Research Article

Mathematical Methods in the Applied Sciences

Received XXXX

(www.interscience.wiley.com) DOI: 10.1002/sim.0000
MOS subject classification: 60J60; 91B30

Sensitivity of the Joint Survival Probability for Reinsurance Schemes

E.E. Roumelioti, M.A. Zazanis*, N.E. Frangos

We model the joint risk process for an insurer and a reinsurer using a diffusion approximation and obtain expressions for the sensitivity of the joint survival probability with respect to parameters of the reinsurance scheme. The approach used leads, more generally, to explicit expressions for the sensitivity of functionals of diffusions in $\mathbb{R}^m$ with constant coefficients, whose drift vector and covariance matrix are differentiable functions of a parameter, in a form suitable for efficient Monte–Carlo simulation. The functionals examined depend on the values of the diffusion at a finite number of time epochs and the sensitivities are calculated using the Likelihood Ratio Method. An extension to dynamic reinsurance schemes is also briefly described and sensitivity estimators are provided using the integration by parts formula of the Malliavin calculus.

Keywords: Reinsurance; Diffusion Approximation; Sensitivity Analysis of Monte Carlo Simulations

1. Introduction

Diffusion models in risk theory have been used for a long time as convenient macroscopic models for the understanding of the qualitative issues in insurance risk problems, e.g. see [10], [8]. They are also a valuable tool for quantitative approximations in the evaluation of reinsurance policies. (Reinsurance is common practice in the insurance business, whereby an insurance company passes part of its liabilities to another firm, the reinsurer, in an attempt to reduce and transfer the risks it undertakes in its portfolio.) For applications of diffusion models to the analysis of reinsurance problems see for instance [1], [2], [9], and the references therein, as well as [12] and [13] who proposed a method based on differential equations for computing sensitivities with respect to several parameters that are included in the model.

In the present paper some aspects of the problem of the joint risk undertaken by a consortium of insurance companies are examined. The framework used is an m–dimensional diffusion process with constant coefficients which models the joint risk process that the companies in the consortium face as a result of mutual reinsurance contracts. The correlation structure of the m–dimensional diffusion process depends on the design of the reinsurance contracts employed. The aim of the paper is to obtain expressions for the *sensitivity of the joint survival probability* of the firms with respect to the parameters involved in the model and in particular with respect to the parameters that are related to the details of the contract e.g. the retention parameter. To this end we will use two different approaches, the Likelihood Ratio Method (essentially a change of measure argument) and the integration by parts formula of the Malliavin calculus. The analysis is carried out first for rather general functionals of multidimensional diffusions with constant coefficients and then applied to the reinsurance problem in question.

The expressions for the sensitivities obtained in this paper may in some simple cases be computed analytically whereas in more complicated situations numerical computation is necessary. In large scale problems they provide efficient estimators in the context of Monte Carlo simulation whereby both the quantities (say survival probabilities) and their sensitivities with respect to parameters are obtained simultaneously. These sensitivities may be used (a) as a first approach towards facing the problem of robustness of the results with respect to the model assumptions and (b) as an extension of the concept of Greeks (i.e. price sensitivities), which is very successfully used in financial risk management, to the problem of reinsurance.

The use of Malliavin calculus for the aim of obtaining sensitivities of survival probabilities in this paper was inspired by the work of Fournié et al. [4] who applied this technique to the problem of providing efficient estimates for price sensitivities in finance, and by Privault and Wei [16] who used Malliavin Calculus techniques to the study of sensitivities of functionals related to the ruin probability of an insurance firm. Alternative approaches to obtaining sensitivity estimates may also be used. One

Department of Statistics Athens University of Economics and Business 76 Patission str. Athens 10434, Greece
* *Correspondence to: Department of Statistics Athens University of Economics and Business 76 Patission str. Athens 10434, Greece,* zazanis@aueb.gr

could use pathwise differentiation (together with smoothing) or measure derivatives (likelihood ratio based techniques). For an overview we refer the interesting reader to Glasserman [6]. For an interesting discussion regarding the connection of Malliavin techniques to pathwise derivatives and likelihood ratio methods see Chen and Glasserman [3].

