = u_0 e^{\alpha t}$ and lower bounding curve $V(t) = v_0 e^{\beta t}$ satisfy the inequalities $v_0 < x_0 < u_0$ and $\beta < \mu < \alpha$. Then*

a) The probability of ever hitting the lower boundary satisfies

$$-\lim_{\epsilon\to0} \epsilon \log \mathbb{P}\left(\inf_{t\ge0} X_t^\epsilon - v_0 e^{\beta t} \le 0\right) =: I_V(\infty) = \frac{x_0^2\mu}{\sigma^2} \frac{1 - e^{-2\mu t_V^o}}{\left(1 + \frac{\beta}{\mu-\beta} e^{-2\mu t_V^o}\right)^2} \tag{2.37}$$

and t_V^o is the unique root of equation (2.11). The optimal path x_* hitting the lower bound is given by

$$x_*(t) = x_0 \frac{e^{-\mu(t_V^o - t)} + \left(\frac{\mu}{\beta} - 1\right) e^{\mu(t_V^o - t)}}{e^{-\mu t_V^o} + \left(\frac{\mu}{\beta} - 1\right) e^{\mu t_V^o}}. \tag{2.38}$$

b) The probability of ever hitting the upper boundary satisfies

$$-\lim_{\epsilon\to0} \epsilon \log \mathbb{P}\left(\inf_{t\ge0} X_t^\epsilon - u_0 e^{\alpha t} \ge 0\right) = I_U(\infty) = \frac{x_0^2\mu}{\sigma^2} \frac{1 - e^{-2\mu t_U^o}}{\left(1 + \frac{\alpha}{\mu-\alpha} e^{-2\mu t_U^o}\right)^2} \tag{2.39}$$

and t_U^o is the unique root of the equation (2.14). The optimal path hitting the upper bound is given by

$$x^*(t) = x_0 \frac{e^{-\mu(t_U^o - t)} - \left(1 - \frac{\mu}{\alpha}\right) e^{\mu(t_U^o - t)}}{e^{-\mu t_U^o} - \left(1 - \frac{\mu}{\alpha}\right) e^{\mu t_U^o}}. \tag{2.40}$$

Proof. Consider first the problem of hitting the upper boundary at some time T_U before hitting the lower boundary. We will obtain low noise logarithmic asymptotics for the probability of hitting the upper boundary (without having first hit the lower). Because in the limit, as $\epsilon \to 0$, the probability of ever hitting either the upper or the lower boundary goes to 0 exponentially (in $\frac{1}{\epsilon}$) we expect that the presence of the lower boundary (and the stipulation to avoid it) does not affect the probability of hitting the upper boundary.

The optimization problem for the action functional becomes

$$\min \int_0^{T_U} F(x, x', t)dt, \quad \text{with } F(x, x', t) = \frac{1}{2\sigma^2}\left(x' - \mu x\right)^2, \tag{2.41}$$

subject to the constraints

$$x(0) = x_0, \quad \text{and } x(T_U) = U(T_U) \tag{2.42}$$

$$V(t) < x(t) < U(t) \ \text{ for } 0 \le t < T_U, \tag{2.43}$$

In the above, both the optimal path x and the horizon T_U are unknowns to be determined. Our approach to dealing with the *inequality path constraint,* (2.43) $x(t) > V(t)$ for all $t \in [0, T)$ will be to initially ignore it and obtain an optimal hitting time t_U^o and an optimal path x_* minimizing the criterion (2.41) and satisfying the boundary conditions (2.42). We will then show that this optimal path satisfies the constraints (2.43).

The necessary conditions for a minimum in the problem *without the path inequality constraint* are

$$\text{Euler-Lagrange Equation:} \quad F_x - \frac{d}{dt}F_{x'} = 0, \tag{2.44}$$

$$\text{Boundary Conditions:} \quad x(0) = x_0, \quad x(T_U) = U(T_U), \tag{2.45}$$

$$\text{Transversality Condition:} \quad F + (U' - x')F_{x'} = 0 \quad \text{at } T_U. \tag{2.46}$$

Taking into account that $F_x = -\mu\sigma^{-2}(x' - \mu x)$, $F_{x'} = \sigma^{-2}(x' - \mu x)$, $\frac{d}{dt}F_{x'} = \sigma^{-2}(x'' - \mu x')$, the Euler-Lagrange equation becomes

$$F_x - \frac{d}{dt}F_{x'} = -\sigma^{-2}\left(x'' - \mu^2 x\right) = 0$$

and thus

$$x'' - \mu^2 x \ = \ 0. \tag{2.47}$$

This has the general solution

$$x(t) = C_1 e^{\mu t} + C_2 e^{-\mu t}. \tag{2.48}$$

Taking into account the boundary conditions (2.45), we obtain

$$x(0) \ = \ C_1 + C_2 \ = \ x_0, \tag{2.49}$$

$$x(T) \ = \ C_1 e^{\mu T_U} + C_2 e^{-\mu T_U} \ = \ u_0 e^{\alpha T_U}. \tag{2.50}$$

The transversality condition (2.46) gives

$$\frac{1}{2\sigma^2}\left(x'(T_U) - \mu x(T_U)\right)^2 + \left(u_0 \alpha e^{\alpha T_U} - x'(T_U)\right)\frac{1}{\sigma^2}\left(x'(T_U) - \mu x(T_U)\right) \ = \ 0$$

or

$$\left(x'(T_U) - \mu x(T_U)\right)\left(-x'(T_U) - \mu x(T_U) + 2u_0 \alpha e^{\alpha T_U}\right) \ = \ 0. \tag{2.51}$$

Taking into account (2.48), it follows that $x'(T_U) - \mu x(T_U) = -2\mu C_2 e^{-\mu T_U}$ and hence, if the first factor of (2.51) were to vanish, this would imply that $C_2 = 0$. This in turn implies, in view of (2.48), (2.49), and (2.50), that $x(T) = x_0 e^{\mu T_U} = u_0 e^{\alpha T_U}$ which is impossible since $x_0 < u_0$ and $\mu < \alpha$. Hence (2.51) implies

$$u_0 e^{\alpha T_U} \ = \ \frac{\mu}{\alpha} C_1 e^{\mu T_U}. \tag{2.52}$$

From (2.45) and (2.52) we obtain

$$C_1 + C_2 \ = \ x_0$$

$$C_1\left(1 - \frac{\mu}{\alpha}\right) e^{\mu T_U} + C_2 e^{-\mu T_U} \ = \ 0$$

whence it follows that

$$C_1 = \frac{x_0 e^{-\mu T_U}}{\left(\frac{\mu}{\alpha}-1\right) e^{\mu T_U} + e^{-\mu T_U}}, \qquad C_2 = \frac{x_0\left(\frac{\mu}{\alpha}-1\right) e^{\mu T_U}}{\left(\frac{\mu}{\alpha}-1\right) e^{\mu T_U} + e^{-\mu T_U}}. \tag{2.53}$$

From (2.48) and (2.52) we obtain the equation

$$\left(\frac{\alpha}{\mu}-1\right) e^{(\mu+\alpha)T_U} - \frac{\alpha}{\mu} e^{(\alpha-\mu)T_U} + \frac{x_0}{u_0} = 0 \tag{2.54}$$

which must be satisfied by the optimal hitting time T_U. This is precisely equation (2.14) and therefore

$$T_U = t_U^o,$$

the unique solution of (2.14). (Indeed, it is easy to see that (2.14) has a unique solution: We have $\phi_U(0) = 1$, $\lim_{t\to\infty}\phi_U(t) = -\infty$, and $\phi_U'(t) = -\frac{\alpha-\mu}{\mu} e^{(\mu+\alpha)t}\left[\mu + \alpha(1-e^{-2\mu t})\right] < 0$ for all $t \geq 0$.) *Note in particular that the value of T_U does not depend on σ as is clear from (2.54).* Also, since T_U is the unique solution of (2.14), it must satisfy the equation $(\mu-\alpha)e^{2\mu T_U} + \alpha = e^{-\alpha T_U}\mu\frac{x_0}{u_0}$. Since the right hand side of this equation is strictly positive, the left must be as well and this implies

$$e^{2\mu T_U} < \frac{\alpha}{\alpha-\mu} \quad \text{or} \quad T_U < \frac{1}{2\mu}\log\frac{\alpha}{\alpha-\mu}. \tag{2.55}$$

An alternative expression for C_1, C_2, taking into account (2.54) is

$$C_1 = u_0\frac{\alpha}{\mu}e^{(\alpha-\mu)T_U}, \qquad C_2 = u_0\left(1-\frac{\alpha}{\mu}\right)e^{(\alpha+\mu)T_U}. \tag{2.56}$$

Thus, by (2.48) and (2.53) or (2.56), the following equivalent expressions describe the optimal path.

$$x^*(t) = x_0\,\frac{\left(\frac{\alpha}{\mu}-1\right)e^{\mu(T_U-t)} - \frac{\alpha}{\mu}e^{-\mu(T_U-t)}}{\left(\frac{\alpha}{\mu}-1\right)e^{\mu T_U} - \frac{\alpha}{\mu}e^{-\mu T_U}} = u_0 e^{\alpha T_U}\left[\frac{\alpha}{\mu}e^{-\mu(T_U-t)} - \left(\frac{\alpha}{\mu}-1\right)e^{\mu(T_U-t)}\right]. \tag{2.57}$$

From the above we obtain the rate function I_U given in (2.39). Alternative expressions for the rate I_U, using (2.54) are, of course, possible. For instance,

$$I_U = \frac{\mu}{\sigma^2}\frac{\left(u_0 e^{(\alpha-\mu)T_U} - x_0\right)^2}{1-e^{-2\mu T_U}} = \frac{\mu}{\sigma^2}u_0^2\left(1-\frac{\mu}{\alpha}\right)^2 e^{2\alpha T_U}\left(e^{2\mu T_U}-1\right). \tag{2.58}$$

There remains to show that the optimal path obtained in (2.57) also satisfies the inequality constraints $v_0 e^{\beta t} < x^*(t) < u_0 e^{\alpha t}$ for $t \in [0, T_U)$. Indeed

$$x^*(t) - x_0 e^{\mu t} = \frac{2\left(1-\frac{\mu}{\alpha}\right)\sinh\mu t}{e^{-2\mu T_U} - 1 + \frac{\mu}{\alpha}} > 0 \quad \text{for } t > 0$$

the denominator in the expression above being positive by (2.55). Since $v_0 e^{\beta t} < x_0 e^{\mu t}$ for all $t > 0$ the above inequality implies $x^*(t) > v_0 e^{\beta t} = V(t)$ for $t \in [0, T_U)$.

Next, we will show that $x^*(t) < U(t)$ for all $t \in [0, T_U)$. Since $\psi(t) := U(t) - x^*(t)$ satisfies $\psi(0) = u_0 - x^*(0) = u_0 - x_0 > 0$, $\psi(T_U) = 0$, and ψ is continuous it suffices to prove that ψ does not vanish in the interval $[0, T_U)$.

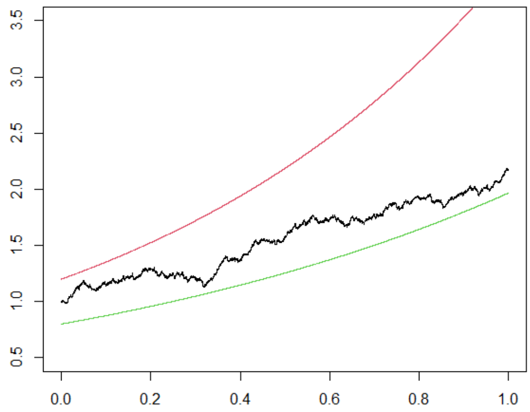

Figure 4: An Ornstein-Uhlenbeck process evolving between an upper and a lower exponential bound.

By (2.57) $\psi(t) = 0$ is equivalent to

$$x_0\big[\big(\frac{\alpha}{\mu} - 1\big)e^{\mu(T_U - t)} - \frac{\alpha}{\mu}e^{-\mu(T_U - t)}\big] = u_0 e^{\alpha t}\big[\big(\frac{\alpha}{\mu} - 1\big)e^{\mu T_U} - \frac{\alpha}{\mu}e^{-\mu T_U}\big]. \tag{2.59}$$

The right hand side of the above can be written as

$$u_0 e^{-\alpha(T_U - t)}\big[\big(\frac{\alpha}{\mu} - 1\big)e^{(\alpha+\mu)T_U} - \frac{\alpha}{\mu}e^{(\alpha-\mu)T_U}\big] = -x_0 e^{-\alpha(T_U - t)}$$

the last equality following from (2.54). Hence setting $T_U - t =: s$ and taking into account (2.59) we obtain

$$\big(\frac{\alpha}{\mu} - 1\big)e^{\mu s} - \frac{\alpha}{\mu}e^{-\mu s} + e^{-\alpha s} = 0.$$

It is easy to see that this equation does not have strictly positive solutions in s and thus there exists no $t \in [0, T_U)$ satisfying (2.59). Therefore the critical path $x^*(t)$ satisfies the inequality $x^*(t) < U(t)$ as well, for all $t \in [0, T)$.

The proof for the corresponding results for the lower boundary are established in an entirely similar manner. □

Intuitively, the uniqueness of the solution of (2.54) makes sense. If T_U is very small the noise factor W_t must exhibit an extremely unlikely behavior in order for the OU process to rise to the level of the upper curve $U(t)$. So having more time available makes the rare event of hitting the upper boundary more likely. But if T_U is too large, because of the difference in the rates of the two processes, again hitting the upper boundary becomes extremely unlikely. Also, in some cases, in the infinite horizon problem, an infimum may exist but no minimum. The rate function I is not "good" and compactness fails. In practical terms, the more time available the more likely it is that the noise term will cause the diffusion path to hit the deterministic boundary curve.

In Figures 6, 7, we consider the OU process $dX_t = X_t + dW_t$, with $X_0 = x_0$, (with the value of the parameters $\mu = 1$, $\sigma = 1$) and the lower and upper bounds $v(t) = 0.5e^{0.5t}$, $u(t) = 2e^{1.3t}$. (Thus $\alpha = 1.3$, $u_0 = 2$, $\beta = 0.5$ and $v_0 = 0.5$.) In Figure 5 the optimal value of T that corresponds to the solution of the optimization problems of section 2.4 (equations (2.11) and (2.14)).

