A queueing system with service phases of random length and vacations

George C. Mytalas
Dept. of Mathematical Sciences, New Jersey Institute of Technology,
University Heights Newark NJ 07102, USA
and
Michael A. Zazanis*
Dept. of Statistics, Athens University of Economics and Business,
Athens 10434, Greece

Abstract

In this paper we examine a queueing model with Poisson arrivals, service phases of random length, and vacations, and its applications to the analysis of production systems in which material handling plays an important role. A second area of application is systems with unreliable servers. The length of a service phase is referred to as a "processing batch" and the analysis is carried out separately for processing batch distributions with bounded and unbounded support. In the first case, standard techniques from the analysis of batch service systems are used involving Rouché's theorem, while in the second the analysis proceeds via Wiener-Hopf factorization techniques. Processing batches with size that is either geometrically distributed or distributed according to a combination of geometric factors lead to particularly simple solutions related to Bernoulli vacation models. In all cases, care is taken in the analysis in order to obtain the steady state distribution of the system under *minimal assumptions*, namely the *finiteness of the first moment of the service and vacation distributions together with the stability condition*. This is in contrast to most of the literature where usually the assumption that the service and vacation distribution is light-tailed is either explicitly stated or tacitly adopted. Applications in manufacturing, materials handling, and reliability are indicated.

KEYWORDS: QUEUEING, MANUFACTURING, BULK SERVICE QUEUES, MATERIALS HANDLING.

1 Model description

We analyze an M/G/1 queue with *service phases of random length* and vacations in the service mechanism. Customers arrive according to a Poisson process with rate $\lambda > 0$ to the system and have i.i.d. service requirements which we will denote by $\{\sigma_n; n \in \mathbb{N}\}$. These are assumed to be independent of the arrival process and their common distribution will be denoted by $B(x) := P(\sigma \leq x)$ with finite mean $E\sigma$. The capacity of the queue is assumed to be infinite. At specific time epochs the server initiates "vacation" periods during which he is unavailable to serve customers, while arriving customers accumulate in the waiting area. Successive vacation periods form a sequence of i.i.d. random variables, independent of the arrival process and service requirements, denoted by $\{G_m; m \in \mathbb{N}\}$, with common distribution, $G(x)$, and finite mean, EG. The server's operation alternates thus between *service phases* and *vacation phases*.

*Corresponding author: zazanis@aueb.gr

At the beginning of each service phase its *size*, i.e. *the number of customers to be served during that phase*, is set. Subsequently, and throughout this paper, we will be referring to the number of customers to be served during a service phase as a *processing batch*. The use of the term *processing batch* in this fashion here is consistent with its use in manufacturing practice. We should emphasize however that in the queueing literature the term service batch is often interpreted to mean that all customers belonging to the same batch are simultaneously admitted to the server and receive service as if they where a single entity. Processing batches are assumed to be i.i.d. random variables, independent of the arrival process, service requirements, and vacation lengths, and will be denoted by $\{\Theta_m\}$. Their common distribution will be denoted by $\theta_k = P(\Theta = k)$, $k = 1, 2, \ldots$, and will be assumed to have finite mean $E\Theta = \sum_{k=1}^{\infty} k\theta_k < \infty$. During the service phase customers are served in a FIFO fashion until either the number of customers served in the phase becomes equal to the processing batch size Θ_m or the queue empties, whichever happens first. In both cases the server initiates a new vacation phase. We shall call this the *partial batch policy*, since it is possible that fewer customers than required by the processing batch are served during the service phase. When the server returns from the vacation a new processing batch is set and a new service phase begins. If, upon returning from a vacation, the server finds the queue empty then we will assume that he immediately takes a new vacation. (We thus allow service phases to have zero duration.) Variations in the behavior of the server when, upon returning from a vacation, he finds the queue empty are possible. For instance we may suppose that in such cases the server waits for a fixed period of time and only if this elapses without arrivals he leaves again, or that he waits until a fixed number of customers arrive etc. Such variations do not burden the analysis but, since they have been studied extensively in the vacations literature, they will not be considered here.

We will also consider the *complete batch policy* according to which, when a service phase is initiated and a processing batch is set, the server remains available, waiting for a customer to arrive if necessary, and works until the processing batch is complete. After this, the vacation phase begins during which the server is unavailable. At the end of the vacation a new cycle begins with a new processing batch determined at random, independently of everything else, from the given distribution θ_k, $k = 1, 2, \ldots$.

The system described above is a type of an M/G/1 queue in a random environment. Under the assumption that the processing batch has fixed size, say $\Theta_m = N$ with probability 1, this system has been studied in Coffman and Gilbert [11]. There, the fixed processing batch is interpreted as the capacity of an output buffer or *a cart*, placed next to the processing station. Finished parts are placed in the cart and when it is full it is taken by the server to its destination. Thus, server vacations in that model correspond to the time it takes the server to deliver the cart. If we suppose that the same cart is used to store the output of two or more stations served by the same server then the need for a cart with stochastic capacity arises naturally.

The model we propose has also applications to queueing systems with unreliable servers. Indeed, suppose that the server is subject to failures. These failures are assumed to manifest themselves at the initiation of service and to be independent of the service requirements of the customers. Under these conditions the random processing batch model proposed constitutes an accurate model. Vacation periods correspond then to down time for the system while the server is being repaired. In this context the complete batch policy described above is more appropriate. (The partial batch policy may be appropriate if we assume that idle periods are used for preventive maintenance. In this case a model with vacations whose duration distribution depends on whether the preceding processing batch has been completed can be used. The analysis of such models will be sketched in section 4.4.)

Throughout the paper the analysis is carried out by distinguishing two cases, according to whether the processing batch distribution has *bounded* or *unbounded support*. In the first case, where the support of the processing batch size distribution is bounded above by a constant N (this could be the cart's capacity in the first model mentioned above) the analysis is based on an argument using Rouché's theorem, typical of the analysis of queues with batch service (see [8]). In this respect attention has been paid in order to establish our results under the natural conditions for the existence of a stationary version of the process i.e. *the finiteness of first moments* plus the stability condition of the system. In contrast, much of the literature of batch service queues either tacitely assumes or explicitly requires the service (and vacation, where appropriate) distributions to be *light tailed.*

The second case, where the distribution of the processing batch size has unbounded support, is harder and in general it can only be dealt with by Wiener-Hopf factorization techniques. We indicate how to carry out this procedure and we also provide explicit solutions for the case of processing batches whose distribution is either geometric or a combination of geometric factors.

In all cases the analysis of the system proceeds by first analyzing an embedded Markov chain by means of generating functions and then using standard results from semi-regenerative processes in order to obtain the stationary distribution of the number of customers in the system. For the most part, throughout the paper, we will use the language of the queue-and-cart model introduced in [11]. In particular, in section 8 we will derive the distribution of the number of customers in the waiting and departing cart, as well as the joint distribution in stationarity for the number of customers in the queue and the cart. Finally, in all cases, we discuss briefly the necessary modifications when customers do not arrive singly but in i.i.d. batches.

2 The embedded chain and the stability condition

We consider the embedded point process of the epochs when the server returns to the queue at the end of a vacation. We denote these points by $\{T_m; m \in \mathbb{Z}\}$. Let us also denote by $\{S_m; m \in \mathbb{Z}\}$ the corresponding epochs when the server leaves the queue to deliver the cart, i.e. the beginnings of vacations. We shall think of the sample path of the process as consisting of cycles. Each cycle comprises a *service phase* where the server is present and serving customers, and a *vacation phase* during which the server is away, delivering the cart to its destination. The number of customers in the system at time t is denoted by X_t and the process $\{X_t; t \in \mathbb{R}\}$ is assumed to have right–continuous sample paths. The mth cycle starts at time T_m, with the end of the $(m-1)$th vacation. We denote by Φ_m the number of customers in the system at epoch T_m, (i.e. $\Phi_m = X_{T_m}$). This means that at the start of the mth cycle, i.e. at the moment when the server returns with the cart to the queue, he finds Φ_m customers waiting for service. Clearly, (T_m, Φ_m), $m \in \mathbb{Z}$, is a Markov–renewal process and $\{X_t; t \in \mathbb{R}\}$ is a *semi–regenerative process* with respect to it. Also denote by Ψ_m the number of customers left behind in the queue at epoch S_m when the server leaves the system to deliver the cart, i.e. $\Psi_m = X_{S_m}$. Finally we will denote by L_m the number of services in the mth cycle which is equal to the contents of the cart when it leaves. Clearly we have $L_m \leq \Theta_m$, and $\Psi_m = 0$ if $L_m < \Theta_m$ since we assume that a partial batch policy is used. Also recall that, according to this policy, if $\Phi_m = 0$ then the server does not stay in the queue at all but immediately takes another vacation. Hence, in that case $S_m = T_m$, and $\Psi_m = \Phi_m = 0 = L_m$. Figure 1 illustrates these definitions.

In the next section we will obtain the stationary distribution of the embedded chain $\{\Phi_m\}$ by analyzing the evolution of the system at subsequent departure epochs. Here we confine ourselves to a qualitative study of the embedded chain $\{\Phi_m\}$ which will be used to derive the stability condition for the system both under the partial and under the complete batch policy. The state space of the embedded chain is the set of nonnegative integers and the chain is clearly irreducible. In what follows we will show that if the condition

$$E\Theta > \frac{\lambda EG}{1 - \lambda E\sigma} \tag{1}$$

is satisfied, then it is also *positive recurrent*. If the above condition holds with equality then we will

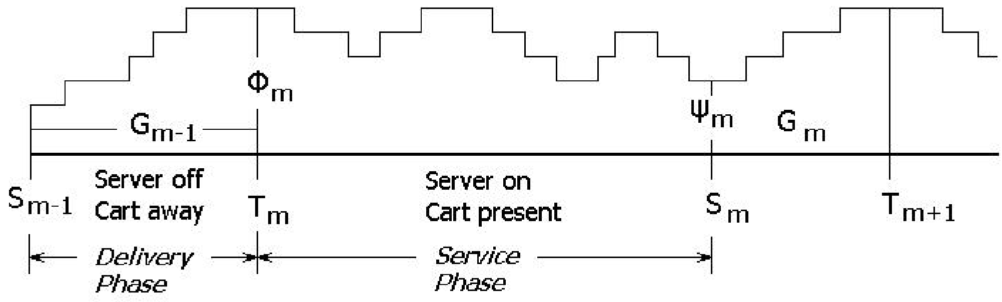

Figure 1: Sample path of the queue.

show that the chain is *null recurrent* whereas if the sense of inequality (1) is reversed then the chain is *transient*. These results have clear implications for the stability of the system as we will see in the sequel.