2. Diffusions with constant coefficients in $\mathbb{R}^m$ as models for reinsurance problems

We consider a consortium consisting of m insurance companies involved in various reinsurance treaties. We describe the free reserves of the ith company over the time period $[0, T]$ by the diffusion process

$$X_i(t) = u_i + \mu_i t + \sum_{r=1}^{d} B_{ir} W_r(t), \qquad 0 \leq t \leq T, \tag{1}$$

where $i = 1, \dots, m$ and $\{W_r(t); t \geq 0\}$, $r = 1, \dots, d$, are independent standard Brownian motions. The coefficients μ_i and B_{ir} are determined by the characteristics of the insurance portfolios and the reinsurance treaties that have been contracted between the companies. We assume that these coefficients depend smoothly on a parameter θ which is some operational characteristic of the reinsurance treaties or the insurance portfolios. (While for simplicity of exposition we will carry out the analysis for scalar θ, in general θ may be a vector parameter.) We further assume $\operatorname{rank}(B) = m$ so that the diffusion $X(t) := (X_1(t), \dots, X_m(t))$ is genuinely m– dimensional.

Suppose now that $0 < t_1 < \cdots < t_n \leq T$ are n times of interest where the free reserves of the insurers will be audited. $X_i(t_k)$ denotes then the free reserves of the ith company at time epoch t_k. Suppose further that $f : \mathbb{R}^{nm} \to \mathbb{R}$ is a (Borel) function of the free reserves of the m companies over the n time periods. For instance, if we are interested in the event that no company has negative free reserves at any of the audit times, then $f(X_i(t_k); i = 1, \dots, m; k = 1, \dots, n) = \prod_{i=1}^{m} \prod_{k=1}^{n} I(X_i(t_k) > 0)$. Another example of practical significance is $f(X_i(t_k); i = 1, \dots, m; k = 1, \dots, n) = -\sum_{i=1}^{m} \min\{0, X_i(t_1), X_i(t_2) \dots, X_i(t_n)\}$ which gives the *aggregate severity of ruin*.

In the sequel we will use the shorthand $[X_i(t_k)] := X_i(t_k); i = 1, \dots, m; k = 1, \dots, n$. We are interested in estimating expectations of cylindrical functionals of X i.e. criteria of the form $\mathbb{E}\Psi$, with

$$\Psi := f([X_i(t_k)]), \tag{2}$$

where f is a Borel function, such that the expectation in (2) exists, as well as their sensitivities with respect to parameters of interest, $\frac{\partial}{\partial\theta}\mathbb{E}\Psi$. For problems of high dimensionality, when m and n are large, Monte Carlo techniques become competitive and even advantageous compared to the numerical evaluation of the multiple integrals involved.

To illustrate how the above diffusion model arises, consider the following risk process involving an insurer and a reinsurer. Claims arrive according to a Poisson process $\{N(t); t \geq 0\}$ with rate λ. The claim sizes, $\{C_k\}$, $k = 1, 2, \dots$, form an i.i.d. sequence of random variables, independent of the Poisson process, with distribution function F. The above risk process is shared by an insurer and a reinsurer according to the following reinsurance contract: If the claim size is y, the insurer retains a part $h_1(y)$ where $h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel function with $0 \leq h_1(y) \leq y$ for all $y \geq 0$, while the reinsurer covers the rest $h_2(y) = y - h_1(y)$. Let u_1, r_1, u_2, r_2, be the initial free reserves and premium rates for the insurer and reinsurer respectively. Denote by $\widetilde{X}(t) = \left(\widetilde{X}_1(t), \widetilde{X}_2(t)\right)$, $t \geq 0$ the free reserves process vector for the insurer and the reinsurer. Then

$$\widetilde{X}_i(t) = u_i + r_i t - \sum_{k=1}^{N(t)} h_i(C_k), \qquad i = 1, 2.$$

A quantity of interest is the probability of failure of the reinsurance scheme, that is the probability that, within some given time horizon $[0, T]$, at least one of the two firms is ruined. Equivalently we can consider the probability that both firms survive, i.e.

$$\mathbb{P}\Big(\inf_{t\in[0,T]} \widetilde{X}_1(t) > 0, \inf_{t\in[0,T]} \widetilde{X}_2(t) > 0\Big). \tag{3}$$

The following theorem (see [15] or [17]) provides the required diffusion approximation.