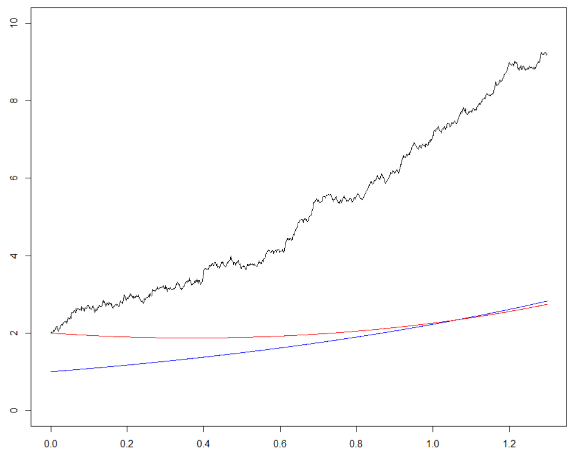

Figure 5: The black line is a typical path of an OU process with $\mu = 1$, $\sigma = 1$ and starting point $x_0 = 2$. The blue curve is the lower exponential bound $v_0 e^{\beta t}$ with $v_0 = 1$ and $\beta = 0.8$. The meeting time T obtained by solving numerically (2.11) is equal to 1.0621. Finally the red optimal (large deviation) path is obtained from (2.38)

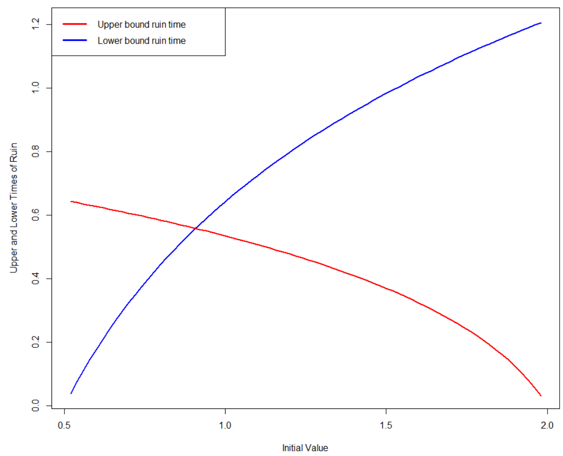

Figure 6: The system under consideration is an OU process with $\mu = 1$, $\sigma = 1$ and initial position x_0. The red line is the "optimal hitting time" for the upper curve $u_0 e^{\alpha t}$ with $u_0 = 2$, $\alpha = 1.3$, i.e. the solution of (2.14). Note that this optimal time decreases to zero as x_0 increases to $u_0 = 2$. Respectively, the blue line is the corresponding "optimal hitting time" for the lower curve $v_0 e^{\beta t}$, $\beta = 0.5$, $v_0 = 0.5$, i.e. the solution of (2.11). In this case the optimal time increases as the distance of x_0 from v_0 increases.

3 More general models

3.1 Ornstein-Uhlenbeck with a general linear drift

Here we consider the Ornstein-Uhlenbeck process with a more general drift. This is important since it arises as a diffusion approximation in the risk models with interest rates considered in the Introduction. Consider

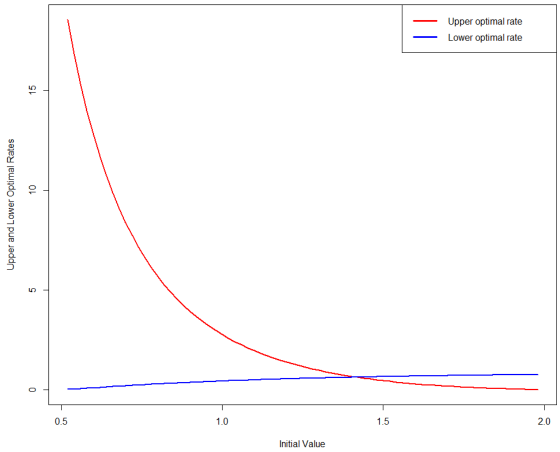

Figure 7: The OU process and the upper and lower curves are as in Figure 6. The red line is a plot of the optimal rate I_U for hitting the upper curve in the infinite horizon problem given by (2.39). Correspondingly, the blue line gives the plot of the optimal rate for hitting the lower curve, I_V, given by (2.37). The point of intersection of the two curves corresponds to the initial condition x_0 for which the exponential rate for the probability of hitting the upper curve is equal to that for the lower curve.

the SDE

$$dX_t = (\mu X_t + r)dt + \sigma dW_t, \quad X_0 = x_0.$$

The upper boundary curve is $U(t) = u_0 e^{\alpha t}$. We assume that $u_0 > x_0$ and $\mu < \alpha$. In the deterministic limit, when $\sigma \to 0$, one obtains the Ordinary Differential Equation $\frac{d}{dt}x(t) = \mu x(t) + r$ which has the solution $x(t) = x_0 e^{\mu t} + \frac{r}{\mu}(e^{\mu t} - 1)$. To ensure that we remain in range of applicability of Large Deviation results we will need to ensure that the deterministic solution remains strictly below the upper bound, $U(t)$ for all $t \geq 0$. Let

$$\phi(t) \ := \ U(t) - x(t) \ = \ u_0 e^{\alpha t} - \left(x_0 + \frac{r}{\mu} \right) e^{\mu t} + \frac{r}{\mu}. \tag{3.1}$$

Then we must have

$$\inf_{t \geq 0} \phi(t) > 0. \tag{3.2}$$

We will make the additional assumption that

$$r \ < \ u_0(\alpha - \mu). \tag{3.3}$$

This is a sufficient condition that ensures that (3.2) holds. Indeed, $\phi(0) = u_0 - x_0 > 0$ and

$$\phi'(t) = e^{\mu t} \left[u_0 \alpha e^{(\alpha-\mu)t} - x_0 \mu - r \right].$$

Then, $u_0 \alpha e^{(\alpha-\mu)t} - x_0\mu - r \geq u_0\alpha - x_0\mu - r > u_0\alpha - x_0\alpha - r > 0$ and hence (3.2) holds.

The action functional is $\frac{1}{2\sigma^2} \int_0^T (x' - \mu x - r)^2 \, du$. The Euler-Lagrange differential equation $F_x - \frac{d}{dt} F_{x'} = 0$ reduces to

$$x'' - \mu^2 x - \mu r = 0$$

with general solution

$$x(t) = C_1 e^{\mu t} + C_2 e^{-\mu t} - \frac{r}{\mu}. \tag{3.4}$$

The boundary conditions are

$$x_0 = C_1 + C_2 - \frac{r}{\mu} \tag{3.5}$$

$$u_0 e^{\alpha T} = C_1 e^{\mu T} + C_2 e^{-\mu T} - \frac{r}{\mu}. \tag{3.6}$$

The transversality condition that must be satisfied by a critical path meeting the curve $U(t) := u_0 e^{\alpha t}$ at T is

$$F + (U'(T) - x'(T))F_{x'} = 0 \quad \text{or} \quad (x' - r - \mu x)\left(-x' - r - \mu x + 2u_0 \alpha e^{\alpha T}\right) = 0$$

which, using (3.4), reduces to

$$C_2 \left(u_0 \alpha e^{\alpha T} - \mu C_1 e^{\mu T}\right) = 0. \tag{3.7}$$

The above equation leads to the examination of two cases:

Case 1. $C_2 = 0$**.** Using this value in (3.5), (3.6), and eliminating C_1 among them gives

$$u_0 e^{\alpha T} - \left(x_0 + \frac{r}{\mu}\right) e^{\mu T} + \frac{r}{\mu} = 0. \tag{3.8}$$

This equation corresponds to the requirement $\phi(T) = 0$ for the function defined in (3.1) which is impossible. Hence $C_2 = 0$ is impossible.

Case 2. $u_0 \alpha e^{\alpha T} - \mu C_1 e^{\mu T} = 0$**.** This, together with (3.6) gives

$$u_0 \left(1 - \frac{\alpha}{\mu}\right) e^{\alpha T} = C_2 e^{-\mu T} - \frac{r}{\mu}. \tag{3.9}$$

Using this, (3.5), (3.6), give

$$C_1 + C_2 = x_0 + \frac{r}{\mu} \tag{3.10}$$

$$C_1 e^{\mu T} + C_2 e^{-\mu T} = u_0 e^{\alpha T} + \frac{r}{\mu}. \tag{3.11}$$

The above system has the solution

$$C_1 = \frac{e^{-\mu T}\left(x_0 + \frac{r}{\mu}\right) - \left(u_0 e^{\alpha T} + \frac{r}{\mu}\right)}{e^{-\mu T} - e^{\mu T}}, \quad C_2 = \frac{u_0 e^{\alpha T} + \frac{r}{\mu} - e^{\mu T}\left(x_0 + \frac{r}{\mu}\right)}{e^{-\mu T} - e^{\mu T}}.$$

Using this, (3.9) reduces to

$$u_0 \left(\frac{\alpha}{\mu} - 1\right) e^{(\alpha+\mu)T} - u_0 \frac{\alpha}{\mu} e^{(\alpha-\mu)T} - \frac{r}{\mu} e^{\mu T} + x_0 + \frac{r}{\mu} = 0. \tag{3.12}$$

Under Assumption (3.3) i.e. if the drift term r is either negative or, if positive, not too large, the above equation has a unique solution which determines T.

Define $f(t) := u_0 \left(\frac{\alpha}{\mu} - 1\right) e^{t(\alpha+\mu)} - u_0 \frac{\alpha}{\mu} e^{(\alpha-\mu)t} - \frac{r}{\mu} e^{\mu t} + x_0 + \frac{r}{\mu}$ and note that $f(0) = x_0 - u_0 < 0$, $\lim_{t\to\infty} f(t) = +\infty$. Also $f'(t) = (\alpha + \mu) u_0 \left(\frac{\alpha}{\mu} - 1\right) e^{t(\alpha+\mu)} - u_0 \frac{\alpha}{\mu}(\alpha - \mu) e^{(\alpha-\mu)t} - r e^{\mu t}$ and $f'(0) = u_0(\alpha - \mu) - r$. Under Assumption (3.3) $f'(0) > 0$. We will show that the assumptions implies in fact

$f'(t) > 0$ for all $t > 0$. Let $e^{-\mu t} f'(t) =: g(t) = (\alpha+\mu)u_0 \left(\frac{\alpha}{\mu} - 1\right) e^{\alpha t} - u_0 \frac{\alpha}{\mu}(\alpha - \mu)e^{(\alpha-2\mu)t} - r$ and thus $g(0) = f'(0) = u_0(\alpha - \mu) - r > 0$. Also $g'(t) = \frac{\alpha}{\mu} e^{\alpha t}(\alpha - \mu)u_0 \left[\alpha + \mu - (\alpha - 2\mu)e^{-2\mu t}\right] > 0$ for all $t \geq 0$. From the above we conclude that $g(t)$ and therefore $f'(t)$ is strictly positive for all t as well and thus that (3.12) has a unique solution. The optimal path is then

$$x^*(t) = \frac{\left(x_0 + \frac{r}{\mu}\right) \sinh(\mu(T-t)) + \left(u_0 e^{\alpha T} + \frac{r}{\mu}\right) \sinh(\mu t)}{\sinh(\mu T)} - \frac{r}{\mu}. \tag{3.13}$$

It is possible to show again that there are two intersction points between the curves $U(t) = u_0 e^{\alpha t}$ and $x^*(t)$, one of which is of course T_U. The condition for this solution to satisfy the inequality constraints as well is

$$x^{*\prime}(T) > u_0 \alpha e^{\alpha T}.$$

This condition is written as

$$\mu \frac{\left(u_0 e^{\alpha T} + \frac{r}{\mu}\right)\left(e^{\mu T} + e^{-\mu T}\right) - 2(x_0 + \frac{r}{\mu})}{e^{\mu T} - e^{-\mu T}} > \alpha u_0 e^{\alpha T}$$

and is is equivalent to

$$\frac{r}{\mu}\left(e^{\mu T} + e^{-\mu T}\right) + u_0 e^{T(\alpha-\mu)} - 2(x_0 + \frac{r}{\mu}) > u_0 e^{\alpha T}\left[\left(\frac{\alpha}{\mu} - 1\right) e^{\mu T} - \frac{\alpha}{\mu} e^{-\mu T}\right] = \frac{r}{\mu}(e^{\mu T} - 1) - x_0$$

the last equation following from (3.12). Hence the condition is equivalent to

$$\frac{r}{\mu} e^{-\mu T} + u_0 e^{(\alpha-\mu)T} > x_0 + \frac{r}{\mu}.$$

This inequality however is true because it is equivalent to $\phi(T) > 0$ for the function ϕ defined in (3.1), which is true.

The optimal rate can be obtained from the fact that $x'(t) - \mu x(t) - r = 2C_2 e^{\mu t}$ and hence

$$I = \frac{\mu}{\sigma^2} \int_0^T 4C_2^2 e^{\mu t} dt = \frac{\mu}{\sigma^2} \frac{\left(u_0 e^{(\alpha-\mu)T} - \frac{r}{\mu}\left(1 - e^{-\mu T}\right) - x_0\right)^2}{1 - e^{-2\mu T}}.$$

Note, of course, that when $r \to 0$ the above reduces to the value of I given in (2.58).

3.2 A ruin problem involving two independent OU processes

Here we generalize the problem examined in the previous section. The lower (or upper) deterministic exponential boundary now is also considered to be stochastic - in fact another, independent, OU process. We may thus study the following pair of SDE's

$$dX_t = \alpha X_t dt + \sigma dW_t, \qquad X_0 = x_0 \tag{3.14}$$

$$dY_t = \beta Y_t dt + b dB_t, \qquad Y_0 = y_0. \tag{3.15}$$

where $\beta < \alpha$ and $y_0 < x_0$. As a result of these inequalities, in the absence of noise, ($\sigma = b = 0$) we would have $Y_t < X_t$ for all t. The presence of noise may cause the two curves to meet however. Again, an exact analysis does not give results in closed form and we obtain low noise logarithmic asymptotics in the

Wentzell-Freidlin framework. Using again Theorem 5.6.7, p. 214 of Dembo and Zeitouni (2010) we obtain a two dimensional version of (2.8) for the action functional to be minimized:

$$I = \int_0^T F(x, x', y, y')dt, \qquad F = \frac{1}{2}\left[\frac{1}{\sigma^2}(x' - \alpha x)^2 + \frac{1}{b^2}(y' - \beta y)^2\right]. \tag{3.16}$$

The boundary conditions $x(0) = x_0$, $y(0) = y_0$, and $x(T) = y(T)$.