Let $\Delta\Phi_m := \Phi_{m+1} - \Phi_m$ denote the increments of the process $\{\Phi_m\}$. We will use standard Foster–Liapunov criteria in order to show that (1) guarantees the positive recurrence of the chain. Indeed, if $A(s,t]$ denotes the number of Poisson arrivals in the interval $(s,t]$, $\Delta\Phi_m = A(T_m, S_m] - L_m + A(S_m, T_{m+1}]$. Suppose that the partial batch policy is used. Since L_m is a stopping time we have in fact that

$$E[\Delta\Phi_m \mid \Phi_m = k] = (\lambda E\sigma - 1)E[L_m \mid \Phi_m = k] + \lambda EG.$$

Also, since the partial batch policy is used, $L_m \le \Theta_m$ w.p. 1 and hence, in view of the independence of Θ_m from Φ_m, we have $E[L_m \mid \Phi_m = k] \le E\Theta$ and

$$\lim_{k\to\infty} E[L_m \mid \Phi_m = k] = E\Theta.$$

Thus, if (1) holds then $E[\Delta\Phi_m \mid \Phi_m = k] < 0$ for all k sufficiently large. This is enough in order to establish the positive recurrence of the Markov chain $\{\Phi_m\}$. Using the same criterion we can show that (1) implies the positive recurrence of $\{\Phi_m\}$ under the complete batch policy. (The argument is somewhat more involved and is relegated to the Appendix.) The positive recurrence of the embedded Markov chain $\{\Phi_m\}$ implies in turn the positive recurrence of the process $\{X_t; t \in \mathbb{R}\}$ since the mean cycle times $E[T_{m+1} - T_m]$ are bounded above by $E\Theta E\sigma + EG$ and thus are finite.

It remains to show that if (1) does not hold then $\{\Phi_m\}$ is either null recurrent (when the inequality in (1) is replaced by equality) or transient (when the inequality is reversed). This analysis is also given in the Appendix.

3 Analysis of the embedded Markov chain of the system under a *partial batch* policy

3.1 Notation

Following the approach of Coffman and Gilbert [11] we let d_k^m be the epoch of the kth service completion during the mth cycle. We will agree to set $d_0^m = T_m$. Clearly, in the mth cycle we have $T_m = d_0^m < d_1^m < d_2^m < \cdots < d_{L_m}^m$. Let $X_{d_k^m}$ be the number of customers left behind at the kth epoch of the mth cycle and in particular note that $X_{d_0^m} = \Phi_m$. We will assume that the system is stationary and we will analyze its behavior over “a typical cycle”. Therefore, without risk of confusion, we will drop the subscript m referring to a particular cycle in what follows. Suppose that the system has been operating in stationarity and that time $t = 0$ coincides with $d_0^m = T_m$ (in other words consider the Palm version of the process with respect to the point process $\{T_m\}$). Note that, under the partial batch policy,

$$\{L \geq k\} = \{X_{d_0} > 0, X_{d_1} > 0, \ldots, X_{d_{k-1}} > 0\} \cap \{\Theta \geq k\} \tag{2}$$

and

$$\{L = k\} = \{X_{d_0} > 0, X_{d_1} > 0, \ldots, X_{d_{k-1}} > 0\} \cap (\{\Theta = k\} \cup \{X_{d_k} = 0\}), \qquad k = 1, 2, \ldots.$$

whereas $\{L = 0\} = \{X_{d_0} = 0\}$. We define the generating functions

$$Q_k(z) = E[z^{X_{d_k}}; L \geq k] \tag{3}$$

and set

$$F_k = Q_k(0) = P(X_{d_k} = 0; L \geq k) \tag{4}$$

or, in view of the fact that under the partial batch policy as soon as the queue empties the server takes the cart to be delivered

$$\begin{aligned} F_k &= P(X_{d_k} = 0; L = k) = P(X_{d_k} = 0; L = k; \Theta \geq k) \\ &= P\left(X_{d_0} > 0, X_{d_1} > 0, \ldots, X_{d_{k-1}} > 0, X_{d_k} = 0; \Theta \geq k\right). \end{aligned}$$

Note that F_k is the probability that the typical service phase consists of precisely k services and that the next vacation phase starts with an empty queue. In section 8 their role in determining the statistics on the cart contents is examined in detail. We also point out that, in view of (2) and (3),

$$Q_k(z) = E[z^{X_{d_k}}; L \geq k \mid \Theta \geq k] = E[z^{X_{d_k}}; L \geq k \mid \Theta = n] \quad \text{for } n = k, k+1, k+1, \ldots. \tag{5}$$

Furthermore, with B denoting the service time distribution and B^* the corresponding Laplace transform,

$$U(z) := B^*(\lambda(1-z)), \tag{6}$$

is the p.g.f. (probability generating function) of the number of arrivals during a service time. Similarly, with G and G^* denoting the distribution and Laplace transform respectively of the vacation period for the server,

$$D(z) := G^*(\lambda(1-z)) \tag{7}$$

is the p.g.f. of the number of arrivals during a server vacation time. We also define for convenience the quantities

$$\alpha(z) := U(z)z^{-1}, \qquad y(z) := \frac{1}{\alpha(z)}. \tag{8}$$

3.2 Random processing batch size with finite support

In this subsection we assume that the processing batch size distribution $\{\theta_n; n \in \mathbb{N}\}$ has bounded support i.e. that $N = \sup\{n : \theta_n > 0\} < \infty$. The “dynamics” of the process during a service period (i.e. during intervals of the form $(S_m, T_{m+1}),\ m \in \mathbb{Z}$) are described by the following basic recursive relationship which involves the generating functions defined in (3) and (8)

$$Q_{k+1}(z) = (Q_k(z) - F_k)\,\alpha(z), \qquad k = 0, 1, \ldots, N-1. \tag{9}$$

This recursion expresses the fact that the number of customers left behind at the end of the $(k+1)$th service completion is equal to the number left behind at the kth service completion minus one plus the number that arrived during this service time, *provided that the queue has not emptied and the processing batch size is equal to* $k+1$ *or greater*. From it we readily obtain

$$Q_n(z) = \alpha(z)^n Q_0(z) - \sum_{k=0}^{n-1} F_k \alpha(z)^{n-k}, \tag{10}$$

$n = 1, 2, \ldots, N$. By definition

$$Q_0(z) = E\left[z^{X_{d_0}}; L \geq 0\right] = E\left[z^{X_{d_0}}\right] = E[z^{\Phi}] \tag{11}$$

is the p.g.f. of the number of customers in the queue at an epoch when the service phase begins. (Of course $P(L \geq 0) = 1$.) Also, the p.g.f. of the number of customers left behind in the queue after the server leaves in order to deliver the cart is given by

$$\Pi(z) := E[z^{\Psi}] = \sum_{n=1}^{n} \theta_n \left(Q_n(z) + \sum_{k=0}^{n-1} F_k \right). \tag{12}$$

Indeed, conditioning on the processing batch size to be equal to n, for the typical cycle in stationarity, F_k, $k = 0, 1, \ldots, n-1$, is the probability that the server leaves behind an empty queue and the cart contains k customers, i.e. a partial processing batch, while $F_n = Q_n(0)$ is the probability that the server leaves behind an empty queue and the cart leaves with a complete batch of n customers. Thus $E[z^{\Psi} \mid \Theta = n] = Q_n(z) + \sum_{k=0}^{n-1} F_k$ and (12) follows by taking expectation over Θ. Taking into account (10) we obtain

$$\Pi(z) = \sum_{n=1}^{n} \theta_n \left(\alpha(z)^n Q_0(z) + \sum_{k=0}^{n-1} F_k \left(1 - \alpha(z)^{n-k}\right) \right). \tag{13}$$

On the other hand the number of customers in the system at the beginning of the typical service phase is equal to the number left behind at the end of the previous service phase plus the number of customers who arrived during the intervening vacation phase. The p.g.f. of the number of these arrivals is $D(z)$ and thus we have, under stationarity,

$$\Pi(z) D(z) = Q_0(z). \tag{14}$$

From the above, in conjunction with (13) we obtain

$$Q_0(z) = \sum_{n=1}^{n} \theta_n \left(\alpha(z)^n Q_0(z) + \sum_{k=0}^{n-1} F_k \left(1 - \alpha(z)^{n-k}\right) \right) D(z). \tag{15}$$

Before proceeding we point out that in the sequel we will occasionally be *dropping the dependence of some generating functions on* z for notational convenience. Thus we will be writing y instead of $y(z)$, D instead of $D(z)$, and so forth. From (10), (15), and (8), we conclude that

$$Q_0 \left(y^n - D \sum_{n=1}^{n} \theta_n y^{N-n} \right) = D \sum_{n=1}^{n} \theta_n \sum_{k=0}^{n-1} F_k \left(y^n - y^{N-n+k} \right) \tag{16}$$

or equivalently

$$\Pi(z) = \frac{\sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n \left(y^n - y^{N-n+k} \right)}{y^n - D \sum_{n=1}^{n} \theta_n y^{N-n}}. \tag{17}$$

The above can also be written as

$$\Pi(z) = \frac{\sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n \left(z^n - z^{N-n+k} U^{n-k}\right)}{z^n - D \sum_{n=1}^{n} \theta_n z^{N-n} U^n}. \tag{18}$$

The N constants, $F_0, F_1, \ldots, F_{N-1}$, can be obtained from Rouché's theorem as follows. It is shown in the Appendix (cf. [11]) that the equation

$$z^n - D(z) \sum_{n=1}^{n} \theta_n z^{N-n} U(z)^n = 0 \tag{19}$$

has N complex roots, $z_0, z_1, \ldots, z_{N-1}$, where $z_0 = 1$ and the remaining $N-1$ roots are within the unit circle, i.e. $|z_i| < 1$ for $i = 1, 2, \ldots, N-1$, provided that the stability condition holds. We thus know that equation (19) has precisely N zeros that satisfy $|z| \leq 1$. One of them is $z = 1$ which obviously satisfies $z^n - D(z) \sum_{n=1}^{n} \theta_n z^{N-n} U(z)^n = 0$ and is a single root. Thus there remain $N-1$ roots of the denominator in the unit disk which we shall call z_i, $i = 1, 2, \ldots, N-1$. Since $Q_0(z)$ does not have any singularities within the unit disk these must also be zeros of the numerator of (18). Hence the N unknown constants, $F_0, F_1, \ldots, F_{N-1}$ must satisfy the $N-1$ equations

$$\sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n \left(z_i^n - z_i^{N-n+k} U(z_i)^{n-k}\right) = 0, \qquad i = 1, 2, \ldots, N-1.$$

Let

$$y_i := \frac{z_i}{U(z_i)}, \qquad i = 1, 2, \ldots, N-1. \tag{20}$$

Considering Q_0 as a function of y, the y_i's must also be zeros of the numerator of (17), or equivalently, taking into account (20), together with the fact that the z_i's satisfy (19), and $U(z_i) \neq 0$ we have

$$\sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n \left(y_i^n - y_i^{N-n+k}\right) = 0, \qquad i = 1, 2, \ldots, N-1.$$

The polynomial in y

$$P(y) := \sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n \left(y^n - y^{N-n+k}\right) \tag{21}$$

has degree N and its roots are $1, y_1, y_2, \ldots, y_{N-1}$. Thus

$$P(y) = C(y-1) \prod_{i=1}^{N-1} (y - y_i). \tag{22}$$

The constant C can be determined by noting that

$$C \prod_{i=1}^{N-1} (1 - y_i) = \lim_{y \to 1} \frac{\sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n \left(y^n - y^{N-n+k}\right)}{y-1} \tag{23}$$

$$= \sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n (n-k), \tag{24}$$

where in the last equation we have used de l' Hospital's rule. The quantity on the right hand side of (23) is obtained by determining the value of $Q_0(z)$ when $z = 1$ as follows. Letting $z \to 1$ (or equivalently $y \to 1$) and applying de l'Hospital's rule in (18), we obtain

$$\sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n (n-k) = E\Theta - \frac{\lambda EG}{1-\rho}. \tag{25}$$

From (22), (24), and (25) we obtain the value of the constant in (22)

$$C = \frac{E\Theta - \frac{\lambda EG}{1-\lambda E\sigma}}{\prod_{i=1}^{N-1}(1-y_i)}. \tag{26}$$

We have thus established the following

Theorem 1. *For a system under the partial batch policy, assuming that the batch size distribution has bounded support, the p.g.f. of the number of customers left behind at the end of a typical service phase in steady state is given by*

$$\Pi(z) \;=\; \frac{E\Theta - \frac{\lambda EG}{1-\lambda E\sigma}}{y^n - D\sum_{n=1}^{n}\theta_n y^{N-n}}(y-1)\prod_{i=1}^{N-1}\frac{y-y_i}{1-y_i} \tag{27}$$

where D is given by (7), y by (8) and the y_i's by (20).