Theorem 2.1 *([15], [17]), The free reserves process $\{\widetilde{X}(t), t \geq 0\}$ can be approximated by means of a diffusion process $\{X(t), t \geq 0\}$, starting at $u = (u_1, u_2)^\top$, with a constant drift vector*

$$\mu = \begin{bmatrix} r_1 - \lambda\mathbb{E}[h_1(C)] \\ r_2 - \lambda\mathbb{E}[h_2(C)] \end{bmatrix} \tag{4}$$

and a constant covariance matrix

$$V = BB^\top = \lambda \begin{bmatrix} \mathbb{E}[h_1^2(C)] & \mathbb{E}[h_1(C)h_2(C)] \\ \mathbb{E}[h_1(C)h_2(C)] & \mathbb{E}[h_2^2(C)] \end{bmatrix}. \tag{5}$$

In the statement of the above theorem C is a "generic" claim having distribution function F. Also, note that the matrix B is not uniquely determined from (5). Hence $\widetilde{X}(t)$ is approximated by $X(t)$ which satisfies the stochastic differential equation

$$dX(t) = \mu\, dt + B dW(t), \quad X(0) = u. \tag{6}$$

It is evident from the statement of Theorem 2.1 that the coefficients in the diffusion approximation will depend on the particular choice of reinsurance scheme. In certain cases, it is possible to obtain an explicit expression for these coefficients in terms of the parameters of the reinsurance scheme as the following example shows.

A reinsurance scheme often used in practice is the *excess of loss scheme* in which $h_1(y) = y \wedge \theta$ and $h_2(y) := \max(y - \theta, 0)$ for some $\theta > 0$. This means that the insurer covers the whole loss up to level θ. The remaining part, if the loss is greater than θ, is covered by the reinsurer. In this case, according to Theorem 2.1, the drift is

$$\mu = \left(r_1 - \lambda \int_0^\theta \bar{F}(y) dy \, , r_2 - \lambda \int_\theta^\infty \bar{F}(y) dy \right)^\top \tag{7}$$

with $\bar{F}(y) := 1 - F(y)$, and the covariance matrix is

$$V \;=\; BB^\top \;=\; \lambda \begin{bmatrix} 2\int_0^\theta y\bar{F}(y)dy & \theta\int_\theta^\infty \bar{F}(y)dy \\ \theta\int_\theta^\infty \bar{F}(y)dy & 2\int_\theta^\infty (y-\theta)\bar{F}(y)dy \end{bmatrix}.$$

For a general reinsurance scheme, we will assume that the coefficients of the diffusion approximation will generally depend on a parameter θ (possibly a vector valued quantity). Therefore the joint risk process may in general be written in the form

$$X(t) = u + \mu(\theta)\, t + B(\theta)\, W(t). \tag{8}$$

We will consider the following sampled version of the non–ruin probability. For given $n \in \mathbb{N}$ and $0 < t_1 < \cdots < t_n \leq T$ let

$$P := \mathbb{P}\left(X_1(t_k) > 0, X_2(t_k) > 0; \, k = 1, 2, \ldots, n\right). \tag{9}$$

This for all practical purposes can be used instead of (3) for the analysis of the reinsurance scheme and, as we shall see, lends itself to computationally simple algorithms. We want to obtain an estimator for the quantity $\frac{\partial}{\partial\theta}\mathbb{E}\left[\prod_{k=1}^n I(X_1(t_k) > 0, X_2(t_k) > 0)\right]$.