We will again tackle the infinite horizon problem directly and solve the moving boundary variational problem using the appropriate transversality conditions. Thus the first order necessary conditions for an extremum are

$$F_x - \frac{d}{dt}F_{x'} = 0, \qquad F_y - \frac{d}{dt}F_{y'} = 0 \tag{3.17}$$

$$x(T) = y(T) \tag{3.18}$$

$$F_{x'} + F_{y'} = 0 \quad \text{at } T, \tag{3.19}$$

$$F - x'F_{x'} - y'F_{y'} = 0 \quad \text{at } T. \tag{3.20}$$

The Euler-Lagrange equations (3.17) give $x'' - \alpha^2 x = 0$ and $y'' - \beta^2 y = 0$ and thus, $x(t) = C_1 e^{\alpha t} + C_2 e^{-\alpha t}$ and $y(t) = C_3 e^{\beta t} + C_4 e^{-\beta t}$ with boundary conditions

$$C_1 + C_2 = x_0, \quad C_3 + C_4 = y_0, \quad \text{and} \quad C_1 e^{\alpha T} + C_2 e^{-\alpha T} = C_3 e^{\beta T} + C_4 e^{-\beta T}. \tag{3.21}$$

The first transversality condition, (3.19) gives

$$\frac{1}{\sigma^2}(x'(T) - \alpha x(T)) + \frac{1}{b^2}(y'(T) - \beta y(T)) = 0 \tag{3.22}$$

or

$$\frac{\alpha}{\sigma^2}C_2 e^{-\alpha T} + \frac{\beta}{b^2}C_4 e^{-\beta T} = 0. \tag{3.23}$$

The second transversality condition (3.20), after routine algebraic manipulations, gives

$$x'F_{x'} + y'F_{y'} - F = \frac{1}{2\sigma^2}(x' - \alpha x)(x' + \alpha x) + \frac{1}{2b^2}(y' - \beta y)(y' + \beta y) = 0.$$

The above, in view of (3.22), becomes

$$\big(x'(T) - \alpha x(T)\big)\,\big(x'(T) + \alpha x(T) - y'(T) - \beta y(T)\big) = 0.$$

If the first factor is zero then, in view of (3.22), we obtain

$$x'(T) - \alpha x(T) = 0, \quad y'(T) - \beta y(T) = 0.$$

In view of the fact that $x'(T) - \alpha x(T) = -2\alpha C_2 e^{-\alpha T}$ this translates into $C_2 = 0$ and similarly $y'(T) - \beta y(T) = -2\beta C_4 e^{-\beta T} = 0$ implies $C_4 = 0$. Hence $x(t) = x_0 e^{\alpha T}$, $y(t) = y_0 e^{\beta T}$, and $x(T) = y(T)$ implies that $x_0 e^{\alpha T} = y_0 e^{\beta T}$ or $e^{(\alpha-\beta)T} = \frac{y_0}{x_0}$. Since $\alpha - \beta > 0$ and $y_0/x_0 < 1$ it is impossible to find $T > 0$ which satisfies this last equation.

The alternative solution is

$$x'(T) + \alpha x(T) = y'(T) + \beta y(T). \tag{3.24}$$

Note that

$$x'(T) + \alpha x(T) = 2\alpha C_1 e^{\alpha T}, \quad y'(T) + \beta y(T) = 2\beta C_3 e^{\beta T}$$

and hence (3.24) gives

$$\alpha C_1 e^{\alpha T} = \beta C_3 e^{\beta T}. \tag{3.25}$$

Determination of the optimal path. Displays (3.21), (3.23), and (3.25) provide the following five equations to determine the five unknown quantities, C_i, $i = 1, \dots, 4$, and T:

$$
\begin{aligned}
C_1 + C_2 &= x_0 && (3.26)\\
\frac{\alpha}{\beta} e^{(\alpha-\beta)T} C_1 - \frac{\alpha}{\beta}\frac{b^2}{\sigma^2} e^{-(\alpha-\beta)T} C_2 &= y_0 && (3.27)\\
C_1 e^{\alpha T} + C_2 e^{-\alpha T} &= \frac{\alpha}{\beta} e^{\alpha T} C_1 - \frac{\alpha}{\beta}\frac{b^2}{\sigma^2} e^{-\alpha T} C_2 && (3.28)\\
C_3 &= \frac{\alpha}{\beta} e^{(\alpha-\beta)T} C_1 && (3.29)\\
C_4 &= -\frac{\alpha}{\beta}\frac{b^2}{\sigma^2} e^{-(\alpha-\beta)T} C_2 && (3.30)
\end{aligned}
$$

From the above we may obtain the values of C_i, $i = 1, \dots, 4$ in terms of T:

$$
\begin{aligned}
C_1 &= x_0 \frac{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right) e^{-\alpha T}}{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right) e^{-\alpha T} + \left(\frac{\alpha}{\beta} - 1\right) e^{\alpha T}}, \quad C_2 = x_0 \frac{\left(\frac{\alpha}{\beta} - 1\right) e^{\alpha T}}{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right) e^{-\alpha T} + \left(\frac{\alpha}{\beta} - 1\right) e^{\alpha T}},\\
C_3 &= y_0 \frac{\left(1 + \frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right) e^{-\beta T}}{\left(1 + \frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right) e^{-\beta T} + \left(\frac{\beta}{\alpha} - 1\right) e^{\beta T}}, \quad C_4 = y_0 \frac{\left(\frac{\beta}{\alpha} - 1\right) e^{\beta T}}{\left(1 + \frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right) e^{-\beta T} + \left(\frac{\beta}{\alpha} - 1\right) e^{\beta T}}.
\end{aligned}
\tag{3.31}
$$

From these we obtain the following expression for the critical path

$$
\begin{aligned}
x(t) &= x_0 \frac{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right) e^{\alpha(t-T)} + \left(\frac{\alpha}{\beta} - 1\right) e^{\alpha(T-t)}}{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right) e^{-\alpha T} + \left(\frac{\alpha}{\beta} - 1\right) e^{\alpha T}},\\
y(t) &= y_0 \frac{\left(1 + \frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right) e^{\beta(t-T)} + \left(\frac{\beta}{\alpha} - 1\right) e^{\beta(T-t)}}{\left(1 + \frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right) e^{-\beta T} + \left(\frac{\beta}{\alpha} - 1\right) e^{\beta T}}.
\end{aligned}
\tag{3.32}
$$

Of course, there remains the task to determine the optimal meeting time T. From the above, when $t = T$ we have

$$
\begin{aligned}
x(T) &= x_0 \frac{\alpha(b^2 + \sigma^2)}{(\alpha - \beta)\sigma^2 e^{\alpha T} + (\beta\sigma^2 + \alpha b^2) e^{-\alpha T}},\\
y(T) &= y_0 \frac{\beta(b^2 + \sigma^2)}{(\beta - \alpha) b^2 e^{\beta T} + (\beta\sigma^2 + \alpha b^2) e^{-\beta T}}.
\end{aligned}
$$

At the meeting time T, $x(T) = y(T)$ and therefore

$$
x_0\alpha \left[(\beta - \alpha) b^2 e^{\beta T} + (\beta\sigma^2 + \alpha b^2) e^{-\beta T}\right] = y_0\beta \left[(\alpha - \beta)\sigma^2 e^{\alpha T} + (\beta\sigma^2 + \alpha b^2) e^{-\alpha T}\right]. \tag{3.33}
$$

Determination of the meeting time T. We will show that the above equation determines uniquely T. To this end, define the function

$$
f(t) := (\alpha - \beta)\left[y_0\beta\sigma^2 e^{\alpha t} + x_0\alpha b^2 e^{\beta t}\right] + (\beta\sigma^2 + \alpha b^2)\left[y_0\beta e^{-\alpha t} - x_0\alpha e^{-\beta t}\right], \qquad t \geq 0.
$$

It holds that

$$
\begin{aligned}
f(0) &= (\alpha-\beta)\left[y_0\beta\sigma^2 + x_0\alpha b^2\right] + (\beta\sigma^2+\alpha b^2)\left[y_0\beta - x_0\alpha\right] \\
&= \alpha\beta(\sigma^2+b^2)(y_0\beta - x_0\alpha) \;<\; 0
\end{aligned}
$$

and also $\lim_{t\to\infty} f(t) = +\infty$. Furthermore

$$
f'(t) \;=\; (\alpha-\beta)\alpha\beta\left[y_0\sigma^2 e^{\alpha t} + x_0 b^2 e^{\beta t}\right] + (\beta\sigma^2+\alpha b^2)\alpha\beta\left[-y_0 e^{-\alpha t} + x_0 e^{-\beta t}\right]
$$

Clearly $f'(t) > 0$ for all $t \geq 0$ since $\left[-y_0 e^{-\alpha t} + x_0 e^{-\beta t}\right] = e^{-\alpha t}\left[-y_0 + x_0 e^{(\alpha-\beta)t}\right] > 0$ because $\alpha > \beta$ and $x_0 > y_0$.

$x(t) > y(t)$ when $t \in [0, T)$. A straight-forward computation (taking into account (3.32), (3.33)) gives

$$
x'(T) - y'(T) \;=\; -\frac{x_0\alpha(\alpha-\beta)(\sigma^2+b^2)}{(\alpha b^2+\beta\sigma^2)e^{-\alpha T} + \sigma^2(\alpha-\beta)e^{\alpha T}} \;<\; 0 \tag{3.34}
$$

Thus it can be seen that the path $x(\cdot)$ starts above $y(\cdot)$ at 0, crosses it from above at T and (since $\alpha > \beta$) crosses it again once more at some $T^* > T$. In particular we note that $x(t) > y(t)$ for $t \in [0, T)$, i.e. x crosses y at T for the first time.

Determination of the rate I. Taking into account that $x'(t) - \alpha x(t) = -2\alpha C_2 e^{-\alpha t}$ and similarly $y'(t) - \beta y(t) = -2\beta C_4 e^{-\beta t}$ the rate function becomes

$$
\begin{aligned}
I &= \frac{1}{2\sigma^2}\int_0^T 4\alpha^2 C_2^2 e^{-2\alpha t}dt + \frac{1}{2b^2}\int_0^T 4\beta^2 C_4^2 e^{-2\beta t}dt \;=\; \frac{\alpha C_2^2}{\sigma^2}\left(1-e^{-2\alpha T}\right) + \frac{\beta C_4^2}{b^2}\left(1-e^{-2\beta T}\right) \\
&= \frac{\frac{\alpha}{\sigma^2}\left(1-e^{-2\alpha T}\right)x_0^2\left(\frac{\alpha}{\beta}-1\right)^2 e^{2\alpha T}}{\left[\left(1+\frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right)e^{-\alpha T} + \left(\frac{\alpha}{\beta}-1\right)e^{\alpha T}\right]^2} + \frac{\frac{\beta}{b^2}\left(1-e^{-2\beta T}\right)y_0^2\left(\frac{\beta}{\alpha}-1\right)^2 e^{2\beta T}}{\left[\left(1+\frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right)e^{-\beta T} + \left(\frac{\beta}{\alpha}-1\right)e^{\beta T}\right]^2}
\end{aligned}
$$

or equivalently

$$
I \;=\; \frac{\alpha(\alpha-\beta)^2\sigma^2 x_0^2\left(e^{2\alpha T}-1\right)}{\left[(\alpha-\beta)\sigma^2 e^{\alpha T} + (\beta\sigma^2+\alpha b^2)e^{-\alpha T}\right]^2} + \frac{\beta\,(\alpha-\beta)^2\,b^2 y_0^2\left(e^{2\beta T}-1\right)}{\left[(\beta-\alpha)\,b^2\,e^{\beta T} + (\beta\sigma^2+\alpha b^2)\,e^{-\beta T}\right]^2}. \tag{3.35}
$$

In particular, when $b = 0$ and $\alpha = \mu$ then the lower OU process becomes a deterministic lower bound and (3.35) reduces indeed to the right hand side of (2.39), as it should.

Again, as in the proof of Theorem 2 we will show that the solution obtained corresponds to a global minimum using the fact that $F : \mathbb{R}^4 \to \mathbb{R}$ is convex and appealing to Theorem 3.16 of Brechtken-Manderscheid (1994), p. 45. To establish the convexity of $F(x, x', y, y') := \frac{1}{2\sigma^2}(x' - \alpha x)^2 + \frac{1}{2b^2}(y' - \beta y)^2$ we note that, for any $(x_0, x_0', y_0, y_0') \in \mathbb{R}^4$,

$$
F(x, x', y, y') - F(x_0, x_0', y_0, y_0') \geq F_x^0\,(x - x_0) + F_{x'}^0\,(x' - x_0') + F_y^0\,(y - y_0) + F_{y'}^0\,(y' - y_0') \tag{3.36}
$$

where F_x^0 is shorthand for $F_x(x_0, x_0', y_0, y_0')$ and similarly for the other three such quantities. The above inequality is equivalent to

$$
\begin{aligned}
&\frac{1}{2\sigma^2}\left(x' - \alpha x\right)^2 + \frac{1}{2b^2}\left(x' - \beta x\right)^2 - \frac{1}{2\sigma^2}\left(x_0' - \alpha x_0\right)^2 - \frac{1}{2b^2}\left(x_0' - \beta x_0\right)^2 \\
&\qquad \geq\; -\frac{\alpha}{\sigma^2}\left(x_0' - \alpha x_0\right)\,(x - x_0) + \frac{1}{\sigma^2}\left(x_0' - \alpha x_0\right)\,(x' - x_0') \\
&\qquad\quad -\frac{\beta}{b^2}\left(y_0' - \beta y_0\right)\,(y - y_0) + \frac{1}{b^2}\left(y_0' - \beta y_0\right)\,(y' - y_0').
\end{aligned}
$$

Elementary algebraic manipulations can show the above inequality to be true and therefore establish inequality (3.36) which implies the convexity of F.

We may summarize the above long derivation in the following

Theorem 4. *Consider the pair of Ornstein-Uhlenbeck SDE's depending on a parameter* $\epsilon > 0$

$$\begin{aligned} dX_t^\epsilon &= \alpha X_t^\epsilon dt + \sqrt{\epsilon}\sigma dW_t, \qquad X_0^\epsilon = x_0, \\ dY_t^\epsilon &= \beta Y_t^\epsilon dt + \sqrt{\epsilon} b dB_t, \qquad Y_0^\epsilon = y_0. \end{aligned}$$

Assume that $0 < y_0 < x_0$ *and* $0 < \beta < \alpha$. *Let* $T^\epsilon := \inf\{t \geq 0 : X_t^\epsilon = Y_t^\epsilon\}$ *(with the standard convention that* $T^\epsilon = +\infty$ *if the set is empty). Then*

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}(T^\epsilon < \infty) \; = \; -I$$

where I *is given by (3.35). If this rare event occurs then the meeting path followed by the two processes is given by (3.32) and the meeting time* T *is the unique solution of (3.33).*

4 Geometric Brownian Motion

In this section an analysis of the problems we examined for the Ornstein-Uhlenbeck process is repeated for the Geometric Brownian motion. The approach followed and the techniques used are analogous to those of section 2. The reason for treating the Geometric Brownian motion in some detail here is due to its great importance in applications but also to the fact that in this case an analytic solution in closed form for the types of ruin problems we consider can be obtained. As a result, the accuracy and merit of the large deviation estimates we obtain may be gauged.