As we saw above, the explicit determination of the N constants, $F_0, \ldots, F_{N-1}$, is not necessary for the determination of $\Pi(z)$. Nonetheless, these constants are useful in order to obtain, among other things, statistics for the cart contents when it is delivered. Their computation is given in the appendix. The detailed analysis of the statistics of the cart's contents is undertaken in section 9. Here we confine ourselves to the observation that the probability that a processing batch is delivered incomplete is equal to $p_e := \sum_{n=1}^{n}\theta_n \sum_{k=0}^{n-1} F_k$. Changing the order of summation and using the above equations we have

$$p_e = \sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n}\theta_n = \frac{E\Theta - \frac{\lambda EG}{1-\lambda E\sigma}}{\prod_{i=1}^{N-1}(y_i-1)} \sum_{i=1}^{N-1} y_i.$$

The expected number of customers in the cart when it is delivered can be computed by first conditioning on the size of the processing batch:

$$E[L \mid \Theta = n] \;=\; \sum_{k=0}^{n-1} kF_k + n\left(1 - \sum_{k=0}^{n-1} F_k\right) \;=\; n - \sum_{k=0}^{n-1} F_k(n-k).$$

Taking expectation over the size of the processing batch, we then have

$$\begin{aligned} EL \;&=\; E\Theta - \sum_{n=1}^{n}\theta_n \sum_{k=0}^{n-1} F_k(n-k) \;=\; E\Theta - \sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{n} \theta_n(n-k) \\ &=\; \frac{\lambda EG}{1-\rho} \end{aligned} \tag{28}$$

where in the last equation we have made use of (25). Note that the expected contents of the cart i.e. the expected "actual processing batch size" is of course less than $E\Theta$ (because of the occurrence of incomplete processing batches when the queue empties) and does not depend on the processing batch size distribution $\{\theta_n\}$, provided that the stability condition (1) holds.

4 Processing batch size with unbounded support

The analysis of the previous section depended on the assumption that the processing batch size had a distribution with finite support. As it will readily become clear, no conceptual difficulties are involved in dropping this assumption. However, from a computational point of view, new difficulties arise as the argument based on Rouché's theorem can no longer be used.

Suppose that the cart capacity is, from transfer to transfer, a random variable with distribution $P(\Theta = n) = \theta_n$, $n = 1, 2, \ldots$ and corresponding generating function $\Theta(z) := \sum_{n=1}^{\infty} \theta_n z^n$. The following theorem provides the counterpart of equation (18) of the previous section.

Proposition 2. *The probability generating function of the number of customers left behind when the cart leaves the queue, denoted by* $\Pi(z) = E\left[z^{\Psi_m}\right]$, *is given by*

$$\Pi(z) \;=\; \frac{\sum_{k=0}^{\infty} F_k \sum_{n=1}^{\infty} \theta_{n+k} \left(1 - \alpha(z)^n\right)}{1 - D(z)\Theta(\alpha(z))}. \tag{29}$$

Proof: The analysis of the previous section applies again, with the same notation as before. Once more the epochs when the server returns after delivering the cart back to queue for the mth time is denoted by T_m while the epoch right after T_m when the server takes the cart (together with any customers that it contains) to be delivered and starts a vacation is denoted by S_m. Here a typical cycle starts, say at T_m, the server serves L_m customers (where $L_m \le \Theta_m$ and Θ_m is the size of the cart during the mth cycle) and then departs to deliver the cart at time S_m. Let, as in the previous section, $Q_0(z) = E\left[z^{\Phi_m}\right]$, $Q_n(z) = E\left[z^{X_{d_n}}; L \ge n\right]$, and $F_n = Q_n(0)$. Then $\Pi(z) = E\left[z^{\Psi_m}\right]$, is given by (cf. equation 27)

$$\Pi(z) \;=\; \sum_{n=1}^{\infty} \left(Q_n(z) + \sum_{k=0}^{n-1} F_k \right) \theta_n. \tag{30}$$

The basic recursion (9) still holds and thus we have (10) for $n = 1, 2, \ldots$ from which we obtain

$$\sum_{n=1}^{\infty} \left(\alpha(z)^n Q_0(z) + \sum_{k=0}^{n-1} F_k \left(1 - \alpha(z)^{n-k}\right) \right) \theta_n \;=\; \Pi(z).$$

Also, (14) still holds as before and using Fubini's theorem to change the order of summation we can rewrite the above expression as

$$\Pi(z) D(z) \Theta(\alpha(z)) + \sum_{k=0}^{\infty} F_k \sum_{n=1}^{\infty} \theta_{n+k} \left(1 - \alpha(z)^n\right) \;=\; \Pi(z)$$

whence we obtain (29). ∎

Note however that the numerator of (29) depends on a whole sequence of unknown constants F_k, $k = 0, 1, 2, \ldots$. Clearly the techniques of the previous section cannot be applied here. In this general case a solution can be obtained, at least in principle, using the Wiener-Hopf decomposition technique as described in the sequel.

4.1 Wiener-Hopf decomposition

From equation (8) we have

$$z = yU(z). \tag{31}$$

Using Lagrange's series expansion (e.g. see Copson [10]) if D is an analytic function in a domain containing the origin then $D(z(y))$ is an analytic function of y with series expansion around the origin given by

$$D(z(y)) = \sum_{n=0}^{\infty} y^n \kappa_n \tag{32}$$

where

$$\kappa_0 = D(0) = G^*(\lambda) \quad \text{and} \quad \kappa_n = \frac{1}{n!}\frac{d^{n-1}}{dt^{n-1}}\left(D'(t)U(t)^n\right)\Big|_{t=0}, \quad n = 1, 2, \ldots. \tag{33}$$

In particular, when $D(z) = z$ the above expression gives

$$z(y) \;=\; \sum_{n=1}^{\infty} \frac{y^n}{n!}\frac{d^{n-1}}{dt^{n-1}}U(t)^n\Big|_{t=0}.$$

Theorem 3. *Let us denote by κ^{*n} the n–fold convolution of the sequence $\{\kappa_m; m = 0, 1, 2, \ldots\}$ with itself, i.e. $\kappa_m^{*1} = \kappa_m$ and $\kappa_m^{*n} = \sum_{l=0}^{m} \kappa_{m-l}^{*(n-1)}\kappa_l$, $m = 0, 1, 2, \ldots$, and similarly let θ^{*n} denote the n–fold convolution of $\{\theta_m\}$ with itself. Then the p.g.f. of the number of customers left behind by a typical cart departure is given by*

$$\Pi(z) \;=\; \exp\left(\sum_{r=1}^{\infty}\left(z^r U^{-r}(z) - 1\right)\sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=1}^{\infty}\kappa_{l+r}^{*n}\theta_l^{*n}\right). \tag{34}$$

Proof: Using the change of variables from z to y and setting $\widetilde{\Pi}(y) := \Pi(z(y))$ equation (29) becomes

$$\widetilde{\Pi}(y) = \frac{\sum_{k=0}^{\infty} F_k \sum_{n=1}^{\infty}\theta_{n+k}\left(1 - y^{-n}\right)}{1 - D(z(y))\Theta(y^{-1})}. \tag{35}$$

Note from (33) that $\kappa_n \geq 0$ (D and U being p.g.f.'s they have non-negative derivatives of all orders) and also from (31) that when $y = 1$ then $z = 1$. Thus, $D(z(1)) = \sum_{n=0}^{\infty}\kappa_n = 1$ and hence κ_n, $n = 0, 1, 2, \ldots$, is a probability distribution on the non–negative integers with corresponding p.g.f. given by

$$K(y) := \sum_{n=0}^{\infty} y^n \kappa_n.$$

We can use the standard Wiener-Hopf decomposition argument as follows. We can write

$$\begin{aligned}
\frac{1}{1 - K(y)\Theta(y^{-1})} &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}K^n(y)\Theta^n(y^{-1})\right) \\
&= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=0}^{\infty}\kappa_m^{*n}y^m\sum_{l=0}^{\infty}\theta_l^{*n}y^{-l}\right) \\
&= \exp\left(\sum_{r=-\infty}^{\infty} y^r \sum_{n=1}^{\infty}\frac{1}{n}\sum_{\{l:m-l=r\}}\kappa_m^{*n}\theta_l^{*n}\right) \\
&= \exp\left(\sum_{r=1}^{\infty} y^r \sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=1}^{\infty}\kappa_{l+r}^{*n}\theta_l^{*n}\right)\exp\left(\sum_{r=0}^{\infty} y^{-r}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=0}^{\infty}\kappa_m^{*n}\theta_{m+r}^{*n}\right)
\end{aligned}$$

and thus

$$\frac{1}{1-K(y)\Theta(y^{-1})} = \frac{J^-(y^{-1})}{J^+(y)} \tag{36}$$

where,

$$J^+(\zeta) := \exp\left(-\sum_{r=1}^{\infty}\zeta^r \sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=1}^{\infty}\kappa_{l+r}^{*n}\theta_l^{*n}\right), \quad J^-(\zeta) := \exp\left(\sum_{r=0}^{\infty}\zeta^r \sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=0}^{\infty}\kappa_m^{*n}\theta_{m+r}^{*n}\right),$$

are two functions that are analytic at least within the unit disk, $|\zeta| \le 1$. Then from (35) and (36) we have

$$\widetilde{\Pi}(y)J^+(y) = J^-(y^{-1})\sum_{k=0}^{\infty}F_k\sum_{n=1}^{\infty}\theta_{n+k}\left(1-y^{-n}\right). \tag{37}$$

Since the left hand side is obviously bounded for $|y| \le 1$ and the right hand side is bounded for $\left|y^{-1}\right| \le 1$ or $|y| \ge 1$ it follows from Liouville's theorem that both sides of (37) are equal to a constant, say Λ. Thus

$$\widetilde{\Pi}(y) = \frac{\Lambda}{J^+(y)}$$

and

$$\Pi(z) = \frac{\Lambda}{J^+(z/U(z))} = \Lambda\exp\left(\sum_{r=1}^{\infty}z^rU^{-r}(z)\sum_{n=1}^{\infty}\frac{1}{n}\sum_{l=1}^{\infty}\kappa_{l+r}^{*n}\theta_l^{*n}\right).$$

Setting $z = 1$ in the above expression we readily determine the value of Λ from the requirement that $\Pi(1) = 1$. Thus we obtain (34). ∎

The above analysis parallels the analysis of M/G/1 queues with bulk service when the batch size has unbounded support. We refer the reader to Prabhu [28, p. 164]. (See also Kemperman [24] and Keilson [19], [20].)