2.1. *Sensitivity analysis for cylindrical functionals of diffusions*

Estimating the sensitivity of the expectation of functionals of the type (2) using Monte Carlo techniques presents serious problems if one attempts to use finite difference estimators. Suppose that we perform N pairs of independent simulation experiments, each pair consisting of a simulation at the nominal value of the parameter, θ, and one at a perturbed value $\theta + \delta$. Thus $(\Psi_i(\theta), \Psi_i(\theta + \Delta\theta))$ are i.i.d. vectors, though the two components of the vectors are not necessarily independent, since one may use variance reduction techniques, such as common random numbers. Then the Finite Difference Monte Carlo estimate for the sensitivity of J with respect to the parameter θ, $J'_\theta = \frac{d}{d\theta}J(\theta)$ becomes

$$\widehat{\frac{\partial J}{\partial\theta}} = \frac{1}{N\delta}\sum_{i=1}^N \Psi_i(\theta+\delta) - \Psi_i(\theta). \tag{10}$$

While the above estimator is easy to implement, the variance and bias properties are not satisfactory. See for instance [6] for a discussion of some of the statistical and numerical issues involved. Suppose that f is a sufficiently smooth function. Let $\{X_t\}$ be the diffusion process (6) with coefficients depending smoothly on a parameter θ defined in (6). In this case a "derivative process" $X'(t) := \frac{\partial}{\partial\theta}X(t)$ can be defined path wise by direct differentiation and is given by $X'(t) = \partial_\theta \mu t + \partial_\theta B W(t)$. When the drift and variance coefficients depend both on the parameter θ and the state of the process X_t this "derivative process" is given by the first variation process which is obtained as a solution of a certain linear SDE (see [14] and [4]). More will be mentioned about this in §4. For a smooth function f an efficient Monte Carlo estimator can be obtained provided that

$$\frac{\partial}{\partial\theta}\mathbb{E}[f(X)] = \mathbb{E}[\nabla f(X) X']. \tag{11}$$

Of course the above relationship hinges upon the interchangeability of expectation and differentiation. If this does not hold, as is the case when f is a discontinuous function, the above approach cannot be used.

There are two widely used techniques available when (11) fails to hold. The first, known as the likelihood ratio method (see [6]) is applicable when the joint density of $(X(t_1), \ldots, X(t_n))$ is known, whereas the second, based on the integration by parts formula of Malliavin calculus (see [4] and the references therein), can be used even when this density is unknown. Both these techniques sidestep the statistical and numerical problems of the finite difference estimators by establishing a relationship of the form

$$\frac{\partial}{\partial\theta}\mathbb{E}[f(X)] = \mathbb{E}[f(X)H] \tag{12}$$

where the *weight* H is a random variable that can be determined from the sample path of $\{X_t\}$. Thus the right hand side of (12) can be estimated by means of ordinary Monte Carlo techniques as an average over many realizations of the process $\{X_t\}$ without having to resort to finite difference estimators. We next show how this is possible in diffusions with constant coefficients when the distribution of $\{X_t\}$ is known. In section §4 we also briefly sketch the Malliavin calculus approach (see also [11]).

A third possibility when the performance criterion in question is of the form $\mathbb{E}[f(X_{t_1})]$, i.e. when the functional of interest depends on the value of $\{X_t; t \geq 0\}$ at a single point, is via the Feynman-Kac equation. We refer the reader to [7] for a description of this approach.

2.2. *Sensitivity via the Likelihood Ratio*

Suppose that $(X_1, \ldots, X_\nu)$ is a ν–dimensional Gaussian vector with mean $\mu \in \mathbb{R}^\nu$ and nonsingular covariance matrix V. Suppose that $f : \mathbb{R}^\nu \to \mathbb{R}$ is a bounded Borel function and assume that μ and V depend smoothly on a real parameter θ. Consider the performance criterion

$$J(\theta) := \mathbb{E}f(X_1, \ldots, X_\nu) = \int_{\mathbb{R}^\nu} f(x_1, \ldots, x_\nu) g(x, \theta) dx \quad \text{where} \quad g(x, \theta) = \frac{1}{(2\pi)^{\nu/2}|V|^{1/2}} e^{-\frac{1}{2}(x-\mu)^\top V^{-1}(x-\mu)} .$$

Then $\partial_\theta J(\theta) = \int_{\mathbb{R}^\nu} f(x)\partial_\theta g(x, \theta) dx = \mathbb{E}[f(X)\partial_\theta \log g(X, \theta)]$ and thus (12) holds with

$$H = \partial_\theta \log g(X, \theta). \tag{13}$$

Taking into account that $\partial_\theta V^{-1} = -V^{-1}\partial_\theta V V^{-1}$ and $\partial_\theta |V| = |V|\mathrm{tr}(V^{-1}\partial_\theta V)$ (see [19]) we have

$$\partial_\theta \log g(x, \theta) = \partial_\theta \mu^\top V^{-1}(x - \mu) + \frac{1}{2}\left((x - \mu)^\top V^{-1}(\partial_\theta V)V^{-1}(x - \mu) \;-\; \mathrm{tr}(V^{-1}\partial_\theta V)\right). \tag{14}$$