4.1 The finite horizon problem

Suppose that $\{X_t; t \geq 0\}$ is a Geometric Brownian motion satisfying the Stochastic Differential Equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \qquad X_0 = x_0 \text{ w.p. } 1. \tag{4.1}$$

As is well known this has the closed form solution

$$X_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{4.2}$$

Let $u_0 > x_0$ and $\alpha > \mu$. Then the event $\{X_t \geq u_0 e^{at} \text{ for some } t \leq T\}$ is an event whose probability goes to 0 as $\sigma \to 0$. Our goal is to obtain low variance Wentzell-Freidlin asymptotics for this finite horizon hitting probability. For reasons of notational compatibility we introduce the *parametrized process*

$$dX_t^\epsilon = \mu X_t^\epsilon dt + \sqrt{\epsilon}\sigma X_t^\epsilon dW_t, \qquad X_0^\epsilon = x_0 \text{ w.p. } 1. \tag{4.3}$$

Theorem 5. *For the parametrized process* $\{X_t^\epsilon\}$,

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\sup_{0 \leq t \leq T} \left(X_t^\epsilon - u_0 e^{\alpha t} \right) \geq 0 \right) \; = \; -I(T). \tag{4.4}$$

The rate function $I(T)$ *is given by*

$$I(T) := \min_{0 \leq t \leq T} J_*(t) \tag{4.5}$$

where $J_*(t)$ is solution to the minimization problem

$$J_*(t) \;=\; \min\left\{J(x,t)\;:\; x\in\mathcal{H},\; x(0)=x_0,\; x(t)=u_0e^{\alpha t},\; x(s)<u_0e^{\alpha s},\; s\in[0,t)\right\}. \tag{4.6}$$

In the above, $\mathcal{H}$ is the Cameron-Martin space of absolutely continuous functions with square integrable derivative (as defined in the statement of Theorem 1) and $J(x,t)$ is the action functional

$$J(x,t) := \frac{1}{2}\int_0^t \left(\frac{x'(s)-\mu x(s)}{\sigma x(s)}\right)^2 ds = \frac{1}{2\sigma^2}\int_0^t \left((\log x(s))'-\mu\right)^2 ds. \tag{4.7}$$

The proof of this theorem is along the lines of the proof of Theorem 2, but a great deal easier. Instead of a proof we give a brief discussion. The minimizing path $x(s)$ can be easily obtained in this case either by using the the Euler-Lagrange differential equation, or simply by observing that the functional $J(x,t)$ is minimized when $(\log x)'$ is constant, say c, or equivalently when $\log x(s) = B + cs$ for some $B\in\mathbb{R}$ and $s\in[0,t]$. This in turn implies that $x(s) = Ke^{cs}$ with $x(0) = x_0 = K$ and $x(t) = x_0e^{ct} = u_0e^{\alpha t}$ whence we conclude that the function that minimizes the action functional under the boundary conditions is

$$x(t) = x_0e^{ct} \quad \text{where} \quad c = \alpha + \frac{1}{t}\log\frac{u_0}{x_0}. \tag{4.8}$$

It is easy to see that the above path satisfies the constraint $x(s) < u_0e^{\alpha s}$ for $s\in[0,t)$. The corresponding minimum action is then

$$J_*(t) \;=\; \frac{t}{2\sigma^2}\left(\alpha-\mu+\frac{1}{t}\log\frac{u_0}{x_0}\right)^2$$

or

$$J_*(t) \;=\; t\,\frac{(\alpha-\mu)^2}{2\sigma^2} + 2\frac{(\alpha-\mu)\log\frac{u_0}{x_0}}{2\sigma^2} + \frac{1}{t}\,\frac{(\log\frac{u_0}{x_0})^2}{2\sigma^2}.$$

The above function is clearly convex on $(0,\infty)$ and achieves its minimum at $t_{\min} = \frac{\log\frac{u_0}{x_0}}{\alpha-\mu}$. This minimum is $J_*(t_{\min}) = \frac{2}{\sigma^2}(\alpha-\mu)\log\frac{u_0}{x_0}$. Thus, since J_* is easily seen to be decreasing in $(0, t_{\min})$ and increasing in $(t_{\min},\infty)$, by (4.5) we conclude that the rate function is

$$I(T) = \begin{cases} \frac{T}{2\sigma^2}\left(\alpha-\mu+\frac{1}{T}\log\frac{u_0}{x_0}\right)^2 & \text{if} \quad T<t_{\min} \\ \frac{2}{\sigma^2}(\alpha-\mu)\log\frac{u_0}{x_0} & \text{if} \quad T\ge t_{\min} \end{cases}. \tag{4.9}$$

In practice, (4.4) gives rise to the approximation

$$-\log\mathbb{P}\left(\sup_{0\le t\le T}(X_t-u_0e^{at})\ge 0\right) \approx I(T) \tag{4.10}$$

which is satisfactory provided that σ is sufficiently small. We assess its quality in the next subsection taking advantage of the fact that an exact, closed form solution also exists in this situation.

4.2 The exact solution

Consider the GBM $X_t = x_0e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}$ and the corresponding finite horizon hitting probability

$$p_T \;:=\; \mathbb{P}\left(\sup_{0\le t\le T}(X_t-u_0e^{at})\ge 0\right)$$

where, as before $\alpha > \mu$ and $0 < x_0 < u_0$. Since the event $(X_t - u_0 e^{\alpha t}) \geq 0$ is the same as $X_t e^{-\alpha t} - u_0 \geq 0$,

$$\begin{aligned} p_T &= \mathbb{P}\left(\sup_{0\leq t\leq T} x_0 e^{(\mu-\frac{1}{2}\sigma^2-a)t+\sigma W_t} \geq u_0\right) = \mathbb{P}\left(\sup_{0\leq t\leq T}(\mu - \frac{1}{2}\sigma^2 - \alpha)t + \sigma W_t \geq \log\frac{u_0}{x_0}\right) \\ &= 1 - \Phi\left(\frac{\log\left(\frac{u_0}{x_0}\right) - (\mu-\alpha-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) + e^{\frac{2}{\sigma^2}(\mu-\alpha-\frac{1}{2}\sigma^2)\log\left(\frac{u_0}{x_0}\right)}\Phi\left(\frac{-\log\left(\frac{u_0}{x_0}\right) - (\mu-\alpha-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \quad (4.11) \end{aligned}$$

Here, $\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$, the standard normal distribution function. The above exact formula for p_T allows us to evaluate the accuracy of the approximation (4.10). Figure 11 shows again $-\sigma^2 \log p_T$ together with the Wentzell-Freidlin asymptotic result when $\sigma \to 0$. Thus approximation (4.10) may be considered satisfactory, provided that σ is small.

4.3 The infinite horizon problem

The exact value of the infinite horizon hitting probability can be obtained from (4.11) by letting $T \to \infty$. This gives

$$\lim_{T\to\infty} p_T \;=:\; p_\infty \;=\; \exp\left(\frac{2}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2 - \alpha\right)\log\frac{u_0}{x_0}\right).$$

Returning to the parametrized version of the problem, concerning the family of processes $\{X_t^\epsilon\}$ defined in (4.3), the corresponding infinite horizon hitting probability is

$$p_\infty^\epsilon \;=\; \exp\left(\frac{2}{\epsilon\sigma^2}\left(\mu - \frac{1}{2}\epsilon\sigma^2 - \alpha\right)\log\frac{u_0}{x_0}\right)$$

and therefore

$$\lim_{\epsilon\to 0} \epsilon \log p_\infty^\epsilon \;=\; -\frac{2}{\sigma^2}(\alpha - \mu)\log\frac{u_0}{x_0}. \quad (4.12)$$

This, as we will see, is the same as the result obtained from Wentzell-Freidlin theory.

Theorem 6. *For the parametrized process* $\{X_t^\epsilon\}$

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\sup_{t\geq 0}\left(X_t^\epsilon - u_0 e^{\alpha t}\right) \geq 0\right) \;=\; -I(\infty) \quad (4.13)$$

where the rate function $I(\infty)$ *is the solution to the infinite horizon variational problem*

$$\inf\left\{J(x,T) \;:\; x \in \mathcal{H},\ x(s) < u_0 e^{\alpha s} \text{ for } 0 \leq s < T,\ x(0) = x_0,\ x(T) = u_0 e^{\alpha T}\right\} \quad (4.14)$$

where $J(x,t) := \frac{1}{2}\int_0^t \left((\log x(u))' - \mu\right)^2 du$ *and* $\mathcal{H}$ *is again the Cameron-Martin space of absolutely continuous functions with square-integrable derivatives. In fact, for the infinite horizon problem,*

$$I(\infty) = 2\frac{\alpha - \mu}{\sigma^2}\log\frac{u_0}{x_0}, \quad (4.15)$$

the optimal time horizon is

$$T = \frac{\log\frac{u_0}{x_0}}{\alpha - \mu}, \quad (4.16)$$

and the optimal path that achieves the minimum *is*

$$x^*(t) = x_0 e^{2\alpha - \mu t}, \quad t \in [0,T]. \quad (4.17)$$

Proof. The optimization problem of Theorem 6 can of course be solved using the finite horizon analysis as a basis. However we prefer to use standard techniques of the calculus of variations for infinite horizon problems with the final value of the path constrained to lie on a prescribed curve using the *transversality conditions*

$$\min \int_0^T F(x, x', t)dt, \qquad \text{with boundary conditions } x(0) = x_0, \text{ and } x(T) = u(T)$$
$$\text{with } F(x, x', t) = \frac{1}{2\sigma^2}\left(\frac{x'}{x} - \mu\right)^2. \tag{4.18}$$

In the above $u(t) = u_0 e^{\alpha t}$ is a given boundary curve with $x_0 < u_0$ and x is a $C^1[0, \infty)$ function which minimizes the "action" integral given the boundary conditions in (4.18). The conditions for a minimum is

$$F_x - \frac{d}{dt}F_{x'} = 0 \tag{4.19}$$
$$x(0) = x_0 \quad \text{and} \quad x(T) = u(T) \tag{4.20}$$
$$F + (u' - x')F_{x'} = 0 \quad \text{at } T. \tag{4.21}$$

The first equation is the Euler-Lagrange DE of the Calculus of Variations. Equation (4.21) is known as the *transversality condition* resulting from the fact that the end time T is not fixed but is itself to be chosen

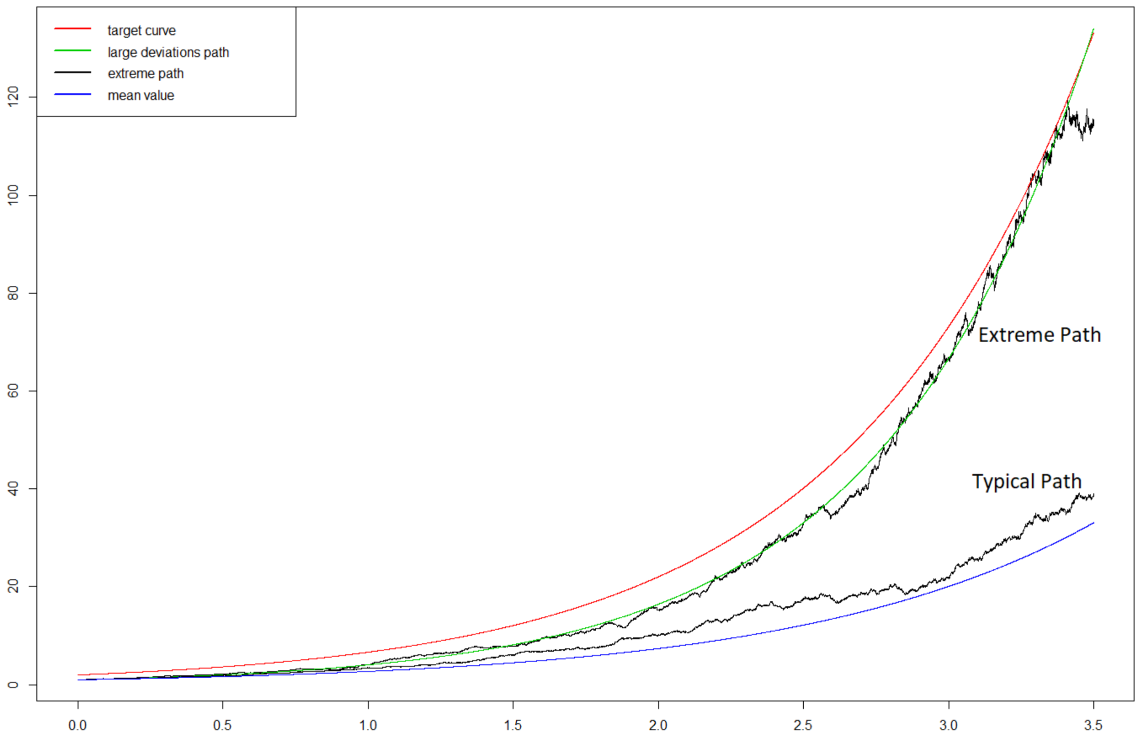

Figure 8: Simulated sample path for $\alpha = 1, x_0 = 1, u_0 = 2$ and $\sigma = 0.15$. The red curve is the exponential target curve $u_0 e^{\alpha t}$. The green curve is the optimal path $x^*(t) = x_0 e^{(2\alpha - \mu)t}t$ given in (4.17). Both a typical path and an extreme path of the Geometric Brownian motion are displayed. The extreme path was generated by simulating a large number of paths ($\approx 10^5$) and selecting one that hit the target, i.e. reached the red curve. As expected it follows closely the green curve. The smaller the variance the smaller the probability of hitting the target and the closer the agreement with the theoretical path.

optimally, under the restriction that $x(T) = u(T)$. Then the Euler-Lagrange equation (4.19) becomes

$$\frac{2}{x^3}\left((x')^2 - x''x\right) = 0$$

or equivalently

$$\frac{x'}{x} = \frac{x''}{x'} \quad \Leftrightarrow \quad \left(\log x'\right)' - (\log x)' = 0 \quad \Leftrightarrow \quad \log x' - \log x = c_1 \quad \Leftrightarrow \quad \frac{x'}{x} = \gamma.$$

Hence

$$x(t) = x_0 e^{\gamma t}. \tag{4.22}$$

The transversality condition (4.21) reduces to

$$\left(\frac{x'(T)}{x(T)} - \mu\right)\left(\frac{x'(T)}{x(T)} - \mu + \left(u_0 \alpha e^{\alpha T} - x'(T)\right)\frac{2}{x(T)}\right) = 0$$

and taking into account (4.22) we obtain either $\mu = \gamma$ or

$$\gamma - \mu + 2\alpha \frac{u_0}{x_0} e^{(\alpha-\gamma)T} - 2\gamma = 0$$

or

$$2\alpha \frac{u_0}{x_0} e^{(\alpha-\gamma)T} = \mu + \gamma. \tag{4.23}$$

Equation (4.20) gives $x_0 e^{\gamma T} = u_0 e^{\alpha T}$ and therefore

$$e^{(\alpha-\gamma)T} = \frac{x_0}{u_0}. \tag{4.24}$$

From (4.23) and (4.24) we have

$$\gamma = 2\alpha - \mu \tag{4.25}$$

The solution of the variational process that minimizes the action functional I and satisfies the boundary conditions yields the optimal path $x_t = x_0 e^{(2\alpha-\mu)t}$ and the rate function

$$I = 2\frac{\alpha - \mu}{\sigma^2} \log \frac{u_0}{x_0} \quad \text{and} \quad T = \frac{\log \frac{u_0}{x_0}}{\alpha - \mu}.$$

□

The exact solution agrees with the Wentzell-Freidlin asymptotic result. Figure 8 illustrates the above result. In particular, the extreme path was selected by simulating a large number of paths and picking the largest among them.