While the above expression gives the p.g.f. in explicit form, in practice even computation of the first moment would be very arduous. The situation however becomes much simpler if we assume that the processing batch size is geometric or a combination of geometric factors. These cases will be examined in the following subsections.

Finally we compute the expected "actual processing batch size" i.e. the expected contents of the cart each time it is delivered. The argument is the same as in the finite support case and thus $EL = E\Theta - \sum_{k=0}^{\infty}F_k\sum_{n=k+1}^{\infty}\theta_n(n-k)$. This expectation is can be explicitly computed from (29) since $\Pi(1) = 1$ by an application of de l'Hospital's rule. Again, $EL = \frac{\lambda EG}{1-\rho}$ regardless of the processing batch distribution, provided that the stability condition holds.

4.2 Geometric processing batch size

As we saw in the previous subsection, the determination of $\Pi(z)$ for a general processing batch distribution is computationally difficult. However, when the processing batch size is geometrically distributed, one can obtain an explicit, computationally tractable solution. One can start in this case with the factorization problem (36) which has a simple solution. Alternatively one could determine the unknown constants F_k, $k = 0, 1, 2, \ldots$ in (29) directly as follows. Suppose that

$$\theta_n = (1-\gamma)\gamma^{n-1}, \quad n = 1, 2, \ldots \tag{38}$$

with $0 < \gamma < 1$ and thus $\Theta(z) = \frac{(1-\gamma)z}{1-\gamma z}$. Define also the generating function of the sequence $\{F_k\}$ as

$$F(z) := \sum_{k=0}^{\infty} F_k z^k. \tag{39}$$

Theorem 4. *When the processing batch distribution is geometric the p.g.f. of the number of customers left behind by a typical cart departure is given by*

$$\Pi(z) = \frac{(z - U(z))F(\gamma)}{z - \gamma U(z) - (1-\gamma)D(z)U(z)} \tag{40}$$

where

$$F(\gamma) = 1 - (1-\gamma)\frac{\lambda EG}{1-\rho}. \tag{41}$$

Proof: From (29) we obtain

$$\Pi(z) = \frac{(1-\gamma)\sum_{k=0}^{\infty} F_k \sum_{n=1}^{\infty} \gamma^{n+k-1}(1-\alpha(z)^n)}{1 - D(z)\frac{(1-\gamma)\alpha(z)}{1-\gamma\alpha(z)}}$$

which, using (39) simplifies into (40). The unknown quantity $F(\gamma)$ in (40) is determined using de l'Hospital's rule and the fact that $\Pi(1) = 1$. ∎

Once $\Pi(z)$ has been determined, it is straightforward to evaluate the steady state distribution for the number of customers in the system as we will see in the sequel. We point out that the above model corresponds to the situation where, after each service completion the server "flips a coin" and with probability γ he decides to serve another customer, if one is available or take a vacation if a customer is not available. With probability $1-\gamma$ the server takes a vacation regardless of whether there are customers waiting in line or not. At the end of each vacation the server returns to the queue and, if empty, he immediately takes another vacation whereas if not then the "coin-flipping procedure" begins again. This is the Bernoulli vacation model (see Keilson and Servi [21] and Doshi [12]).

4.3 Linear combination of geometric factors

More generally, we may assume that the processing batch size is a linear combination of geometric factors, i.e.

$$\theta_n = \sum_{s=1}^{S} c_s(1-\gamma_s)\gamma_s^{n-1}, \qquad n = 1, 2, \ldots, \tag{42}$$

where $0 < \gamma_s < 1$, the γ_s's are assumed to be different from each other, and the c_s's are such that $c_s \neq 0$, $\sum_{s=1}^{S} c_s = 1$, and $\theta_n \geq 0$, $\forall n \in \mathbb{N}$. Then

$$\Theta(z) = \sum_{s=1}^{S} c_s \frac{(1-\gamma_s)z}{1-\gamma_s z}$$

and with the definition (39) we have the following

Theorem 5. *The p.g.f. of the number of customers left behind is given by*

$$\Pi(z) = \frac{(z-U(z))\sum_{s=1}^{S} F(\gamma_s)c_s \prod_{r\neq s}(z-\gamma_r U(z))}{\prod_{s=1}^{S}(z-\gamma_s U(z)) \; - \; D(z)U(z)\sum_{s=1}^{S} c_s(1-\gamma_s)\prod_{r\neq s}(z-\gamma_r U(z))}. \tag{43}$$

The denominator of the above expression has precisely S roots inside the unit disk, $|z| < 1$, say $z_1, z_2, \ldots, z_s$, and the S unknown constants $F(\gamma_s)$, $s = 1, 2, \ldots, S$, are obtained by the solution to the following system

$$\sum_{s=1}^{S} F(\gamma_s) \frac{c_s}{z_t - \gamma_s U(z_t)} = 0, \quad t = 1, 2, \ldots, S-1, \tag{44}$$

$$\sum_{s=1}^{S} F(\gamma_s) \frac{c_s}{1 - \gamma_s} = E\Theta - \frac{\lambda E G}{1 - \rho}. \tag{45}$$

Proof: Substituting (42) into (29) we obtain

$$\begin{aligned} \Pi(z) &= \frac{\sum_{k=0}^{\infty} F_k \sum_{n=1}^{\infty} \sum_{s=1}^{S} c_s (1-\gamma_s) \gamma_s^{n+k-1} (1 - \alpha(z)^n)}{1 - D(z) \sum_{s=1}^{S} c_s \frac{(1-\gamma_s)\alpha(z)}{1-\gamma_s \alpha(z)}} \\ &= \frac{(z - U(z)) \sum_{s=1}^{S} F(\gamma_s) \frac{c_s}{z - \gamma_s U(z)}}{1 - D(z) \sum_{s=1}^{S} c_s \frac{(1-\gamma_s)U(z)}{z - \gamma_s U(z)}} \end{aligned}$$

whence (43) follows after some simplifications. The S unknown constants, $F(\gamma_s)$, $s = 1, 2, \ldots, S$, can be determined from a standard argument using Rouché's theorem as follows. If we set

$$f(z) := \prod_{s=1}^{S} (z - \gamma_s U(z))$$

and

$$g(z) := - D(z) U(z) \sum_{s=1}^{S} c_s (1 - \gamma_s) \prod_{r \neq s} (z - \gamma_r U(z)) = -D(z) U(z) \Theta(z) f(z)$$

then, it is easy to see that the function f has precisely S roots within the disc $|z| < 1$. Indeed, when $\gamma_s \in (0, 1)$ the equation $z = \gamma_s U(z)$ has a unique, real solution $r_s \in (\gamma_s, 1)$. On the circle $|z| = 1 - \varepsilon$ (where ε is chosen so small that the contour contains $r_1, \ldots, r_s$) $|g(z)| \leq |D(z)| \, |U(z)| \, |\Theta(z)| \, |f(z)| \leq (1 - \varepsilon)^3 \, |f(z)| < |f(z)|$, thus Rouché's theorem applies. Hence the denominator of (43) has precisely S roots within the circle $|z| = 1 - \varepsilon$, say $z_1, z_2, \ldots, z_s$. These must also be roots of the numerator of (43). The equation $z = U(z)$ has precisely two roots, 1, and a real root greater than 1, when $U'(1) = \rho < 1$. Thus the factor $(z - U(z))$ in the numerator of (43) cannot vanish inside the circle $|z| \leq 1 - \varepsilon$. Furthermore,

$$\prod_{s=1}^{S} (z_t - \gamma_s U(z_t)) \neq 0 \quad \text{for } t = 1, 2, \ldots, S. \tag{46}$$

Indeed, if $\prod_{s=1}^{S} (z_{t_1} - \gamma_s U(z_{t_1})) = 0$ for some t_1, then $z_{t_1} - \gamma_{s_1} U(z_{t_1}) = 0$ for some s_1. Since z_{t_1} is a root of the denominator of (43),

$$D(z_{t_1}) U(z_{t_1}) \sum_{s=1}^{S} c_s (1 - \gamma_s) \prod_{r \neq s} (z_{t_1} - \gamma_r U(z_{t_1})) = 0$$

and hence, $c_{s_1} (1 - \gamma_{s_1}) \prod_{r \neq s_1} (z_{t_1} - \gamma_r U(z_{t_1})) = 0$. This implies in turn that $z_{t_1} - \gamma_{s_2} U(z_{t_1}) = 0$ for some $s_2 \neq s_1$. But then $z_{t_1} - \gamma_{s_1} U(z_{t_1}) = 0 = z_{t_1} - \gamma_{s_2} U(z_{t_1})$ which implies $\gamma_{s_1} = \gamma_{s_2}$ which is impossible. Thus, dividing the numerator with the left hand side of (46) we have

$$\sum_{s=1}^{S} F(\gamma_s) \frac{c_s}{z_t - \gamma_s U(z_t)} = 0, \qquad t = 1, 2, \ldots, S.$$

One of the above equations is in fact redundant and has to be replaced by the condition obtained by the requirement that $\Pi(1) = 1$ which, applying de l'Hospital's rule, gives

$$\sum_{s=1}^{S} F(\gamma_s) \frac{c_s}{1 - \gamma_s} = E\Theta - \frac{\lambda EG}{1 - \rho}.$$

From (44) and (45) we obtain the values of the S unknown constants $F(\gamma_s)$, $s = 1, 2, \ldots, S$. ∎

4.4 Vacation length depending on whether the processing batch is complete

Here we examine a variation of the above model according to which the distribution of the vacation length depends on whether the server completed the processing batch that preceded it or whether it was incomplete. In the context of the server failure model we suppose that, if $L_m = \Theta_m$ then a failure has occurred and therefore the subsequent vacation period has distribution G (corresponding to full repair) whereas if $L_m < \Theta_m$ this means that the subsequent vacation period will have distribution G_{inc}. One easily sees that (30) still holds while now

$$Q_0(z) = \sum_{n=1}^{\infty} \theta_n \left(D_{\text{inc}}(z) \sum_{k=1}^{n-1} F_k + D(z) Q_n(z) \right). \tag{47}$$

which we can also write as

$$Q_0(z) = (D_{\text{inc}}(z) - D(z)) \sum_{n=1}^{\infty} \theta_n \sum_{k=1}^{n-1} F_k + D(z)\Pi(z)$$

Thus we have

$$\sum_{n=1}^{\infty} \left(\alpha(z)^n Q_0(z) + \sum_{k=0}^{n-1} F_k \left(1 - \alpha(z)^{n-k}\right) \right) \theta_n = \Pi(z).$$

$$\left((D_{\text{inc}}(z) - D(z)) \sum_{n=1}^{\infty} \theta_n \sum_{k=1}^{n-1} F_k + D(z)\Pi(z) \right) \Theta(\alpha(z)) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} F_k \left(1 - \alpha(z)^{n-k}\right) \theta_n = \Pi(z)$$

or

$$\Pi(z) = \frac{\Theta(\alpha(z)) \left(D_{\text{inc}}(z) - D(z)\right) \sum_{n=1}^{\infty} \theta_n \sum_{k=1}^{n-1} F_k + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} F_k \left(1 - \alpha(z)^{n-k}\right) \theta_n}{1 - \Theta(\alpha(z)) D(z)}.$$

In the case of geometric processing batches (i.e. constant probability of failure) where θ_n is given by (38) we obtain

$$\Pi(z) = \frac{((1-\gamma)U(z)(D_{\text{inc}}(z) - D(z)) + z - U(z))F(\gamma)}{z - \gamma U(z) - (1-\gamma)D(z)U(z)}.$$

The unknown $F(\gamma)$ is again determined by de l'Hospital's rule and is seen to be equal to

$$F(\gamma) = \frac{1 - \rho - (1-\gamma)\lambda EG}{1 - \rho + (1-\gamma)\lambda \left(EG_{\text{inc}} - EG\right)}.$$

Of course, in a reliability context, preventive maintenance would be useless in this case and hence $D_{\text{inc}}(z) = 1$ and $EG_{\text{inc}} = 0$. The case of combination of geometric batches as well as the general approach via the Wiener-Hopf decomposition can be treated by adopting the analysis of sections 4.3 and 4.1 *mutatis mutandis*.