3. Sensitivity analysis of cylindrical functionals of multidimensional diffusions with constant coefficients

We now use the above ideas in the context of diffusions (1) discussed in section 2 and obtain estimators for the sensitivity of the expectations of the functionals considered there. We proceed to give an explicit representation for weight H in this case.

Theorem 3.1 *If $\Psi = f([X_i(t_k)])$ is a cylindrical functional of the multidimensional diffusion with constant coefficients defined in (1) whose coefficients are continuously differentiable functions of θ and f a Borel function then the sensitivity of $\mathbb{E}\Psi$ is given by the expression*

$$\frac{d}{d\theta} J(\theta) = \mathbb{E}[f(X(t_1), \ldots, X(t_n))H] \tag{15}$$

with

$$H = \mu_\theta^\top (BB^\top)^{-1} BW_{t_n} + \sum_{k=1}^{n}\left(\frac{(W_{t_k} - W_{t_{k-1}})^\top B_\theta^\top (BB^\top)^{-1}(W_{t_k} - W_{t_{k-1}})}{t_k - t_{k-1}} - tr\left(B^\top (BB^\top)^{-1} B_\theta\right)\right).$$

In the above μ_θ and B_θ denote the derivatives of the drift vector and variance matrix B with respect to the parameter θ

The proof can be obtained directly using (14) noting that $V = BB^\top$.

As a simple illustration of the above theorems consider the case where $m = r = n = 1$. Then $\Psi = f(X(t))$ where $X(t) = u + \mu(\theta)t + \sigma(\theta)W(t)$ and, applying (15), we have

$$\frac{\partial}{\partial\theta}\mathbb{E}\Psi = \frac{\sigma_\theta}{t\sigma}\mathbb{E}[\Psi W^2(t)] + \frac{\mu_\theta}{\sigma}\mathbb{E}[\Psi W(t)] - \frac{\sigma_\theta}{\sigma}\mathbb{E}[\Psi].$$

When $\Psi = I(X(t) > 0)$ the above expression becomes

$$\frac{\partial}{\partial\theta}\mathbb{P}(X(t) > 0) = \mathbb{E}\left[I(X(t) > 0)\frac{\mu_\theta(\theta)}{\sigma(\theta)}W(t)\right] + \mathbb{E}\left[I(X(t) > 0)\frac{\sigma_\theta(\theta)}{t\sigma(\theta)}(W^2(t) - t)\right].$$

Evaluating the above expectations we obtain

$$\frac{\partial}{\partial\theta}\mathbb{P}(X(t) > 0) = \frac{\sigma(\theta)\mu_\theta(\theta)t - \sigma_\theta(\theta)(u + \mu(\theta)t)}{\sigma^2(\theta)\sqrt{2\pi t}}\, e^{-\frac{(u+\mu(\theta)t)^2}{2\sigma^2(\theta)}} .$$

This result can of course also be obtained by differentiating with respect to θ the explicit expression $\mathbb{P}(X(t) > 0) = \Phi\left(\frac{u+\mu(\theta)t}{\sigma(\theta)\sqrt{t}}\right)$, where Φ is the standard normal distribution function.

The above proposition (with a convenient choice for B), or (14) used directly, also provides the appropriate weight for the sensitivity of the survival probability in the insurer and the reinsurer problem discussed in §2. Consider the joint risk process, after the adoption of a reinsurance scheme, parameterized by a parameter M, as modeled by the bivariate diffusion process (8). Here we apply the results of the previous section in order to obtain an estimator of the sensitivity of the joint survival probability with respect to the parameter θ i.e.