4.4 Two correlated Geometric Brownian Motions

Suppose that W_t, V_t, are independent standard Brownian motions and $\rho \in [-1, 1]$. Set $B_t = \rho W_t + \sqrt{1-\rho^2}\, V_t$. Then (W_t, B_t) are correlated Brownian motions with correlation ρ. Consider now the processes

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma X_t dW_t, \quad X_0 = x_0, \\ dY_t &= \beta Y_t dt + b Y_t dB_t, \quad Y_0 = y_0. \end{aligned}$$

We will assume that $\alpha > \beta$ and $x_0 > y_0 > 0$. Thus, in the absence of noise one would have $X_t > Y_t$ for all $t > 0$. In the presence of noise however the probability that $X_T = Y_T$ for some $T > 0$ is non-zero. The second equation can be written equivalently as

$$dY_t = \beta Y_t dt + \rho b Y_t dW_t + \sqrt{1-\rho^2} b Y_t dV_t.$$

Using once more Theorem 5.6.7, p. 214 of Dembo and Zeitouni (2010) we obtain again a two dimensional version of (2.8) for the action functional to be minimized:

$$I = \frac{1}{2}\int_0^T \left(\frac{x' - \alpha x}{x\sigma}\right)^2 + \frac{1}{1-\rho^2}\left(\frac{y' - \beta y}{yb} - \rho\frac{x' - \alpha x}{x\sigma}\right)^2 dt \tag{4.26}$$

This of course can be justified by appealing to the multidimensional version of (2.8) as we have already seen. Set

$$F = \frac{1}{2\sigma^2}\left(\frac{x'}{x} - \alpha\right)^2 + \frac{1}{2(1-\rho^2)}\left(\frac{1}{b}\left(\frac{y'}{y} - \beta\right) - \frac{\rho}{\sigma}\left(\frac{x'}{x} - \alpha\right)\right)^2 \tag{4.27}$$

The conditions for minimum are

$$F_x - \frac{d}{dt}F_{x'} = 0 \tag{4.28}$$
$$F_y - \frac{d}{dt}F_{y'} = 0 \tag{4.29}$$
$$x(T) = y(T) \tag{4.30}$$
$$F_{x'} + F_{y'} = 0 \quad \text{at } T, \tag{4.31}$$
$$F - x'F_{x'} - y'F_{y'} = 0 \quad \text{at } T. \tag{4.32}$$

Then, after some routine algebraic operations, (4.28) becomes

$$\frac{1}{x}\left[b^2\left(\frac{x''}{x} - \left(\frac{x'}{x}\right)^2\right) - \rho b\sigma\left(\frac{y''}{y} - \left(\frac{y'}{y}\right)^2\right)\right] = 0$$

which gives $b^2(\log x)'' - \rho b\sigma(\log y)'' = 0$. Similarly (4.29) gives $\sigma^2(\log y)'' - \rho b\sigma(\log x)'' = 0$. These equations together imply that $(\log x)'' = (\log y)'' = 0$ whence we obtain $\frac{x'}{x} = c_1$ and $\frac{y'}{y} = c_2$ for arbitrary c_1, c_2, and hence

$$x(t) = x_0 e^{c_1 t}, \qquad y(t) = y_0 e^{c_2 t}. \tag{4.33}$$

Condition (4.30) gives

$$x_0 e^{c_1 T} = y_0 e^{c_2 T}. \tag{4.34}$$

Taking into account that $\frac{x'}{x} = c_1$ and similarly $\frac{y'}{y} = c_2$, condition (4.31) gives

$$\frac{1}{x_0 e^{c_1 T}}\left[b^2(c_1 - \alpha) - \rho b\sigma(c_2 - \beta)\right] + \frac{1}{y_0 e^{c_2 T}}\left[\sigma^2(c_2 - \beta) - \rho b\sigma(c_1 - \alpha)\right] = 0.$$

Setting

$$u_1 = c_1 - \alpha, \quad u_2 = c_2 - \beta, \tag{4.35}$$

and taking into account (4.34) the above equation implies $b^2 u_1 - \rho b\sigma u_2 + \sigma^2 u_2 - \rho b\sigma u_1 = 0$. This gives

$$u_2 = \lambda u_1 \quad \text{with} \quad \lambda = \frac{b}{\sigma}\frac{\rho\sigma - b}{\sigma - \rho b} \tag{4.36}$$

Finally, from (4.32),

$$b^2 u_1^2 + \sigma^2 u_2^2 - 2\rho b\sigma u_1 u_2 - 2c_1\left[b^2 u_1 - \rho b\sigma u_2\right] - 2c_2\left[\sigma^2 u_2 - \rho b\sigma u_1\right] = 0$$

or, taking into account (4.36) and (4.35)

$$-u_1^2\left[b^2+\sigma^2\lambda^2-2\rho b\sigma\lambda\right]+2u_1\left[-\alpha b^2+\beta\rho b\sigma-\lambda\beta\sigma^2+\lambda\alpha b\sigma\rho\right]=0.$$

Besides the solution $u_1 = 0$ which means ($c_1 = \alpha$), we obtain

$$u_1=-\frac{2}{b^2+\sigma^2\lambda^2-2\rho b\sigma\lambda}\left(\alpha b^2+a\lambda\sigma^2-\rho b\sigma(a+\lambda\alpha)\right)$$

and a corresponding expression for u_2. After routine algebraic manipulations we obtain [1]

$$u_1=2(\beta-\alpha)\frac{\sigma(\sigma-\rho b)}{\sigma^2+b^2-2\rho b\sigma},\qquad u_2=2(\beta-\alpha)\frac{b(\rho\sigma-b)}{\sigma^2+b^2-2\rho b\sigma}. \tag{4.37}$$

From (4.27) and (4.37), together with the definition of u_1, u_2,

$$F \;=\; \frac{1}{2b^2\sigma^2(1-\rho^2)}\left[b^2u_1^2+\sigma^2u_2^2-2\rho b\sigma u_1u_2\right] \;=\; \frac{2(\beta-\alpha)^2}{\sigma^2+b^2-2\rho b\sigma}. \tag{4.38}$$

Thus, since

$$T=\frac{1}{\alpha-\beta}\log\left(\frac{x_0}{y_0}\right),$$

the optimal rate is

$$I=\frac{2(\alpha-\beta)\log\left(\frac{x_0}{y_0}\right)}{\sigma^2+b^2-2\rho b\sigma}. \tag{4.39}$$

4.4.1 Exact analysis for two correlated Geometric Brownian Motions

An exact analysis is again possible here. Suppose

$$X_t^\epsilon \;=\; x_0e^{\left(\alpha-\frac{1}{2}\sigma_\epsilon^2\right)t+\sigma_\epsilon W_t},\qquad Y_t^\epsilon \;=\; y_0e^{\left(\beta-\frac{1}{2}b_\epsilon^2\right)t+b_\epsilon B_t},$$

are two families of Geometric Brownian Motions, indexed by a positive parameter ϵ. We will assume that $\sigma_\epsilon=\sigma\sqrt{\epsilon}$ and, similarly, $b_e=b\sqrt{\epsilon}$. Assuming that $\alpha>\beta$ and $x_0>y_0$ and that $\{W_t\}$, $\{B_t\}$. are standard Brownian motions with correlation ρ as in section 4.4, we are interested in obtaining an expression for the probability

$$\mathbb{P}(T_\epsilon<\infty)\quad\text{where } T_\epsilon=\inf\{t>0: Y_t^\epsilon\ge X_t^\epsilon\}. \tag{4.40}$$

The condition $Y_t^\epsilon \ge X_t^\epsilon$ is equivalent to

$$\left(\alpha-\beta+\frac{1}{2}(b_\epsilon^2-\sigma_\epsilon^2)\right)t+\sigma_\epsilon W_t-b_\epsilon B_t\le\log\frac{y_0}{x_0}.$$

Set $\log\frac{y_0}{x_0}=-u$, $\gamma_\epsilon:=\alpha-\beta+\frac{1}{2}(b_\epsilon^2-\sigma_\epsilon^2)$ and $\theta_\epsilon:=\sqrt{\sigma_\epsilon^2+b_\epsilon^2-2\rho b_\epsilon\sigma_\epsilon}$. If $\{\tilde{W}_t\}$ is standard Brownian motion, then (4.40) becomes

$$\mathbb{P}(T_\epsilon<\infty) \;=\; \mathbb{P}\left(\inf_{t\ge0}(\gamma_\epsilon t+\theta_\epsilon\tilde{W}_t)<-u\right). \tag{4.41}$$

[1] Note that, if $\sigma<b$ then $\rho\in(-1,\frac{\sigma}{b})\Rightarrow u_1>0,\ u_2<0$ whereas $\rho\in(\frac{\sigma}{b},1)\Rightarrow u_1<0,\ u_2<0$.
If $\sigma>b$ then $\rho\in(-1,\frac{b}{\sigma})\Rightarrow u_1>0,\ u_2<0$ whereas $\rho\in(\frac{\sigma}{b},1)\Rightarrow u_1>0,\ u_2>0$.
If $\sigma=b$ then $u_1>0,\ u_2<0$ for all values of ρ.

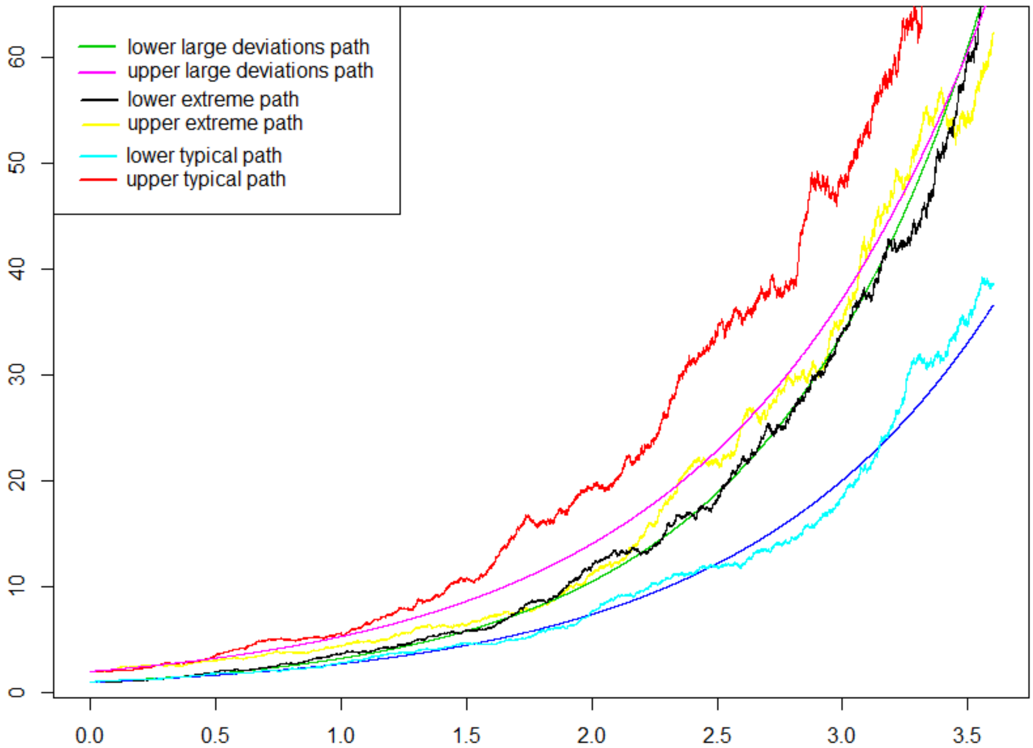

Figure 9: Two independent Geometric Brownian Motions. The red and blue paths are typical upper and lower paths. The yellow and black paths are extreme upper and lower paths that meet, following the corresponding magenta and green large deviation optimal paths. They were picked by simulating a large number of paths until a successful instance, where an upper and a lower path met, was obtained.

Since $\alpha > \beta$, when ϵ is sufficiently small, $\gamma_\epsilon > 0$ regardless of the values of σ and b. Therefore (see Cox and Miller, 1977) (4.41) becomes

$$\mathbb{P}(T_\epsilon < \infty) \;=\; e^{-u\frac{2\gamma_\epsilon}{\theta_\epsilon^2}} \;=\; e^{\log\frac{y_0}{x_0}\,\frac{2(\alpha-\beta)+(b_\epsilon^2-\sigma_\epsilon^2)}{\sigma_\epsilon^2+b_\epsilon^2-2\rho b_\epsilon\sigma_\epsilon}}.$$

It therefore follows that

$$\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}(T_\epsilon<\infty) \;=\; \log\frac{y_0}{x_0}\lim_{\epsilon\to 0}\frac{2(\alpha-\beta)+(b_\epsilon^2-\sigma_\epsilon^2)}{\epsilon^{-1}\left(\sigma_\epsilon^2+b_\epsilon^2-2\rho b_\epsilon\sigma_\epsilon\right)} \;=\; \log\frac{y_0}{x_0}\,\frac{2(\alpha-\beta)}{\sigma^2+b^2-2\rho b\sigma}.$$

This result of course agrees with (4.39).

5 Appendix

5.1 The paths $x(\cdot)$ and $V(\cdot)$.

Here we refer to part 3 of the proof of Theorem 2. The comparison between the slope of the optimal path $x(\cdot)$ and $V(\cdot)$ at the intersection point t is given by the following

Proposition 7.

$$\operatorname{sgn}(x'(t)-V'(t)) = \begin{cases} -1 & \textit{if} \quad t<t_2 \\ 0 & \textit{if} \quad t=t_2 \\ +1 & \textit{if} \quad t>t_2 \end{cases} \tag{5.1}$$

where t_2 is the unique solution of the equation $\phi_2(t) \;=\; 2\frac{x_0}{v_0}$ with

$$\phi_2(s) \;:=\; \left(1-\frac{\beta}{\mu}\right)e^{(\beta+\mu)s}+\left(1+\frac{\beta}{\mu}\right)e^{(\beta-\mu)s}. \tag{5.2}$$

Also,

$$t_1 < t_V^o < t_2 \tag{5.3}$$

where t_1 is defined in (2.31) and t_V^o in (2.28).