5 Time-stationary distribution of the number of customers in the queue and sojourn times

As we saw in the previous section (T_m, Φ_m), $m \in \mathbb{Z}$, is a Markov–renewal process and that the process $\{X_t; t \in \mathbb{R}\}$ is semi–regenerative with respect to this Markov–renewal process. Furthermore, it is possible to see that, under the stability condition (1), the Markov chain $\{\Phi_m; m \in \mathbb{Z}\}$ is positive recurrent.

Consider the basic epochs $\{T_m\}$ when the server leaves the queue in order to deliver the cart and a vacation period begins. Under the stability condition, it is clear that there exists a steady–state regime since this is a semi–regenerative system. (Alternatively, we could identify ordinary regeneration cycles corresponding to the epochs when the server leaves the queue *empty* to deliver the cart.) It is also possible to show that these regenerative cycles have finite mean and thus there exists a steady state random variable, say X_∞, such that $X_t \stackrel{d}{\to} X_\infty$ as $t \to \infty$ (where $\stackrel{d}{\to}$ denotes convergence in distribution). We shall establish the following

Theorem 6. *The stationary number of customers in the system when the server uses a partial batch policy has p.g.f. given by*

$$Ez^{X_\infty} = \Pi(z) G_i^*(\lambda(1-z)) \frac{(1-\rho)B^*(\lambda(1-z))}{1-\rho B_i^*(\lambda(1-z))}, \tag{48}$$

where $\Pi(z)$ *is the p.g.f. of the number of customers present in the system at the beginning of a typical vacation. Depending on whether the processing batch size distribution has bounded or unbounded support,* $\Pi(z)$ *is given by (27) or by (34).*

Proof: We will establish the theorem assuming that the processing batch size distribution does not necessarily have bounded support. We begin with a version of the process which satisfies the following conditions: *(i)* The time origin coincides with the beginning of a "typical" cycle, i.e. $\cdots < T_{-2} < T_{-1} < T_0 = 0 < T_1 < T_2 < \cdots$ and *(ii)* $\Phi_0 = X_{T_0} = X_0$ is distributed according to the (jump) stationary distribution of the Markov Chain $\{\Phi_m; m \in \mathbb{Z}\}$. If we denote by λ^* the rate of the process $\{T_m\}$ we then have the following formula connecting the distribution of X_∞ to that of $\{X_t; t \in [T_0, T_1)\}$. For any bounded function $f : \mathbb{N} \to \mathbb{R}$,

$$Ef(X_\infty) = \lambda^* E \int_{T_0}^{T_1} f(X_s) ds.$$

In particular, if we take $f(x) = z^x$ (where $0 \le z \le 1$) we have the following expression for the p.g.f. of the time stationary distribution of the number of customers in the queue:

$$Ez^{X\infty} = \lambda^* E \int_{T_0}^{T_1} z^{X_s} ds. \tag{49}$$

The formulae above can be thought of as consequence of the semi-regenerative nature of the system (see [7]). Alternatively, if one is willing to use the language of stationary processes these are special cases of the Palm inversion formula (see Baccelli and Brémaud [3]). The integral on the right hand side of (49) can be split into two parts,

$$I_1 := \int_{T_0}^{S_0} z^{X_s} ds; \qquad \text{and} \qquad I_2 := \int_{S_0}^{T_1} z^{X_s} ds.$$

The first term is analyzed by conditioning on the size of the processing batch. On the event $\{\Theta = n\}$ it splits into a sum of n terms as follows

$$I_1 = \sum_{i=0}^{n-1} \mathbf{1}(L > i, \Theta = n) \int_{d_i}^{d_{i+1}} z^{X_s} ds.$$

Since $X_s = X_{d_i} + A(d_i, s]$ where $A(d_i, s]$ is the number of Poisson arrivals in the interval $(d_i, s]$, we can write

$$EI_1 = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} E\left[\mathbf{1}(L > i, \Theta = n) \int_{d_i}^{d_{i+1}} z^{X_{d_i} + A(d_i,s]} ds\right].$$

Note that, because of the independent increments property of the Poisson arrival process,

$$\begin{aligned} E\left[\mathbf{1}(L > i, \Theta = n) \int_{d_i}^{d_{i+1}} z^{X_{d_i}+A(d_i,s]} ds\right] &= E\left[\mathbf{1}(L > i, \Theta = n) z^{X_{d_i}} \int_{d_i}^{d_{i+1}} e^{-\lambda(s-d_i)(1-z)} ds\right] \\ &= E\left[\mathbf{1}(L > i, \Theta = n) z^{X_{d_i}} \frac{1 - e^{-\lambda(d_{i+1}-d_i)(1-z)}}{\lambda(1-z)}\right] \\ &= \frac{1 - B^*(\lambda(1-z))}{\lambda(1-z)} E\left[\mathbf{1}(L > i, \Theta = n) z^{X_{d_i}}\right] \end{aligned}$$

where, in the above derivation we have used the fact that $E[e^{-s(d_{i+1}-d_i)} \mid L > i, \Theta = n] = B^*(s)$ and $d_{i+1} - d_i$ is independent of X_{d_i} on $\{L > i\}$. Also, taking into account (2), (3), (4), (5), and the fact that $i < n$ we have that

$$\begin{aligned} E\left[\mathbf{1}(L > i, \Theta = n) z^{X_{d_i}}\right] &= E\left[\mathbf{1}(L \geq i, \Theta = n) z^{X_{d_i}}\right] - E\left[\mathbf{1}(L = i, \Theta = n) z^{X_{d_i}}\right] \\ &= E\left[\mathbf{1}(L \geq i) z^{X_{d_i}}\right] P(\Theta = n) - F_i P(\Theta = n) \\ &= (Q_i(z) - F_i)\, \theta_n. \end{aligned}$$

Hence, taking into account (6) and (8) we have

$$EI_1 = \frac{1 - zy^{-1}}{\lambda(1-z)} \sum_{n=1}^{\infty} \theta_n \sum_{i=0}^{n-1} (Q_i(z) - F_i)\,. \tag{50}$$

Using (9) (which as we saw holds regardless of whether the processing batch size has bounded support or not) we obtain

$$\sum_{i=0}^{n-1} (Q_i(z) - F_i) = \sum_{i=1}^{n} Q_i(z) y.$$

Elementary manipulations yield

$$\sum_{i=1}^{n} Q_i(z) \;=\; Q_n(z) \frac{y^n - 1}{y - 1} \;+\; \sum_{j=1}^{n-1} F_j \frac{y^j - 1}{y - 1}$$

and thus (50) can be written as

$$EI_1 \;=\; \frac{y-z}{\lambda(1-z)(y-1)}\sum_{n=1}^{\infty}\theta_n\left(Q_n(z)\,(y^n-1)+\sum_{j=1}^{n-1}F_j\,(y^j-1)\right).$$

Using (16) in the above expression, we can rewrite EI_1 after some algebraic manipulations as

$$EI_1 \;=\; (D-1)\frac{y-z}{\lambda(1-z)(y-1)}\sum_{n=1}^{\infty}\theta_n\left(Q_n(z)+\sum_{j=0}^{n-1}F_j\right) \;=\; \frac{1-D}{\lambda(1-z)}\frac{y-z}{1-y}\Pi(z)$$

where, in the second equation we have used (30).

On the other hand, the expectation of I_2, is given by

$$EI_2 = E[z^{\Psi}\int_0^G z^{A(0,s]}ds] = \Pi(z)\frac{1-D(z)}{\lambda(1-z)}. \tag{51}$$

Thus adding the two equations above term by term we have

$$EI_1 + EI_2 = \Pi(z)\frac{1-D(z)}{\lambda(1-z)}\left(1+\frac{y-z}{1-y}\right).$$

From the above, after some elementary manipulations we obtain

$$Ez^{X_\infty} = \lambda^*(EI_1+EI_2) = \lambda^*\Pi(z)G_i^*(\lambda(1-z))EG\frac{B^*(\lambda(1-z))}{1-\rho B_i^*(\lambda(1-z))},$$

where the rate λ^* can be computed from the normalization requirement by setting $z=1$ in the above relationship. Indeed,

$$\lambda^* = \frac{1-\rho}{EG} \tag{52}$$

and this completes the proof of the theorem. ∎

Remark: The representation of the p.g.f. of the number of customers in stationarity can be interpreted as a *decomposition into three parts* of the type one should expect in view of the well known properties of M/G/1 queues with vacations (see [17] and also [12], [16], and [23]). The term $\frac{(1-\rho)B^*(\lambda(1-z))}{1-\rho B_i^*(\lambda(1-z))}$ is of course the p.g.f. the number of customers in a steady state M/G/1 queue without vacations, the term $G_i^*(\lambda(1-z))$ is the p.g.f. of the number of Poisson arrivals during the forward recurrence time of a typical vacation, and finally $\Pi(z)$ is the p.g.f. of the number of customers present in the system at the beginning of a typical vacation. Of course, this decomposition holds because of the partial batch policy used.