$$\frac{\partial}{\partial\theta}\mathbb{E}\left[\prod_{k=1}^{n} I\big(X_1(t_k) > 0, X_2(t_k) > 0\big)\right] = \mathbb{E}\left[\left[\prod_{k=1}^{n} I\big(X_1(t_k) > 0, X_2(t_k) > 0\big)\right] H\right]$$

with H given either by theorem 1 or, equivalently, by (14) with μ and V given by (4) and (5) respectively and

$$\partial_\theta\mu = \lambda\left(-\bar{F}(\theta)\ ,\ \lambda\bar{F}(\theta)\right), \qquad \partial_\theta V = \lambda\begin{bmatrix} 2\theta\bar{F}(\theta) & \int_\theta^\infty \bar{F}(y)dy - \theta\bar{F}(\theta) \\ \int_\theta^\infty \bar{F}(y)dy - \theta\bar{F}(\theta) & -2\int_\theta^\infty \bar{F}(y)dy \end{bmatrix}$$

4. Sensitivity estimates via the Malliavin Calculus for dynamic reinsurance contracts

The estimates derived above were obtained by taking advantage of the fact that the joint density of $X(t_1), \ldots, X(t_n)$ is known (in fact, gaussian). Thus a change of measure argument leads to an integrating factor that could be directly estimated from the sample path. In this section we give an alternative approach based on the integration by parts formula of the Malliavin calculus (see [14], [11], [4]) that can be applied even in situations where the above joint density is unknown.

The Malliavin calculus approach does not lead to a unique sensitivity estimator. The estimators obtained typically depend on arbitrary functions or processes typically depend on arbitrary functions or processes (see [4], [11]). Typically these are chosen so as to reduce the variance of the resulting estimators or to simplify their form and reduce their computational requirements. Perhaps not surprisingly it can be shown (see [5]) that the (generally unknown) weight $\frac{\partial}{\partial\theta}\log g_\theta$ is the one that minimizes the variance of the estimator.

Suppose that the reinsurance contracts between the members of the consortium depend not only of the size of the claims but also on the free reserves of the companies. More specifically, assume that there exist Borel functions $h_i : \mathbb{R}_+^{m+1} \to \mathbb{R}$, $i = 1, \ldots, m$, such that $h_i(c, x) \geq 0$ and $\sum_{i=1}^m h_i(x, c) = c$ for all $c \geq 0$, $x = (x_1, \ldots, x_m) \in \mathbb{R}_+^m$. Assume further that $h_i(c, x)$ is continuous in x for all c. For each $x \in \mathbb{R}_+^m$ let $V(x)$ be the $m \times m$ matrix with elements $V_{ij}(x) = \lambda\mathbb{E}[h_i(C, x)h_j(C, x)]$. We will assume that the reinsurance contracts h_i are such that V satisfies the following assumption (see [4]).

Assumption 4.1 (Uniform Ellipticity) *There exists $\kappa > 0$ such that $y^\top V(x)y \geq \kappa y^\top y$ for all $y \in \mathbb{R}^m$ and $x \in \mathbb{R}_+^m$.*

While this assumption is not necessary in order to obtain sensitivity estimates using the Malliavin calculus (see [18]) it simplifies the form of the estimates. To illustrate its significance note that $y^\top V(x)y = \sum_{i=1}^m\sum_{j=1}^m \lambda y_i y_j \mathbb{E}[h_i(C, x)h_j(C, x)]$ and consider zone reinsurance contracts defined as follows.

Let b_i, $i = 0, \ldots, n$, be smooth real functions defined on $\mathbb{R}_+^m$ such that, for each $x \in \mathbb{R}_+^m$, $0 \equiv b_0(x) < b_1(x) < b_2(x) < \cdots < b_n(x)$ and let $h_i(c, x) = (b_i(x) \wedge c - b_{i-1}(x))^+$. Thus, the ith reinsurer becomes involved only when the claim size c exceeds b_{i-1} and then pays only up to the amount $b_i - b_{i-1}$. In the model examined here the levels b_i are assumed to depend smoothly on the size of the free reserves of the companies, x. Taking into account the representation $h_i(c, x) = (b_i(x) \wedge c - b_{i-1}(x))^+ = \int_{b_{i-1}(x)}^{b_i(x)} I(c > u)du$ we see using Fubini's theorem that $V_{ij}(x) = \int_{b_{i-1}}^{b_i}\int_{b_{j-1}}^{b_j} \mathbb{E}[I(C > u)I(C > v)]dudv$ which for $i < j$ reduces to $V_{ij}(x) = (b_i - b_{i-1})\int_{b_{j-1}}^{b_j} \overline{F}(u)du$, $i < j$. (To simplify the notation we have dropped the dependence on x.) The diagonal elements of the matrix are given by