Proof. Taking into account (2.23), $x'(t)-V'(t) \;=\; \mu\frac{v_0e^{\beta t}\left(e^{\mu t}+e^{-\mu t}\right)-2x_0}{e^{\mu t}-e^{-\mu t}} - \beta v_0 e^{\beta t}$ and hence

$$x'(t)-V'(t)<0 \;\Leftrightarrow\; \left(1-\frac{\beta}{\mu}\right)e^{(\beta+\mu)t}+\left(1+\frac{\beta}{\mu}\right)e^{(\beta-\mu)t}<2\frac{x_0}{v_0}. \tag{5.4}$$

Defining the function ϕ_2 via (5.2) we note that $\phi_2'(s) \;=\; (\mu^2-\beta^2)\frac{2}{\mu}e^{\beta s}\sinh(\mu s) \;>\; 0$ for all $s \geq 0$ and $\phi_2(0) = 2$. Hence, the equation $\phi_2(s) \;=\; 2\frac{x_0}{v_0}$ has a unique, positive solution, say t_2. Since the function $\phi_2(s)$ is continuous and strictly increasing this establishes (5.1).

Next we will show that

$$t_V^o \;<\; t_2. \tag{5.5}$$

Indeed, using the definition of ϕ_1 and t^o,

$$\begin{aligned}\phi_2(t_V^o) \;&=\; \left(1-\frac{\beta}{\mu}\right)e^{(\beta+\mu)t_V^o}+\left(1+\frac{\beta}{\mu}\right)e^{(\beta-\mu)t_V^o}\\ &=\; \phi_1(t_V^o)\;+\;e^{(\beta-\mu)t_V^o}<\frac{x_0}{v_0}+1\;<\;2\frac{x_0}{v_0}\;=\;\phi_2(t_2).\end{aligned}$$

where we have used the fact that $\beta - \mu < 0$ and that $x_0 > v_0$. Then (5.5) follows from the fact that ϕ_2 is increasing.

Finally note that $\phi_2(t_1) = \frac{x_0}{v_0}\left(1 - \frac{\beta}{\mu} + \left(1 + \frac{\beta}{\mu}\right)e^{-2\mu t}\right) < 2\frac{x_0}{v_0}$ which implies, since ϕ_2 is strictly increasing, (5.3). □

Define the function $h(s) := x(s) - V(s)$. We have $h(0) = x(0) - V(0) = x_0 - v_0 > 0$. Also $h(t) = x(t) - V(t) = 0$. We will show that, when $t > t_1$, there are precisely two zeros of the function h on $[0, \infty)$, t and $\tau(t)$. When $t \in (t_1, t_2)$ $\tau(t) > t$ whereas when $t > t_2$, $\tau(t) < t$. In the special case $t = t_2$, $\tau(t_2) = t_2$ is the single zero of h at which h' also vanishes.

We have

$$h'(s) \;=\; \mu c_1 e^{\mu s} - \mu c_2 e^{-\mu s} + \beta v_0 e^{\beta s}. \tag{5.6}$$

The following proposition gives some qualitative properties of this function.

Proposition 8. *Suppose $t > t_1$. Then there exists $s_1(t) > 0$ such that $h'(s) < 0$ when $s < s_1(t)$, $h'(s_1) = 0$, and $h'(s) > 0$ when $s > s_1(t)$. Also $\lim_{s\to\infty} h'(s) = +\infty$ and the following holds: There are precisely two values for which the function h vanishes. One is t while the second we denote by $\tau(t)$. If $t < s_1(t)$ then $\tau(t) > t$ while if $t > s_1(t)$ then $\tau(t) < t$. When $t = s_1(t)$ then $t = \tau(t)$ and $h(t) = h'(t) = 0$.*

Proof. First we will show that $h'(0) < 0$. Indeed,

$$\begin{aligned}
h'(0) &= \mu(c_1 - c_2) - \beta v_0 \;=\; \mu\,\frac{2v_0 e^{\beta t} - x_0(e^{\mu t} + e^{-\mu t})}{e^{\mu t} - e^{-\mu t}} - \beta v_0 \\
&= \frac{2\mu}{e^{2\mu t} - 1}\, v_0 e^{(\beta+\mu)t} - v_0\beta - \mu x_0\,\frac{e^{2\mu t} + 1}{e^{2\mu t} - 1} \\
&= \frac{2\mu}{e^{2\mu t} - 1}\, v_0\left(e^{(\beta+\mu)t} - 1\right) - v_0\beta + v_0\frac{2\mu}{e^{2\mu t} - 1} - \mu x_0 - x_0\frac{2\mu}{e^{2\mu t} - 1} \\
&= \frac{\int_0^t e^{(\beta+\mu)\xi} d\xi}{\int_0^t e^{2\mu\xi} d\xi}\,(\beta + \mu)v_0 - v_0\beta + v_0\frac{2\mu}{e^{2\mu t} - 1} - \mu x_0 - x_0\frac{2\mu}{e^{2\mu t} - 1}
\end{aligned}$$

The ratio of integrals above is seen to be less than one (since $\beta < \mu$) and hence

$$\begin{aligned}
h'(0) &\leq (\beta + \mu)v_0 - v_0\beta + v_0\frac{2\mu}{e^{2\mu t} - 1} - \mu x_0 - x_0\frac{2\mu}{e^{2\mu t} - 1} \\
&= (v_0 - x_0)\left(\mu + \frac{2\mu}{e^{2\mu t} - 1}\right) < 0.
\end{aligned}$$

From (5.6) we see that $h'(s) = e^{\mu s} h_1(s)$ with $h_1(s) := \mu c_1 - \mu c_2 e^{-2\mu s} + \beta v_0 e^{-(\mu-\beta)s}$. Clearly $h'(s)$ and $h_1(s)$ have the same sign. Also, $h_1(0) = h'(0) < 0$ and since $c_1 > 0$, $c_2 > 0$, (the first because $t > t_2$) and $\mu > \beta$, it follows that $h_1(s)$ is strictly increasing in s and satisfies $h_1(s) \uparrow \mu c_1 > 0$ as $s \uparrow \infty$. Therefore there exists a unique $s_1 > 0$ such that $h_1(s_1) = 0$.

We have course $h(t) = 0$. Since the value of s_2 determined in Proposition 7 depends on t we will use the notation $s_2(t)$. Then,

- If $t \in (t_1, t_2)$ then, from Proposition 7, $h'(t) < 0$ which implies, in view of the above analysis that $s_2(t) > t$. This in turn means that $\tau(t) > s_2(t)$ and hence that $t < \tau(t)$.
- If $t = t_2$ then $h'(t_2) = 0$ which implies that $s_2(t_2) = t_2$.

- If $t > t_2$ then $h'(t) > 0$ which implies that $s_2(t) < t$ and hence that $\tau(t) < s_2(t)$. Thus in this case $\tau(t) < t$.

This concludes the proof of the proposition. $\square$

5.2 A time-change approach to the Ornstein-Uhlenbeck ruin problem

Consider the two sided problem

$$dX_t \; = \; \mu X_t dt + \sigma dW_t, \quad X_0 = x_0$$

with an upper boundary given by the curve $U(t) := u_0 e^{\alpha t}$ and a lower boundary given by $V(t) := v_0 e^{\beta t}$. We assume that $0 < v_0 < x_0 < u_0$ and $0 < \beta < \mu < \alpha$. We are interested in the hitting time $T = \inf\{t \geq 0 : X_T \geq U(T) \text{ or } X_T \leq V(T)\}$. (Of course, if the set is empty, the hitting time is equal to $+\infty$ corresponding to the case where the process never exits from one of the two boundary curves.) Ignoring the parametrization by ϵ since here we want to discuss exact results, the SDE (2.6) has the solution

$$X_t \; = \; x_0 e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} dW_s \tag{5.7}$$

The condition

$$V(t) < X_t < U(t)$$

is equivalent to $e^{-\mu t}V(t) < e^{-\mu t}X_t < e^{-\mu t}U(t)$ or

$$v_0 e^{-(\mu-\beta)t} < x_0 + \sigma \int_0^t e^{-s\mu} dW_s < u_0 e^{(\alpha-\mu)t}. \tag{5.8}$$

The stochastic integral $\xi(t) := \sigma \int_0^t e^{-s\mu} dW_s$ is a Gaussian process with independent intervals and variance function

$$\mathsf{Var}(\xi(t)) = \sigma^2 \int_0^t e^{-2\mu s} ds = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right).$$

Note that the limit $\lim_{t\to\infty} \mathsf{Var}(\xi(t)) = \frac{\sigma^2}{2\mu}$ is finite. Consider the time change function $\tau(t)$ defined by

$$\tau(t) = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right), \qquad t \in [0, \infty) \tag{5.9}$$

The inverse function (which necessarily exists since $\mathsf{Var}(\xi(t))$ is an increasing function) is

$$t(\tau) = \log\left(1 - \frac{2\mu\tau}{\sigma^2}\right), \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right) \tag{5.10}$$

Applying this change of time to the double inequality (5.8) we obtain

$$v_0 e^{-(\mu-\beta)\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)} < x_0 + \sigma \int_0^{\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)} e^{-s\mu} dW_s < u_0 e^{(\alpha-\mu)\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)}, \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right).$$

However, $\tilde{W}_\tau := \sigma \int_0^{\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)} e^{-s\mu} dW_s$ is standard Brownian motion. (It can easily be seen that it is a continuous martingale with quadratic variation function $\langle \tilde{W} \rangle_\tau = \tau$.) Thus we have the equivalent problem

$$v_0 \left(1 - \frac{2\mu\tau}{\sigma^2}\right)^{\frac{\mu-\beta}{2\mu}} \; < \; x_0 \; + \; \tilde{W}_\tau \; < \; u_0 \left(1 - \frac{2\mu\tau}{\sigma^2}\right)^{-\frac{\alpha-\mu}{2\mu}}, \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right). \tag{5.11}$$

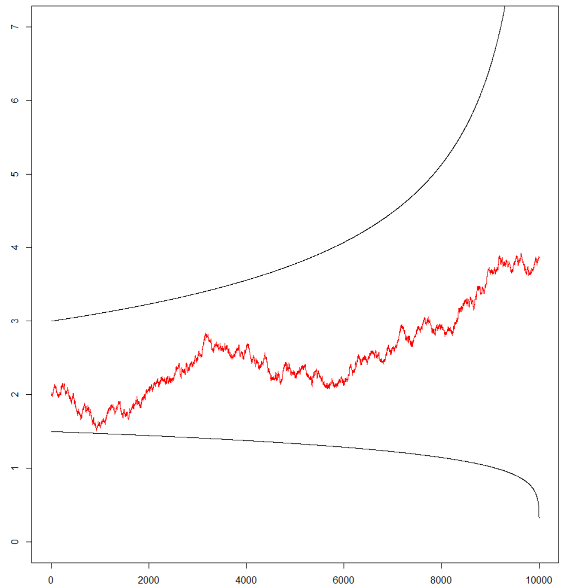

Figure 10: Time-change in an Ornstein-Uhlenbeck ruin problem.

In general, the passage time – hitting probability problem associated with (5.11) must be solved numerically. Of course the time change transformation may have computational advantages. There is a great deal of work, both theoretical and applied, regarding passage times and hitting probabilities of Brownian motion with curving boundaries. In the special case where $\alpha = \beta = \mu$ an exact solution exists. In general we have not been able to obtain closed form expressions even with a single boundary even in the few cases where exact solutions are known, such as for a parabolic boundary: When $\beta = 0$ then the time-changed lower bound is $v_0\sqrt{1 - \frac{2\mu\tau}{\sigma^2}}$. While this is a parabolic boundary, the results that have obtained for this case, Durbin and Williams (1992), Durbin (1985), apply when it acts as an *upper* and not a lower boundary. The exact solution in the case we are examining is not known, to the best of our knowledge. (See also Novikov, 1981, Herrmann and Tanré, 2016, and the references therein.)

A two–boundary case: $\alpha = \beta = \mu$. In that case (5.11) becomes

$$v_0 - x_0 \; < \; \tilde{W}_\tau \; < \; u_0 - x_0, \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right).$$

The exact probability of never exiting either boundary, can be obtained from the well known expression for the density of standard Brownian motion (starting at zero) with absorbing boundaries at a, b, $(a, b > 0)$. If $p(x,t)dx := \mathbb{P}\left(W_t \in (x, x+dx);\ -b < W_s < a,\ 0 \le s \le t\right)$, then, (see Cox and Miller, 1977, p. 222)

$$p(x,t) \;=\; \sum_{n=1}^{\infty} \frac{2}{a+b} \sin\left(\frac{n\pi b}{a+b}\right) e^{-\lambda_n t} \sin\left(n\pi \frac{x+b}{a+b}\right),$$
$$\text{where } \lambda_n = \frac{1}{2}\frac{n^2\pi^2}{(a+b)^2}, \quad n = 1, 2, \ldots.$$

Then

$$\mathbb{P}\left(-b < W_s < a, \;\; \text{for } 0 \le s \le t\right) = \int_{-b}^{a} p(x,t)dx,$$

and in our case $-b = v_0 - x_0$, $a = u_0 - x_0$, $t = \frac{\sigma^2}{2\mu}$. Hence,

$$\mathbb{P}\left(-b < W_s < a,\ 0 \le s \le \frac{\sigma^2}{2\mu}\right) \tag{5.12}$$
$$= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \exp\left(-\frac{(2k+1)^2\pi^2\sigma^2}{2(u_0-v_0)^2\mu}\right) \sin\frac{(2k+1)\pi(x_0-v_0)}{u_0-v_0}.$$

5.3 Exact asymptotics via Gaussian process theory

The rich and extensive theory of extreme value analysis for Gaussian processes may be used in some of the problems considered in this paper in order to obtain exact (as opposed to logarithmic) asymptotics. For a review of results in this area and applications to ruin problems we refer the reader to the monograph by Piterbarg (1996) and to Hüsler and Piterbarg (1999), Hashorva and Hüsler (2000), and Dębicki, Hashorva and Li (2015). The ruin problems discussed in Section 2 involve Ornstein-Uhlenbeck processes which are Gaussian but are not stationary or self-similar, nor do they have stationary increments. However, with appropriate scaling one may obtain a Gaussian process with a single point of maximum variance and the technique described in Piterbarg (1996) p. 19 and Hüsler and Hashorva (2000) may be applied.