Corollary 7. *In particular, when the processing batch size is geometric, i.e.* $\theta_n = (1-\gamma)\gamma^{n-1}$, $n = 1,2,3,\ldots$, *the p.g.f. of the number of customers in the system in steady state is given by*

$$Ez^{X_\infty} = \frac{1}{\lambda EG}\frac{(1-\rho)U(z)F(\gamma)(1-D(z))}{z-\gamma U(z)-(1-\gamma)D(z)U(z)} \tag{53}$$

where $F(\gamma)$ *is given by (41).*

Proof: Use (40) for $\Pi(z)$ in theorem 2. ∎

5.1 Sojourn time distribution

The sojourn time is obtained easily from the above formula via the distributional version of Little's law (see [4], [22], [23], and [33].) Indeed, setting $s = \lambda(1-z)$ in (48) we obtain

$$T(s) = \Pi(1 - s/\lambda) G_i^*(s) \frac{(1-\rho)B^*(s)}{1-\rho B_i^*(s)}, \tag{54}$$

where, $\Pi(1 - s/\lambda)$ can be computed from (27). After the necessary simplifications, taking into account that $\alpha(1 - s/\lambda) = B^*(s)\frac{\lambda}{\lambda+s}$, we have

$$\Pi(1 - s/\lambda) = \frac{\sum_{k=0}^{N-1} F_k \sum_{n=k+1}^{N-1} \theta_n \left(1 - B^*(s)^{n-k}\left(\frac{\lambda}{\lambda-s}\right)^{n-k}\right)}{1 - G^*(s)\sum_{n=1}^{n} \theta_n B^*(s)^n \left(\frac{\lambda}{\lambda-s}\right)^n}.$$

It should be pointed out that (54) gives the total time from the moment a customer enters the queue to the moment he enters the cart. The additional delay due to the time the customer has to wait until the cart is delivered is not included. In fact it is not possible to do this using the distributional version of Little's law, since the total sojourn time of a customer in this case depends on future arrivals as well.

In the case of the geometric batch transfer size, setting $z = 1 - s/\lambda$ in (53) and carrying out the necessary simplifications we obtain

$$T(s) = \frac{(1-\rho)G_i^*(s)B^*(s)F(\gamma)}{1 - \rho B_i^*(s) - (1-\gamma)\rho_G B^*(s) G_i^*(s)}.$$

6 The "complete batch" policy

So far we have carried out the analysis assuming a partial batch policy. Alternative strategies can also be analyzed, as in [11]. In this section we sketch the analysis for the *complete batch policy*. According to this policy, each time the server returns with the cart to the system, a random variable representing the processing batch size is realized. The server keeps serving customers until this processing batch size is completed (waiting for new arrivals if the queue empties) and as soon as the batch is completed he departs to deliver the cart thus initiating a vacation period. Upon returning to the system, a new processing batch is set and the whole process repeats itself. The starting point in our analysis is to realize that, with the given policy, each service phase consists of a complete batch so that $L_m = \Theta_m$. If we define

$$R_k(z) := E\left[z^{X_{d_k}} | \Theta \geq k\right] = E\left[z^{X_{d_k}} | \Theta = k\right],$$

the system dynamics in this case are described by

$$R_k(z) = \left(\frac{R_{k-1}(z) - H_{k-1}}{z} + H_{k-1}\right) U(z), \tag{55}$$

where $H_k := R_k(0)$. This in turn with the notation of (8), upon iteration, gives

$$\alpha^n R_0(z) = R_n(z) + (1-z)\left(H_0\alpha^n + H_1\alpha^{n-1} + \cdots + H_{n-1}\alpha\right). \tag{56}$$

Since we still have $\Pi(z) = \sum_{n=1}^{\infty} \theta_n R_n(z)$ and

$$R_0(z) = \Pi(z) D(z), \tag{57}$$

from the above we obtain

$$\Pi(z) = (z-1)\frac{\sum_{n=1}^{\infty}\theta_n \sum_{j=0}^{n-1} H_j \alpha^{n-j}}{1 - D(z)\Theta(\alpha)}. \tag{58}$$

When the batch size distribution has finite support, say the set $\{1, 2, \ldots, N\}$, then the denominator is the same as in the corresponding expression for the partial batch policy in section 3.2. Thus the N unknown constants, $H_0, H_1, \ldots, H_{N-1}$, on which $\Pi(z)$ depends in this case are obtained by Rouché's theorem, as before.

When the batch size distribution has infinite support, in general one has to resort to Wiener-Hopf factorization techniques in order to determine $\Pi(z)$. Of course one can analyze easily the case where the batch size distribution is a combination of geometric factors as in section 4.3. Here we will restrict ourselves to the analysis of the case of geometric batches, i.e. $\theta_n = (1-\gamma)\gamma^n$, $n = 1, 2, \ldots$. Then, arguing as in section 4.2 we see that

$$\Pi(z) = \frac{(z-1)U(z)(1-\gamma)H(\gamma)}{z - \gamma U(z) - (1-\gamma)D(z)U(z)}$$

where

$$H(\gamma) = \frac{1-\rho}{1-\gamma} - \lambda EG$$

as can be seen from an argument using the fact that $\Pi(1) = 1$ and de l'Hospital's rule.

Finally we determine the stationary distribution of the number of customers in the system (excluding the cart) under the complete batch policy. We indicate the differences in this case, illustrating the case of geometric processing batches. With the notation of the section 5 we have

$$\begin{aligned}
EI_1 &= \sum_{n=1}^{\infty}\theta_n \sum_{i=0}^{n-1} E\left[\int_{d_i}^{d_{i+1}} z^{X_s} ds \big| \, \Theta = n\right] \\
&= \sum_{n=1}^{\infty}\theta_n \sum_{i=0}^{n-1}\left(\lambda^{-1}\left(1 + z\tfrac{1-U(z)}{1-z}\right) P(X_{d_i} = 0 | \, \Theta = n) + E\left[z^{X_{d_i}}\mathbf{1}(X_{d_i} > 0)\big| \, \Theta = n\right]\lambda^{-1}\tfrac{1-U(z)}{1-z}\right) \\
&= \sum_{n=1}^{\infty}\theta_n \sum_{i=0}^{n-1}\left(\lambda^{-1}\left(1 + z\frac{1-U(z)}{1-z} - \frac{1-U(z)}{1-z}\right) H_i + R_i(z)\lambda^{-1}\frac{1-U(z)}{1-z}\right) \\
&= \sum_{n=1}^{\infty}\theta_n \sum_{i=0}^{n-1}\left(\lambda^{-1}U(z)H_i + R_i(z)\lambda^{-1}\frac{1-U(z)}{1-z}\right)
\end{aligned}$$

Recall that, by definition $R_i(z) := E[z^{X_{d_i}}|\Theta = i]$ and $H_i = P(X_{d_i} = 0 \mid \Theta = i)$. Rewrite (55) as $yR_i = R_{i-1} + (z-1)H_{i-1}$ and obtain

$$\sum_{i=0}^{n-1} R_i(z) = \frac{y}{1-y}\left(R_n(z) - R_0(z)\right) - \frac{z-1}{1-y}\sum_{i=0}^{n-1} H_i.$$

Thus we have

$$\begin{aligned}
EI_1 &= \lambda^{-1}\left(U(z) + \frac{1-U(z)}{1-y}\right)\sum_{n=1}^{\infty}\theta_n \sum_{i=0}^{n-1} H_i + \lambda^{-1}\frac{1-U(z)}{1-z}\sum_{n=1}^{\infty}\theta_n \frac{y}{1-y}\left(R_n(z) - R_0(z)\right) \\
&= \lambda^{-1}U\frac{1-z}{U-z}C' + \lambda^{-1}\frac{1-U(z)}{1-z}\frac{z}{U-z}\sum_{n=1}^{\infty}\theta_n\left(R_n(z) - R_0(z)\right) \\
&= \lambda^{-1}U\frac{1-z}{U-z}C' + \lambda^{-1}\frac{1-U}{1-z}\frac{z}{U-z}\left(1-D\right)\Pi(z)
\end{aligned}$$

where we have set $C' := \sum_{n=1}^{\infty} \theta_n \sum_{i=0}^{n-1} H_i$ and taken into account that $\Pi(z) = \sum_{n=1}^{\infty} \theta_n R_n(z)$ and $R_0(z) = \Pi(z)D(z)$. On the other hand (51) still holds and thus

$$\lambda^*(EI_1 + EI_2) \;=\; \frac{\lambda^*}{\lambda} U \frac{1-z}{U-z} C' + \frac{\lambda^*}{\lambda} \Pi(z) \frac{1-D}{U-z}. \tag{59}$$

The value of C' can be determined from (58) using the observation that $\Pi(1) = 1$ and de l' Hospital's rule:

$$C' = (1-\rho)E\Theta - \lambda EG.$$

Since, as in section 5 $Ez^{X_\infty} = \lambda^*(EI_1 + EI_2)$, we can determine λ^* by setting $z = 1$ in (59) and using once more de l' Hospital's rule. Thus we obtain

$$\frac{\lambda}{\lambda^*} = E\Theta. \tag{60}$$

Putting things together we obtain

$$\begin{aligned} Ez^{X_\infty} &= (1-\rho)U\frac{1-z}{U-z}\left(1 - \frac{\lambda EG}{1-\rho}\right) + \frac{1}{E\Theta}\Pi(z)\frac{1-D}{U-z} \\ &= (1-\rho)U\frac{1-z}{U-z}\left(1 - \frac{\lambda EG}{E\Theta(1-\rho)}\right) + \frac{\lambda EG}{E\Theta(1-\rho)}\Pi(z)\frac{1-D}{\lambda EG(1-z)}(1-\rho)\frac{1-z}{U-z} \end{aligned}$$

or

$$Ez^{X_\infty} \;=\; (1-p)(1-\rho)U\frac{1-z}{U-z} + p\Pi(z)G_i^*(\lambda(1-z))(1-\rho)\frac{1-z}{U-z} \tag{61}$$

where $p = \frac{\lambda EG}{E\Theta(1-\rho)}$, $\Pi(z)$ as given in (58) is the p.g.f. of the number of customers left behind at the end of the typical service phase, $G_i^*(\lambda(1-z))$ the p.g.f. of the number of Poisson arrivals during the residual service time of a vacation period and finally $(1-\rho)U\frac{1-z}{U-z}$ is the p.g.f. of the stationary number of customers in the corresponding M/G/1 system without vacations (in that case the size of the processing batch becomes irrelevant). Note that the second term on the right hand side of (61) includes the term $(1-\rho)\frac{1-z}{U-z}$ which is the generating function of the number of Poisson arrivals during the *waiting time* in the corresponding M/G/1 system without vacations.

7 Bulk arrivals

There are no significant changes in the above analysis if we assume that customers arrive not singly but in batches. Arrival epochs are still Poisson (λ) and the arriving batches are an i.i.d. sequence of random variables $\{\beta_n\}$, independent of the Poisson arrival process, with common distribution $P(\beta = k) = b_k$, $k = 1, 2, 3, \ldots$. The corresponding p.g.f. will be denoted by $b(z) := \sum_{k=1}^{\infty} b_k z^k$ and the mean batch size by $m_b = \sum_{k=1}^{\infty} k b_k$. In order not to obscure the main features of the problem we will introduce here the simplifying assumption that the processing batch size sequence $\{\Theta_m\}$ is deterministic and equal to the cart capacity N. In this case, the stability condition becomes $N > \frac{\lambda EG}{1-\lambda E\beta E\sigma}$.