$$V_{ii}(x) = \int_{b_{i-1}}^{b_i}\int_{b_{i-1}}^{b_i} \mathbb{E}[I(C > u)I(C > v)]dudv = 2\int_{b_{i-1}}^{b_i}\int_{b_{i-1}}^{u} \mathbb{E}[I(C > u)]dvdu = 2\int_{b_{i-1}}^{b_i}(u - b_{i-1})\overline{F}(u)du.$$

It can easily be shown that for such a reinsurance policy Assumption 4.1 is satisfied. Proportional reinsurance schemes on the other hand, i.e. contracts for which $h_i(c, x) = cg_i(x)$ (for at least two values of the index i) fail to satisfy the assumption. As an example, when the claims are exponentially distributed with rate $a > 0$ the elements of the covariance matrix become

$$V_{ij} = \frac{\lambda}{a}(b_i - b_{i-1})\left(e^{-ab_{j-1}} - e^{-ab_j}\right), \quad \text{for } i < j \text{ and } V_{ii} = \frac{\lambda}{a^2}\left(e^{-ab_{i-1}} - (1 + a(b_i - b_{i-1}))e^{-ab_i}\right).$$

Arguing as in §2 we see that the drift vector is given by $\mu_i(x) = r_i - \lambda\mathbb{E}[h_i(C, x)]$. Choose B so that $BB^\top = V$. Then the diffusion process in $\mathbb{R}^m$ that arises as the solution of the SDE

$$dX_t = \mu(X_t)dt + B(X_t)dW_t, \qquad X_0 = u \tag{16}$$

can be used to model the free reserves of the reinsurance consortium. $\mu : \mathbb{R}^m \to \mathbb{R}^m$ and $B : \mathbb{R}^m \to \mathbb{R}^{m\ \ m}$ are assumed continuous and bounded ($\sup_{x\in\mathbb{R}^m} \|\mu(x)\| < \infty$ and $\sup_{x\in\mathbb{R}^m} \|B(x)\| < \infty$).

Under the above assumptions the first variation process $\{\Phi_t; t \geq 0\}$ of (16) (see [4]) is the unique solution of

$$d\Phi_t = M(X_t)\Phi_t dt + \sum_{j=1}^m S_j(X_t)\Phi_t dW_j(t), \qquad \Phi_0 = I \tag{17}$$

4.1. Sensitivity with respect to the initial free reserves vector u

Let $\gamma \in \mathbb{R}^m$ and consider the family of solutions of (16) parametrized by ϵ, $\{X_t^\epsilon; t \geq 0\}$ as the solution of (16) with initial condition $X_0 = u + \epsilon\gamma$. For each given ϵ this represents the perturbed path of diffusion when the initial free reserves vector u has been changed in the direction γ. The integration by parts formula of the Malliavin calculus can be used to provide the appropriate weight H for the sensitivity of cylindrical funtionals [4].

Theorem 4.1 *Let $\alpha : [0, T] \to \mathbb{R}$ be any function in $L^2[0, T]$ such that $\int_0^{t_i} \alpha(s)ds = 1$, $i = 1, \ldots, n$. Under the above assumptions*

$$\frac{d}{d\epsilon}\mathbb{E}[f(X_{t_1}^{u+\epsilon\gamma}, \ldots, X_{t_n}^{u+\epsilon\gamma})] = \mathbb{E}\left[f(X_{t_1}, \ldots, X_{t_n}) \int_0^T \alpha(s)\Phi_s^\top (B^\top(X_s))^{-1} dW_s\right]. \tag{18}$$

Note that the weight obtained depends on an arbitrary function α. This can be chosen so as to minimize the variance of estimator obtained. It should be pointed out that even though an explicit expression for the first variation process $\{\Phi_s\}$ is difficult to obtain in closed form (except in the scalar case $m = 1$) it can easily be obtained numerically, in parallel with the diffusion process $\{X_s\}$. Thus the value of the Itô integral in (18) can be readily obtained for each sample path, leading to an efficient sensitivity estimator. Finally we point out that Malliavin calculus techniques for estimating the sensitivity of cylindrical functionals of diffusions with respect to parameters of the drift or the variance coefficients are provided in [4] (see also [18]). These typically result to weights that involve stochastic integrals in the sense of Skorokhod (see [14]).