We illustrate this briefly in one of the problems of Section 2.3. Write the solution of the SDE (2.6) as

$$X_t^\epsilon = x_0 e^{\mu t} + \epsilon^{1/2}\sigma \int_0^t e^{\mu(t-u)} dW_u.$$

Then

$$\mathbb{P}\Big(\sup_{t\in[0,T]} X_t^\epsilon > u_0 e^{\alpha t}\Big) \;=\; \mathbb{P}\Big(\sup_{t\in[0,T]} Z_t > \epsilon^{-1/2}\Big)$$

where $\{Z_t; t \ge 0\}$ is the centered Gaussian process defined by

$$Z_t \;:=\; \frac{\sigma}{u_0 e^{\alpha t} - x_0 e^{\mu t}} \int_0^t e^{\mu(t-u)} dW_u.$$

(Thus we pass from low noise asymptotics to high threshold asymptotics.) The covariance function of the process $\{Z_t; t \ge 0\}$ can be seen by a simple comptutation to be

$$R(s,t) \;=\; \frac{\sigma^2}{2\mu} \frac{e^{\mu(t+s)} - e^{\mu(t-s)}}{\Big(u_0 e^{\alpha s} - x_0 e^{\mu s}\Big)\Big(u_0 e^{\alpha t} - x_0 e^{\mu t}\Big)}, \quad 0 \le s \le t \le T.$$

and in particular the variance function $S^2(t) := \mathsf{Var}(Z_t) = R(t,t)$ is given by

$$S^2(t) \;=\; \frac{\sigma^2}{2\mu} \frac{e^{2\mu t} - 1}{\Big(u_0 e^{\alpha t} - x_0 e^{\mu t}\Big)^2}. \tag{5.13}$$

Denote by $\mathcal{H}^1_{x_0,U(t)}$ the set of absolutely continuous functions $x : [0,t] \to \mathbb{R}$ with square integrable functions, such that $x(0) = x_0$ and $x(t) = U(t) := u_0 e^{\alpha t}$ (c.f. the definition of $\mathcal{H}^1_{x_0,V(t)}$ in the proof of Part 1 of Theorem 2) and set

$$J_*(t) \;:=\; \inf\{J(x;t) : x \in \mathcal{H}^1_{x_0,U(t)}\}$$

for the infimum of the action functional for paths in $\mathcal{H}^1_{x_0,U(t)}$. An analysis paralleling step for step the derivation of (2.24) shows that the minimum value of the action functional $J(x;t)$ is

$$J_*(t) \;=\; \frac{\mu}{\sigma^2} \frac{\Big(u_0 e^{\alpha t} - x_0 e^{\mu t}\Big)^2}{e^{2\mu t} - 1}. \tag{5.14}$$

Furthermore, if we allow t to vary in $[0,T]$, $J_*(t)$ has a strict global minimum at t_0^U which is the unique positive solution of equation (2.14) (c.f. (2.12) and (2.13)). Assuming that $T > t_0^U$, the rate function is $I_U(T) = J_*(t_0^U)$. However, by comparing (5.13) and (5.14), we see that $S^2(t) = (2J_*(t))^{-1}$ and hence t_0^U is a unique point of maximum variance for the Gaussian process $\{Z_t; t \geq 0\}$ with corresponding value

$$S^2(t_0^U) = \frac{1}{2J_*(t_0^U)} = \frac{1}{2I_U(T)}. \tag{5.15}$$

Define now the process $\{\widetilde{Z}_t; t \geq 0\}$ via

$$\widetilde{Z}_t = \sqrt{2I_U(T)}\, Z_t.$$

This is a centered Gaussian process with a.s. continuous paths and variance function

$$\widetilde{S}^2(t) := \mathsf{Var}(\widetilde{Z}_t) = S^2(t) 2I_U(T) \leq 1$$

with equality holding only at $t = t_0^U$. We claim that his process satifies conditions E.1, E.2, and E.3 in Piterbarg (1996), p.19. Since $(S^2)'(t_0^U) = 0$ we can see by a Taylor expansion around t_0^U that

$$\widetilde{S}(t) \;=\; 1 - \kappa_1 \big(t - t_0^U\big)^2 + o\big((t - t_0^U)^2\big) \tag{5.16}$$

where

$$\kappa_1 = 2I_U(t_0^U) \cdot (S^2)''(t_0^U). \tag{5.17}$$

(This is condition E.1.) The covariance function of $\{\widetilde{Z}_t\}$ is $\widetilde{R}(s,t) := \frac{I_U(T)}{2} R(s,t)$ and (by a simple but tedious calculation) it can be shown that it is *locally stationary* at t_0^U, i.e. that it satisfies the condition

$$\widetilde{R}(s,t) \;=\; 1 - \kappa_2 |t-s| (1 + o(1)), \quad \text{as } |t - t_0^U| \to 0, |s - t_0^U| \to 0. \tag{5.18}$$

where

$$\kappa_2 = \frac{\mu}{e^{2\mu t_0^U} - 1}. \tag{5.19}$$

(This is condition E.2. See also Hashorva and Hüsler, 2000.) Finally one can also check by a straighforward computation that $\mathbb{E}\big(\widetilde{Z}(t) - \widetilde{Z}(s)\big)^2 \leq G|t-s|$ for some $G > 0$, i.e. that the regularity condition E.3 in Piterbarg (1996) is satisfied. Therefore, applying Theorem D3 of Piterbarg (1996) (p.19) (choosing appropriately $\alpha = 1$, $\beta = 2$ in that theorem).

$$\begin{aligned}
\mathbb{P}(\sup_{t\in[0,T]} X_t^\epsilon > u_0 e^{\alpha t}) &= \mathbb{P}\Big(\sup_{t\in[0,T]} Z_t > \epsilon^{-1/2}\Big) = \mathbb{P}\Big(\max_{t\in[0,T]} \widetilde{Z}_t > \epsilon^{-1/2} (2I_U(T))^{1/2}\Big) \\
&= \frac{\kappa_2 \mathcal{H}_1 \Gamma(1/2)}{\kappa_1^{1/2}} \frac{\epsilon^{1/2}}{2I_U(T)} \Psi\big((2I_U(T)/\epsilon)^{1/2}\big) \big(1 + o(1)\big).
\end{aligned} \tag{5.20}$$

In the above, $\Psi(u) := \mathbb{P}(\mathcal{N} > u)$ where $\mathcal{N}$ is a standard normal r.v. and $\mathcal{H}_1$ is the value of Pickands' constant $\mathcal{H}_\alpha$ for $\alpha = 1$ which is equal to 1.

Taking into account the fact that $\lim_{u\to\infty} \sqrt{2\pi}\, u\, e^{u^2/2} \Psi(u) = 1$, the *exact asymptotics* for the probability of hitting the upper curve *when the maximum variance point* t_0^U *is an interior point of* $[0,T]$ are expressed as

$$\mathbb{P}(\sup_{t\in[0,T]} X_t^\epsilon > u_0 e^{\alpha t}) \;=\; \frac{\kappa_2}{\sqrt{2\kappa_1}} e^{-\frac{1}{\epsilon} I_U(T)} (1 + o(1)). \tag{5.21}$$

This agrees with the logarithmic asymptotics in (2.12) and (2.13) for the case where $t_0^U < T$.

The case $t_0^U \geq T$ corresponds to the case where the variance of the Gaussian process $\{Z_t; t \geq 0\}$ has a maximum at the boundary point of the interval, T and the exact asymptotics are obtained by multiplying (5.21) by a factor of $1/2$. (See Konstant and Piterbarg, 1993, Theorem 2.2.)

References

[1] Asmussen, S. and M. Albrecher. *Ruin Probabilities*, Second Edition. World Scientific. 2010.

[2] Azencott, R. "Petites perturbations aléatoires des systèmes dynamiques: développements asymptotiques." *Bulletin des sciences mathématiques*, Vol 109, Num 3, pp 253-308, 1985.

[3] Azencott R. *Grandes deviations et applications*, in Ecole d'été de Probabilité de St. Four VIII-1978, Lect. Notes Math. 774, Springer, 1-176, 1980.

[4] Baldi, P. "Exact Asymptotics for the Probability of Exit from a Domain and Applications to Simulation." *The Annals of Probability*, 23, 4. 1644-1670, 1995.

[5] Baldi, P. and L. Caramellino. "General Freidlin–Wentzell Large Deviations and positive diffusions." *Statistics and Probability Letters,* 81, 1218-1229, 2011.

[6] Bisewski, K., K. Dębicki, and N. Kriukov. "Simultaneous ruin probability for multivariate Gaussian risk model." *Stochastic Processes and their Applications*, 160, 386–408, 2023.

[7] Bobrovsky, B-Z. and O. Zeitouni. "Some results on the problem of exit from a domain". *Stochastic Processes and their Applications*, 41, 241-256, 1992.

[8] Brechtken-Manderscheid, U. *Introduction to the Calculus of Variations*. Chapman & Hall. 1994.

[9] Cottrell, M., J-C. Fort, and G. Malgouyres. "Large Deviations and Rare Events in the Study of Stochastic Algorithms." *IEEE Transactions on Automatic Control,* AC-28, 9, 907-920, 1983.

[10] Cox, D.R. and H.D. Miller. *The Theory of Stochastic Processes.* Chapman and Hall/CRC, 1977.

[11] Day, M.V. "On the asymptotic relation between equilibrium density and exit measure in the exit problem," *Stochastics* 12:3-4, 303-330, 1984.

[12] Dębicki K. "Ruin probability for Gaussian integrated processes." *Stochastic Processes and their Applications*, 98, 151–174, 2002.

[13] Dębicki, K., E. Hashorva, and L. Ji. "Gaussian risk models with financial constraints." *Scandinavian Actuarial Journal*, Vol. 2015, No. 6, 469–481, 2015.

[14] Dębicki, K., E. Hashorva, and L. Wang. "Extremes of vector-valued Gaussian processes." *Stochastic Processes and their Applications*, 130, 5802–5837, 2020.

[15] Dembo A. and O. Zeitouni. *Large Deviations Techniques and Applications.* Second Edition. Springer 2010.

[16] Dieker A.B. "Extremes of Gaussian processes over an infinite horizon." *Stochastic Processes and their Applications,* **115**, 207–248, 2005.

[17] Dupuis P. and H.J. Kushner. "Large deviations estimates for systems with small noise effects, and applications to stochastic systems theory". *SIAM J. Control and Optimization* 24, 5, 979-1008, 1986.

[18] Durbin J. "The First-Passage Density of a Continuous Gaussian Process to a General Boundary." *Journal of Applied Probability,* 22, 1, 99-122, 1985.

[19] Durbin J. and D. Williams. "The First-Passage Density of the Brownian Motion Process to a Curved Boundary." *Journal of Applied Probability,* 29, 2, 291-304, 1992.

[20] Fleming, W.H. "Exit probabilities and optimal stochastic control." *Appl.Math. Optim.* 4, 329-346. 1978.

[21] Freidlin, M.I. and A.D. Wentzell, *Random Perturbations of Dynamical Systems*, 3rd edition, Springer Science & Business Media, 2012.

[22] Freidlin, M., L. Koralov, and A. Wentzell, "On the Behavior of Diffusion Processes with Traps," *The Annals of Probability*, 45(5), 3202–3222, 2017.

[23] Gerber, H.U. and E.S.W. Shiu, "Geometric Brownian Motion Models for Assets and Liabilities: From Pension Funding to Optimal Dividends," *North American Actuarial Journal*, 7(3), 37-51, 2003.

[24] Harrison, M.J. "Ruin Problems with Compounding Assets," *Stochastic Processes and their Applications,* 5, 67-79, 1977.

[25] Hashorva, E. and J. Hüsler. "Extremes of Gaussian Processes with Maximal Variance near the Boundary Points." *Methodology and Computing in Applied Probability*, **2**:3, 255–269, 2000.

[26] He, X. and Y. Hu. "Ruin Probability for the Integrated Gaussian Process With Force of Interest," *Appl. Prob.* 44, 685–694, 2007.

[27] Herrmann, S. and E. Tanré. "The first-passage time of the brownian motion to a curved boundary: an algorithmic approach," *SIAM J. Sci. Comput.* 38, 1, pp. A196–A215, 2016.

[28] Hill K., J. Zanetell, and J. A. Gemmer. "Most probable transition paths in piecewise-smooth stochastic differential equations," *Physica D: Nonlinear Phenomena,* 439, article 133424, 2022.

[29] Hüsler, J. "Extreme values and high boundary crossings of locally stationary Gaussian processes." *The Annals of Probability*, **18**, 3, 1141–1158, 1990.

[30] Hüsler, J. and V.I. Piterbarg. "Extremes of a certain class of Gaussian processes." *Stochastic Processes and their Applications* 83, 257-271, 1999.

[31] Hüsler, J. and V.I. Piterbarg. "On the ruin probability for physical fractional Brownian motion." *Stochastic Processes and their Applications*, 113, 315–332, 2004.

[32] Hüsler, J. and V.I. Piterbarg. "A limit theorem for the time of ruin in a Gaussian ruin problem." *Stochastic Processes and their Applications* 118, 2014–2021, 2008.

[33] Kifer, Ju. I. "On small random perturbations of some smooth dynamical systems," *Math. USSR Izv.* 8, 1083-1107, 1974.

[34] Kobelkov, S.G. "The ruin problem for the stationary gaussian process." *Theory Probab. Appl. SIAM* Vol. 49, No. 1, pp. 155–163, 2005 .

[35] Konstant, D.G. and V.I. Piterbarg. "Extreme Values of the Cyclostationary Gaussian Random Process." *Journal of Applied Probability,* **30**, 1, pp. 82–97, 1993.

[36] Kushner, H. J. "Robustness and Approximation of Escape Times and Large Deviations Estimates for Systems with Small Noise Effects," *SIAM J. Appl. Math.* 44, 1, 160-182, 1984.

[37] Ludwig, D. "Persistence of Dynamical Systems under Random Perturbations," *SIAM Review,* 17, 4, 605-640, 1975.

[38] Maier, R.S. and D.L. Stein. "Limiting Exit Location Distributions in the Stochastic Exit Problem," *SIAM J. Appl. Math.* 57, 3, 752-790, 1997.

[39] Matkowsky, R. J. and Z. Schuss. “The Exit Problem for Randomly Perturbed Dynamical Systems,” *SIAM J. Appl. Math.* 33, 2, 365-382, 1977.

[40] Novikov, A.A. “On estimates and the asymptotic behavior of nonexit probabilities of a wiener process to a moving boundary,” *USSR Sb.* 38, 495-505, 1981.

[41] Piterbarg, V.I. *Asymptotic Methods in the Theory of Gaussian Processes and Fields.* American Mathematical Society, Providence RI, 1996.

[42] Pham H. “Some Applications and Methods of Large Deviations in Finance and Insurance,” in *Paris-Princeton Lectures on Mathematical Finance 2004*, Carmona, R. et al. Eds, pp. 191-244, Springer, 2007.

[43] Priouret P. *Remarques sur les petites perturbations de systèmes dynamiques*, Seminaire de Probabilité XVI, Lect. Notes Math. 921, Springer, 184-200, 1982.

[44] Robertson, S. “Sample path Large Deviations and optimal importance sampling for stochastic volatility models.” *Stochastic Processes and their Applications,* 120, 66-83, 2010.

[45] Schilder, M. “Asymptotic formulas for Wiener integrals.” *Trans. Amer. Math. Soc.* 125, 63–85, 1966.