We can analyze this system in precisely the same way as the single customer arrival case. Indeed, equations (10) and (15) hold unchanged, if we substitute for $U(z)$ and $D(z)$ the p.g.f.'s

$$U_b(z) := B^*(\lambda(1-b(z))), \quad D_b(z) := G^*(\lambda(1-b(z))).$$

Then $\Pi_b(z)$, the p.g.f. of the number of customers left behind in the queue at a typical vacation start is given by the relationship

$$\Pi_b(z) = \frac{\sum_{k=0}^{N-1} F_{b,k} z^k \left(z^{N-k} - U_b(z)^{N-k}\right)}{z^n - D_b(z)U_b(z)^n}. \tag{62}$$

where, as before, the N constants $F_{b,k}$, $k = 0, 1, 2, \ldots, N-1$ are obtained by Rouché's theorem.

An analysis entirely analogous to that of section 5 gives the following expression for the p.g.f. of the stationary number of customers in the system, X_∞.

$$Ez^{X_\infty} = \Pi_b(z) G_i^*(\lambda(1-b(z))) \frac{(1-\rho m_b)B^*(\lambda(1-z))}{1-\rho B_i^*(\lambda(1-z))},$$

which assumes again the form of a three way decomposition. The term $\frac{(1-\rho m_b)B^*(\lambda(1-z))}{1-\rho B_i^*(\lambda(1-z))}$ is the p.g.f. of the time-stationary number of customers in an M/G/1 queue with bulk arrivals and without vacations, the term $G_i^*(\lambda(1-b(z)))$ is the p.g.f. of the total number of arrivals during the forward recurrence time of a typical vacation; and finally $\Pi_b(z)$ is the p.g.f. of the number of customers present in the system at the beginning of a typical vacation.

8 The contents of the cart when it is delivered

When the partial batch policy is used the contents of the cart when it is delivered or "actual processing batch size" is a random variable stochastically smaller than the processing batch size. Its distribution in stationarity is given by the following

Theorem 8. *The typical contents of the cart when it is delivered has distribution given by*

$$P(L=n) = \begin{cases} F_0 & \text{if } n = 0 \\ \theta_n \left(1 - \sum_{k=0}^{n-1} F_k\right) + F_n P(\Theta > n) & \text{if } n \geq 1 \end{cases}. \tag{63}$$

Proof: Define the generating functions $\Upsilon(z,w) := \sum_{k=0}^{\infty} Q_k(z) w^k$ and $F(w) := \sum_{k=0}^{\infty} F_k w^k = \Upsilon(0,w)$. Then, from (9) it follows that

$$\Upsilon(z,w) - Q_0(z) = \sum_{k=0}^{\infty} Q_{k+1}(z) w^{k+1} = w\alpha \left(\sum_{k=0}^{\infty} Q_k(z) w^k - \sum_{k=0}^{\infty} F_k w^k \right)$$

or, recalling definition (8), after some elementary manipulations,

$$\Upsilon(z,w) = \frac{zQ_0(z) - F(w)wU(z)}{z - wU(z)}. \tag{64}$$

The above expression involves the unknown function $F(w)$ which can be determined as follows. Suppose that $|w| < 1$. The equation $z - wU(z) = 0$ has for each fixed value of w in the unit disk a unique solution $\zeta(w)$. (This can be seen by an application of Rouché's theorem, see Takács [30]). In fact

$$\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n!} w^n \left. (d/dt)^{n-1} U^n(t) \right|_{t=0}$$

according to the Lagrange inversion formula. The numerator of (64) must also vanish when $z = \zeta(w)$ and thus $\zeta(w)Q_0(\zeta(w)) = F(w)wU(\zeta(w))$ or $F(w) = Q_0(\zeta(w))$. Taking into account (14) as well we have

$$F(w) = \Pi(\zeta(w))D(\zeta(w)). \tag{65}$$

As we saw in section 4.1, $D(\zeta(w)) = \sum_{n=0}^{\infty} \kappa_n w^n$ with $\kappa_0 = D(0)$ and $\kappa_n = \frac{1}{n!}\left.(d/dt)^{n-1} D'(t) U^n(t)\right|_{t=0}$, $n = 1, 2, \ldots$. Using (34) and the fact that $\zeta(w)/U(\zeta(w)) = w$ we thus obtain the generating function for the sequence $\{F_n\}$ as follows

$$F(w) = \exp\left(\sum_{r=1}^{\infty} (w^r - 1) \sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=1}^{\infty} \kappa_{l+r}^{*n} \theta_l^{*n}\right) \sum_{n=0}^{\infty} \kappa_n w^n. \tag{66}$$

Once the sequence $\{F_n\}$ has been determined, the number of customers in the actual processing batch size is obtained by first conditioning on the processing batch size as follows. We have

$$E\left[w^L \,|\Theta = n\right] = \sum_{k=0}^{n-1} w^k F_k + w^n \left(1 - \sum_{k=0}^{n-1} F_k\right) = w^n + \sum_{k=0}^{n-1} F_k \left(w^k - w^n\right)$$

and thus

$$\begin{aligned} Ew^L &= \Theta(w) + \sum_{k=0}^{\infty} w^k F_k \sum_{n=k+1}^{\infty} \theta_n - \sum_{n=1}^{\infty} w^n \theta_n \sum_{k=0}^{n-1} F_k \\ &= \Theta(w) + \sum_{k=0}^{\infty} w^k F_k P(\Theta > k) - \sum_{n=1}^{\infty} w^n \theta_n \sum_{k=0}^{n-1} F_k \\ &= F_0 + \sum_{n=1}^{\infty} w^n \left(\theta_n \left(1 - \sum_{k=0}^{n-1} F_k\right) + F_n P(\Theta > n)\right). \end{aligned} \tag{67}$$

which gives (63). ∎

Things of course become simpler when the processing batch size is geometric, as in section 4.2. Then, setting $K(w) := D(\zeta(w)) = \sum_{n=0}^{\infty} \kappa_n w^n$ (65) becomes

$$F(w) = K(w) \frac{(\zeta(w) - U(\zeta(w)))F(\gamma)}{\zeta(w) - \gamma U(\zeta(w)) - (1-\gamma)K(w)U(\zeta(w))} = \frac{F(\gamma)K(w)(1-w)}{-w + \gamma + (1-\gamma)K(w)}$$

with $F(\gamma)$ given by (41). The mean of the probability distribution $\{\kappa_n\}$ is given by $K'(0) = D'(0)\zeta'(0)$ and of course $\zeta'(0) = U(0) = 1$, thus $K'(0) = \lambda EG$. Hence, if we define the distribution function

$$K_i(w) := \frac{1}{K'(0)} \frac{1 - K(w)}{1 - w}$$

we have

$$F(w) = \frac{F(\gamma)K(w)}{1 - (1-\gamma)\lambda EG K_i(w)}. \tag{68}$$

When the processing batch size is geometric (67) simplifies into the following expression

$$Ew^L = \frac{1 - \gamma + (1-w)F(\gamma w)}{1 - \gamma w}$$

which, together with (68) gives the generating function of the number of items delivered.

9 The contents of the cart in steady state

In this section we will use the interpretation that refers to the queue and cart system and we will assume that the cart has random capacity. We are interested in the cart as long as it is "next to the server", receiving customers, so we will suppose that the number-in-the-cart process, $\{Y_t; t \in \mathbb{R}\}$, becomes equal to zero as soon as the server takes the cart to deliver it (see figure 2). Random cart capacity processing batches can be analyzed in a similar fashion as in the previous sections. In the first subsection we examine the marginal distribution of the cart contents under the partial batch policy, in the second the marginal distribution of the cart contents under the complete batch policy, while in the third the joint distribution of the number of customers in the queue and the cart under the partial batch policy. The joint distribution under the complete batch policy can be obtained by similar arguments and this derivation is omitted.

9.1 The marginal distribution under the partial batch policy

If we denote by Y_∞ a random variable with the steady state distribution of the number of customers in the cart

Proposition 9. *The steady–state number of customers in the cart is given by*

$$P(Y_\infty = n) = \begin{cases} (1-\rho)\left(1+\dfrac{E\sigma}{EG}\right) & \text{if } n = 0 \\ \rho \sum_{m=n}^{\infty}\left(\theta_m\left(1-\sum_{k=0}^{m-1}F_k\right)+F_m P(\Theta > m)\right) & \text{if } n \geq 1 \end{cases}.$$

The expected number of customers in the cart in steady state is then equal to

$$EY_\infty = \sum_{n=1}^{\infty}\theta_n \sum_{k=1}^{n-1} k\frac{\rho}{EL}\left(1-\sum_{i=0}^{k}F_i\right) = \rho\frac{E[L(L-1)]}{2EL}.$$

Proof: Denote by $C(w) := Ew^{Y_\infty}$ the p.g.f. of the number of customers in the cart in steady–state. The cycle formula gives

$$C(w) = Ew^{Y_\infty} = \frac{E\int_{T_0}^{T_1} w^{Y_s}ds}{E(T_1-T_0)} = \frac{EG + E\sigma E\left[\sum_{k=1}^{L} w^{k-1}\right]}{EG+ELE\sigma}. \tag{69}$$

An application of Fubini's theorem gives

$$E\left[\sum_{k=1}^{L} w^{k-1}\right] = E\sum_{k=1}^{\infty}\mathbf{1}(L \geq k)w^{k-1} = \sum_{m=0}^{\infty} w^m P(L > m).$$

Also, taking into account that $EL = \frac{\lambda EG}{1-\lambda E\sigma}$ (this is equation 28) and hence $\frac{EG}{EG+ELE\sigma} = 1-\rho$ and $\frac{E\sigma}{EG+ELE\sigma} = \rho\dfrac{1}{EL} = (1-\rho)\frac{E\sigma}{EG}$ we obtain

$$C(w) = 1-\rho+\rho\sum_{m=0}^{\infty} w^m \frac{P(L>m)}{EL}. \tag{70}$$

The above, together with (69) and (63), establishes the proposition. ∎

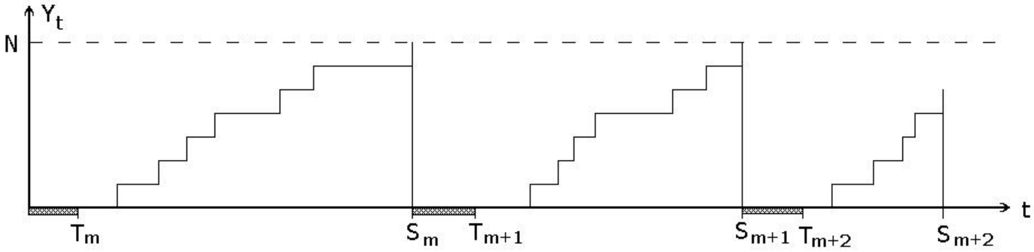

Figure 2: Sample path of cart contents.