5. Conclusion

We have studied the problem of sensitivity of functionals of a reinsurance scheme, in the diffusion approximation, with respect to the parameters of the scheme, using likelihood ratio techniques and the techniques of Malliavin calculus. In particular we have studied the sensitivity of the joint survival probability of m firms, with respect to the contract design, and provided expressions for it. The sensitivities can be understood as analogues of the greeks in financial contracts, thus providing the reinsurer and the insurer with a feeling of the risk undertaken by entering the contract. The sensitivity estimates where obtained in explicit form suitable for direct implementation using either numerical techniques or Monte Carlo methods. Future work along these lines includes extending these results to more general functionals using Girsanov's theorem instead of the elementary change of measure arguments used in this paper and further investigation of the possibilities provided by the Malliavin calculus techniques.

References

1. Asmussen, S. and M. Taksar. Controlled diffusion models for optimal divident pay–out. *Insurance: Mathematics and Economics,* 20, 1–15, 1997.
2. Asmussen S., B. Hojgaard, and M. Taksar. Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation *Finance Stochast.* 4, 299–324, 2000.
3. Chen, N. and P. Glasserman. Malliavin Greeks without Malliavin calculus, *Stochasitc Process. Appl.* 117, 1689–1723, 2007.
4. Fournié, E., J.M. Lasry, J. Lebuchoux, P.L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance, *Finance and Stochastics,* 3, 391–412, 1999.
5. Fournié, E., J.M. Lasry, J. Lebuchoux, and P.L. Lions, Applications of Malliavin Calculus to Monte-Carlo Methods in Finance. II, *Finance and Stochastics,* 5, 201–236, 2001.
6. Glasserman P. *Monte Carlo Methods in Financial Engineering*. Springer–Verlag, New York, 2004.
7. Gobet, E. and R. Munos. Sensitivity Analysis Using Itô-Malliavin Calculus and Martingales, and Application to Stochastic Optimal Control. *SIAM J. of Control and Optimization,* 43, 1676-1713, 2005.
8. Harrison, M. Ruin problems with compounding assets. *Stochastic Process. Appl,* 5, 67–79, 1977.
9. Hojgaard, B. and M. Taksar. Optimal proportional reinsurance policies for diffusion models. *Scand. Act. J.*, 166–180, 1998.
10. Iglehart, D.L. Diffusion approximations in collective risk theory. *J. Appl. Probab.* 6, 285–292, 1969.
11. Kohatsu-Higa, A. and Montero, M. Malliavin calculus in Finance, *Handbook of Computational Finance.* Birkhauser, 2004.
12. Norberg, R. Ruin problems with assets and liabilities of diffusion type, *Stoch. Processes Appl.* 81, 255–269, 1999.
13. Norberg, R. Sensitivity analysis in insurance and finance, Information Processes, Kalashnikov Memorial Seminar 2, 240–240, 2002.
14. Nualart, D. *Malliavin Calculus and its Applications*, Springer 2nd Ed. 2006.
15. Pinsky, M. Homogenization in stochastic differential geometry, *Publ. Res. Inst. Math. Sci.,* 17, 1, 235–244, 1981.
16. Privault, N. and Wei, X. A Malliavin calculus approach to sensitivity analysis in insurance, *Insurance: Mathematics and Economics,* 35, 679–690, 2004.
17. Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J.L., *Stochastic Process for Insurance and Finance*, Wiley, Chichester, 1999.
18. Roumelioti, E.E., Zazanis, M.A., Frangos, N.E., and A.N. Yannacopoulos. Sensitivity Analysis for Multidimensional Diffusions via the Malliavin Calculus, in preparation 2013.
19. Schott, J.R. *Matrix Analysis for Statistics*. Wiley, 2005.