[46] Simonian, A. “Asymptotic Distribution of Exit Times for Small-Noise Diffusion.” *SIAM J. Appl. Math.* 55, 3, 809-826. 1995.

[47] Williams, R. G. “The Problem of Stochastic Exit.” *SIAM J. Appl. Math.* 40, 2, 208-223. 1981.

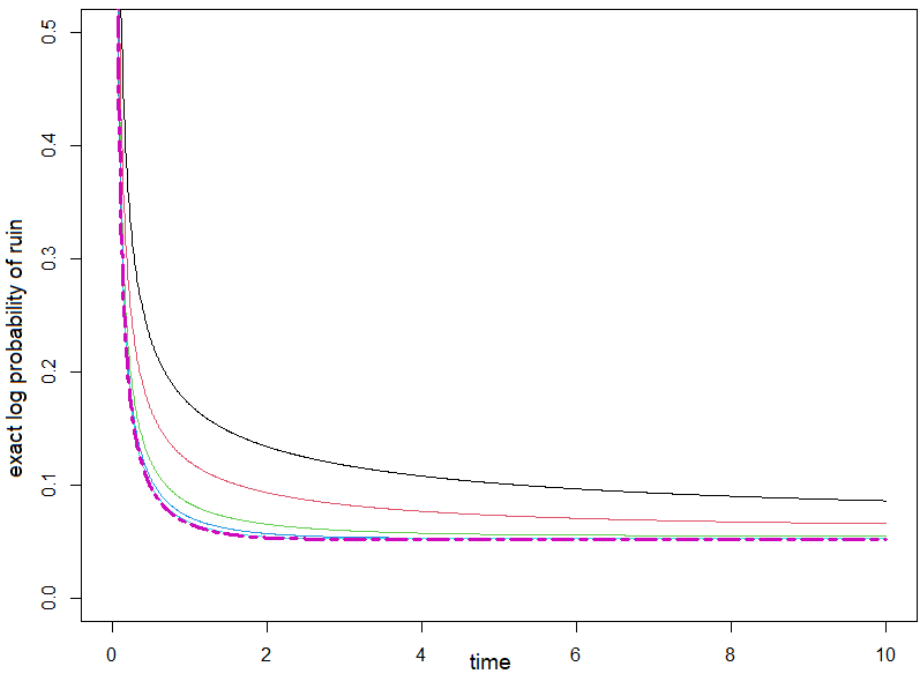

Figure 11: Negative logarithm of hitting probability for $\sigma = 3, 2, 1, 0.5, 0.2$ and comparison with the Wentzell–Freidlin low variance limit.

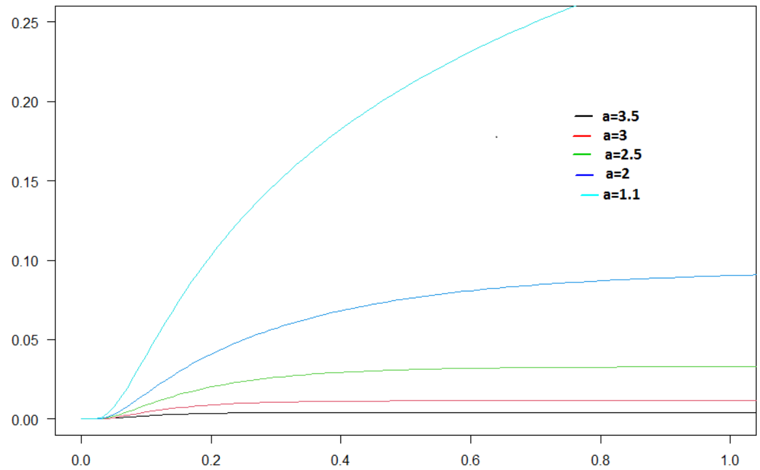

Figure 12: Probability of hitting the upper boundary as a function of time horizon based on the exact solution (4.11). Here $\sigma = 0.5$, $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$. The function is plotted for $\alpha = 1.1, 2, 2.5, 3, 3.5$.

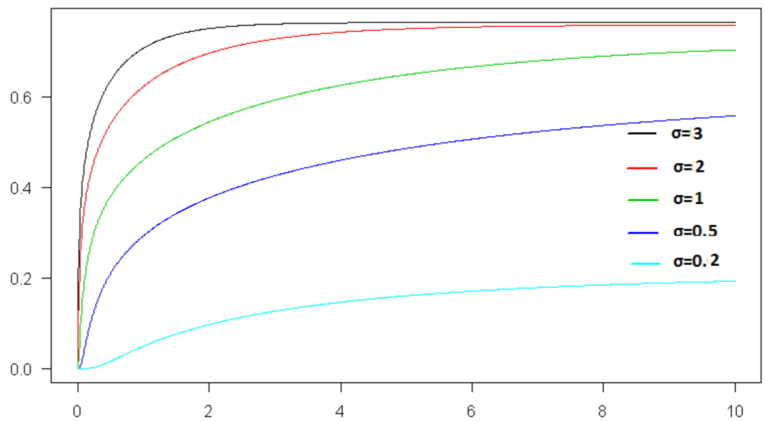

Figure 13: Probability of hitting the upper boundary as a function of time horizon based on the exact solution (4.11). Here $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$ $\alpha = 1.1$. The function is plotted for $\sigma = 0.2, 0.5, 1, 2, 3$.

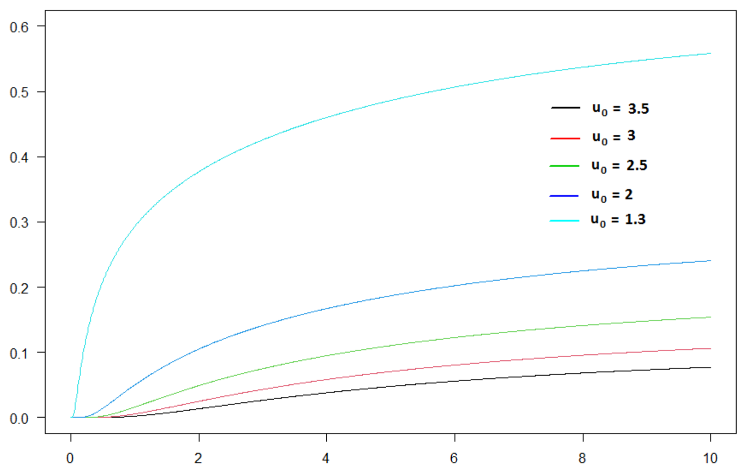

Figure 14: Probability of hitting the upper boundary as a function of time horizon based on the exact solution (4.11). Here $x_0 = 1$, $\alpha = 1.1$, $\mu = 1$ $\sigma = 0.5$. The function is plotted for $u_0 = 1.3, 2, 2.5, 3, 3.5$.

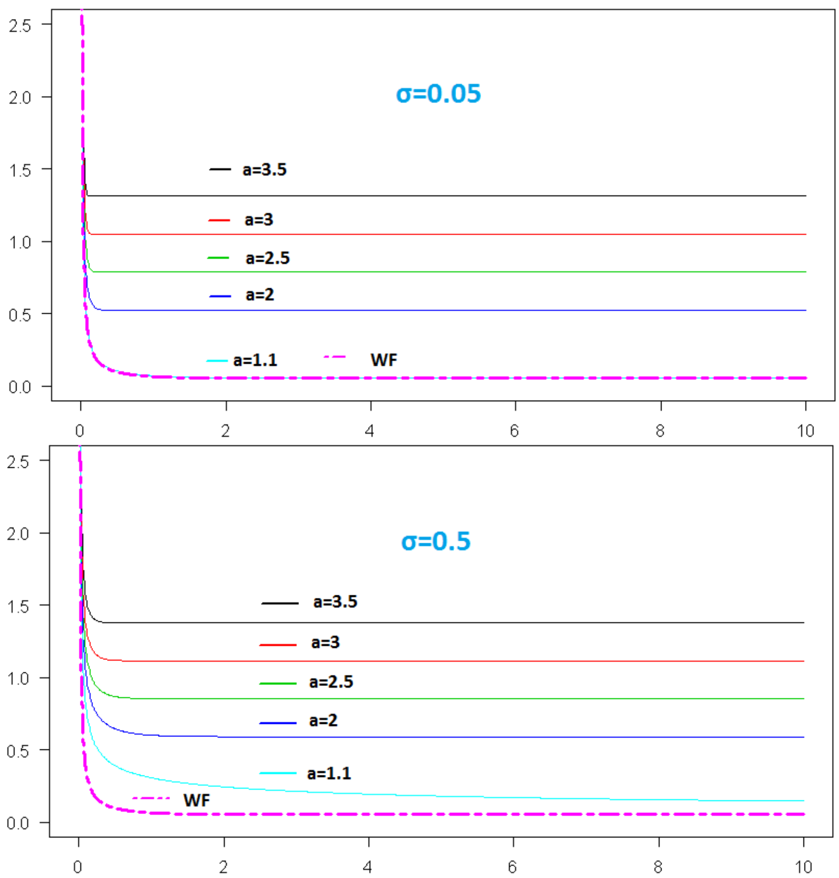

Figure 15: $-log$ Probability of hitting the upper boundary based on the exact solution (4.11). Here $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$. The upper graph was obtained for $\sigma = 0.05$ while the lower for $\sigma = 0.5$. The magenta dotted line gives the value of (the exponent of) the Wentzell-Freidlin approximation.

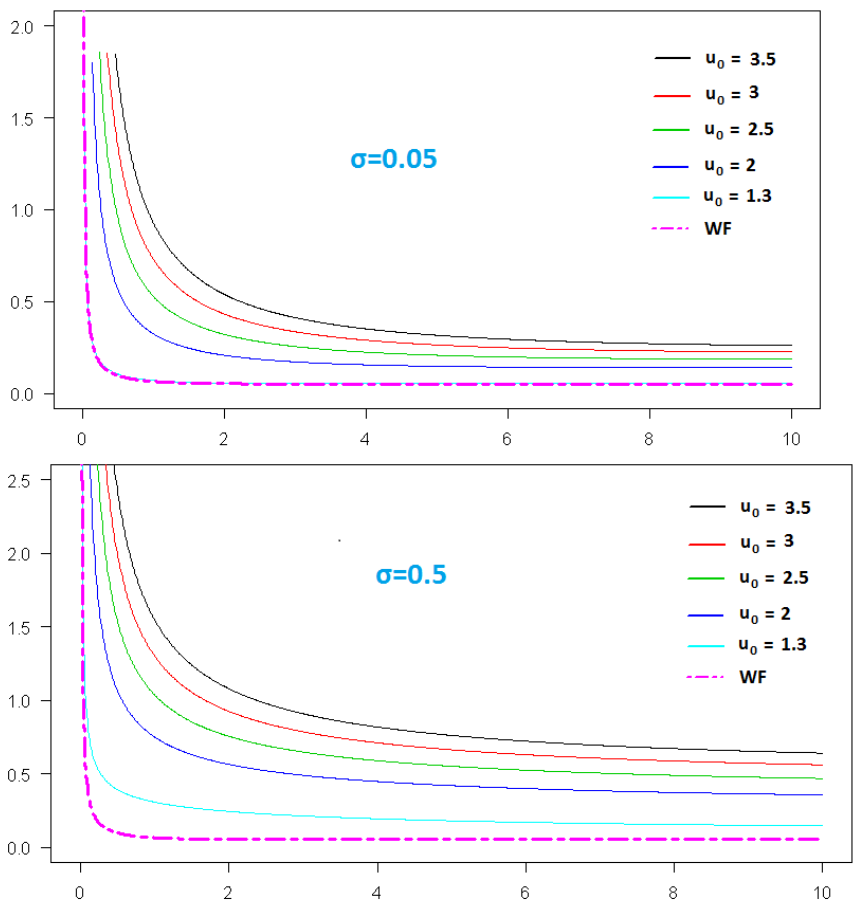

Figure 16: $-log$ Probability of hitting the upper boundary based on the exact solution (4.11). Here $x_0 = 1$, $\mu = 1$, $\alpha = 1.3$. The upper graph was obtained for $\sigma = 0.05$ while the lower for $\sigma = 0.5$. The magenta dotted line gives the value of (the exponent of) the Wentzell-Freidlin approximation.

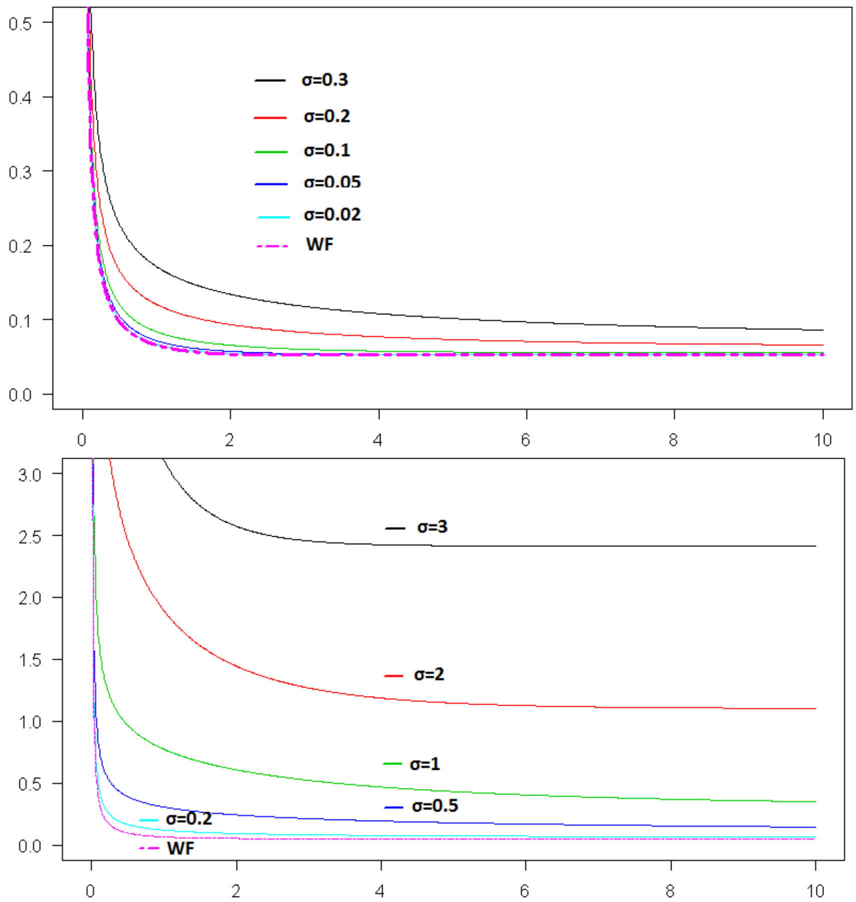

Figure 17: $-log$ Probability of hitting the upper boundary based on the exact solution (4.11). Here $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$, $\alpha = 1.1$. The magenta dotted line gives the value of (the exponent of) the Wentzell-Freidlin approximation.