9.2 The marginal distribution under the complete batch policy

Under the complete batch policy we have the following

Proposition 10. *The steady–state number of customers in the cart is given by*

$$P(Y_\infty = k) = \begin{cases} \dfrac{1}{E\Theta}\left(\lambda EG + \rho + H_0\right) & \text{if } k = 0 \\ \\ \dfrac{1}{E\Theta}\left(\rho + H_k\right) P(\Theta > k) & \text{if } k \geq 1 \end{cases} .$$

Proof: With the same notation as in section 9.1 we have

$$C(w) = Ew^{Y_\infty} = \frac{E\int_{T_0}^{T_1} w^{Y_s} ds}{E(T_1 - T_0)}. \tag{71}$$

The numerator above is given by

$$E\int_{T_0}^{T_1} w^{Y_s} ds = EG + E\sigma \sum_{n=1}^{\infty} \theta_n \sum_{k=0}^{n-1} w^k + \lambda^{-1} \sum_{n=1}^{\infty} \theta_n \sum_{k=0}^{n-1} H_k w^k \tag{72}$$

while the denominator is

$$E(T_1 - T_0) = EG + \sum_{n=1}^{\infty} \theta_n n E\sigma + \lambda^{-1} \sum_{n=1}^{\infty} \theta_n \sum_{k=0}^{n-1} H_k = \lambda^{-1} E\Theta \tag{73}$$

where we have taken account that $\sum_{n=1}^{\infty} \theta_n \sum_{k=0}^{n-1} H_k = (1-\rho)E\Theta - \lambda EG$. Substituting (71) and (72) into (73) we obtain

$$C(w) = \frac{\lambda EG}{E\Theta} + \frac{1}{E\Theta} \sum_{n=1}^{\infty} \theta_n \sum_{k=0}^{n-1} w^k (\rho + H_k)$$

or

$$C(w) = \frac{\lambda EG}{E\Theta} + \frac{1}{E\Theta} \sum_{k=0}^{\infty} P(\Theta > k) w^k (\rho + H_k).$$

By expanding this last expression for the p.g.f. of the stationary number of customers in the cart completes the proof of the proposition. ∎

9.3 Joint distribution of the number of customers in the queue and the cart under the partial batch policy

Proposition 11. *The joint p.g.f. of the queue and cart contents is given by*

$$\begin{aligned} V(z,w) &= (1-\rho)\Pi(z)\left(G_i^*(\lambda(1-z)) + \frac{E\sigma}{EG}B_i^*(\lambda(1-z))D(z)w\alpha\frac{1-\Theta(w\alpha)}{1-w\alpha}\right) \\ &\quad -(1-\rho)\frac{E\sigma}{EG}B_i^*(\lambda(1-z))\sum_{j=0}^{\infty}F_j w^j\sum_{k=1}^{\infty}(w\alpha)^k P(\Theta\geq k+j). \end{aligned} \tag{74}$$

Proof: Arguing as above, we can obtain with a little more effort the joint distribution of the number of customers in the queue and the cart, $V(z,w) := Ez^{X_\infty}w^{Y_\infty}$ by using the same method as in the analysis of §5. With the notation of §5 we have

$$V(z,w) = \lambda^*\left(E\int_{T_0}^{S_0} z^{X_t}w^{Y_t}dt + E\int_{S_0}^{T_1} z^{X_t}w^{Y_t}dt\right). \tag{75}$$

The integral over the vacation phase, where $Y_t = 0$, is

$$E\int_{S_0}^{T_1} z^{X_t}w^{Y_t}dt = \Pi(z)\frac{1-G^*(\lambda(1-z))}{\lambda(1-z)},$$

where, as in §5, $\Pi(z) = \sum_{n=1}^{\infty}\theta_n(Q_n(z) + \sum_{j=0}^{n-1}F_j)$. Taking into account that at the beginning of a cycle, when the server returns with the cart to the queue and starts serving, $Y_{T_0} = 0$ (i.e. the cart is empty) we have

$$\begin{aligned} E\int_{T_0}^{S_0} z^{X_t}w^{Y_t}dt &= E\sum_{n=1}^{\infty}\theta_n\sum_{k=0}^{n-1}\mathbf{1}(L>k)w^k\int_{d_k}^{d_{k+1}} z^{X_t}dt \\ &= \frac{1-U(z)}{\lambda(1-z)}\sum_{n=1}^{\infty}\theta_n\sum_{k=0}^{n-1}w^k(Q_k(z)-F_k) \\ &= \frac{1-U(z)}{\lambda(1-z)}\sum_{n=1}^{\infty}\theta_n\sum_{k=0}^{n-1}w^kQ_{k+1}(z)y \end{aligned} \tag{76}$$

Using also the recursion (9) together with (8) we can write the last sum in the above equation as $\sum_{k=0}^{n-1}w^k yQ_{k+1}(z) = w^{-1}\sum_{k=1}^{n}w^k yQ_k(z)$. Also, from (10) $Q_k(z) = \alpha^kQ_0(z) - \sum_{j=0}^{k-1}F_j\alpha^{k-j}$ and hence the right hand side of (76) can be written as

$$\begin{aligned} \frac{y-z}{w\lambda(1-z)}\sum_{n=1}^{\infty}\theta_n\sum_{k=1}^{n}w^kQ_k &= \frac{y-z}{w\lambda(1-z)}\sum_{n=1}^{\infty}\theta_n\sum_{k=1}^{n}w^k\left(\alpha^kQ_0 - \sum_{j=0}^{k-1}F_j\alpha^{k-j}\right) \\ &= \frac{y-z}{w\lambda(1-z)}\left(\Pi(z)D(z)w\alpha\frac{1-\Theta(w\alpha)}{1-w\alpha} - \sum_{j=0}^{\infty}F_jy^j\sum_{l=j+1}^{\infty}(w\alpha)^lP(\Theta\geq l)\right) \end{aligned}$$

where we have used the fact that $y = \alpha^{-1}$ and (14). Upon substitution in (75), taking into account (52), we obtain after some simplifications, (74). ∎

10 Appendix

10.1 Stability

In this section of the Appendix we complete the discussion on the stability of the system by furnishing the proofs of the assertions made in the last paragraphs of section 2. We begin by recalling Foster's criterion for positive recurrence (see [2, p.19])

Theorem 12. *Suppose that P_{ij} is the transition probability matrix of a discrete time Markov chain with countable state space E which is irreducible and let E_0 be a finite subset of E. Then:*

(i) the chain is recurrent if there exists a function $h : E \to \mathbb{R}$ which is not bounded on E and satisfies

$$\sum_{k\in E} P_{jk}h(k) \leq h(j), \qquad j \notin E_0.$$

(ii) the chain is positive recurrent if for some $h : E \to \mathbb{R}$ and some $\epsilon > 0$ we have $\inf_{x\in E} h(x) > -\infty$ *and*

$$\begin{aligned}\sum_{k\in E} P_{jk}h(k) &< \infty, \qquad j \in E_0\\ \sum_{k\in E} P_{jk}h(k) &< h(j)-\epsilon, \qquad j \notin E_0.\end{aligned}$$

We will apply the above theorem to the embedded Markov chain $\{\Phi_m\}$ of the number of customers at the beginning of each active period which has state space $\mathbb{N}$ and is clearly irreducible. Taking $h(x) = x$ to be the identity function we shall establish that $\{\Phi_m\}$ is positive recurrent by showing that $E[\Phi_1 - \Phi_0 \,|\Phi_0 = k] < 0$ for all k greater than some k_0.

10.1.1 Stability under the complete batch policy

We first show that when the stability condition (1) is satisfied then the system under the complete batch policy is stable. To this end, as we have already seen, it suffices to show that the embedded Markov chain $\{\Phi_m\}$ is positive recurrent.

With the notation of section 3 let $X_{d_n^m}$ be the number of customers in the system immediately after the nth service completion of the mth cycle. Let us set $X_{d_n^m} := \chi_n^m$ and denote by ξ_n^m the number of arrivals during the nth service time of the mth cycle (but excluding the arrival that initiates the service time if the server happens to be idle and waiting for a new arrival). Also, let ζ_m denote the number of Poisson arrivals during the vacation phase of the mth cycle. Then we clearly have

$$\begin{aligned}\chi_{n+1}^m &= (\chi_n^m - 1)^+ + \xi_n^m \qquad \text{for } n = 0,1,2,\ldots,\Theta_m - 1,\\ \chi_0^m &= \Phi_m,\end{aligned} \tag{77}$$

and hence

$$\Phi_{m+1} = \sum_{n=1}^{\infty} \mathbf{1}(\Theta_m = n)\chi_n^m + \zeta_m. \tag{78}$$

From (77) we have $\chi^m_{n+1} - \chi^m_n = -\mathbf{1}(\chi^m_n > 0) + \xi^m_n$ and hence (78) can be written as

$$\begin{aligned} \Phi_{m+1} &= \Phi_m + \sum_{n=1}^{\infty} \mathbf{1}(\Theta_m \geq n)(\chi^m_n - \chi^m_{n-1}) + \zeta_m \\ &= \Phi_m + \zeta_m + \sum_{n=1}^{\infty} \mathbf{1}(\Theta_m \geq n) \left(\xi^m_{n-1} - \mathbf{1}(\chi^m_{n-1} > 0)\right). \end{aligned} \tag{79}$$

Thus

$$\begin{aligned} E[\Delta\Phi_m \mid \Phi_m = k] &= \lambda EG + \sum_{n=1}^{\infty} P(\Theta_m \geq n) \left(\rho - E[\mathbf{1}(\chi^m_{n-1} > 0) \mid \Phi_m = k]\right) \\ &= \lambda EG + \rho E\Theta - \sum_{n=0}^{\infty} P(\Theta_m > n) E[\mathbf{1}(\chi^m_n > 0) \mid \Phi_m = k]. \end{aligned}$$

Now, using the Dominated Convergence Theorem, we have

$$\begin{aligned} \lim_{k\to\infty} \sum_{n=1}^{\infty} P(\Theta_m > n) E[\mathbf{1}(\chi^m_n > 0) \mid \Phi_m = k] &= \sum_{n=0}^{\infty} P(\Theta_m > n) \lim_{k\to\infty} E[\mathbf{1}(\chi^m_n > 0) \mid \Phi_m = k] \\ &= \sum_{n=0}^{\infty} P(\Theta_m > n) = E\Theta \end{aligned} \tag{80}$$

where in the last equation we have used the fact that $\lim_{k\to\infty} E[\mathbf{1}(\chi^m_n > 0) \mid \Phi_m = k] = 1$. Thus, (80) together with (1) implies that there exists $k_0 \in \mathbb{N}$ such that $E[\Delta\Phi_m \mid \Phi_m = k] < 0$ for all $k \geq k_0$. This in turn implies the positive recurrence of $\{\Phi_m\}$. Again, the fact that the expected cycle time is finite implies the stability of the system itself.

It remains to show that, when the inequality in (1) is reversed, then the system is unstable. To establish this it is enough to show that the Markov chain $\{\Phi_m\}$ is transient. We do this by means of a stochastic dominance argument as follows. Consider an auxiliary Markov chain $\{\widetilde{\Phi}_m\}$ defined by means of the recursion

$$\widetilde{\Phi}_{m+1} = \left(\widetilde{\Phi}_m + \sum_{n=1}^{\Theta_m} \xi^m_n - \Theta_m\right)^+ + \zeta_m. \tag{81}$$

We will now argue inductively that, if $\widetilde{\Phi}_0 = \Phi_0$ with probability 1, then

$$\widetilde{\Phi}_m \leq \Phi_m \text{ w.p. 1 for each } m \in \mathbb{N}. \tag{82}$$

Indeed, suppose that (82) holds for a given value of m. Note that (79) can be written also as

$$\Phi_{m+1} = \Phi_m + \zeta_m + \sum_{n=1}^{\Theta_m} \xi^m_{n-1} - \sum_{n=1}^{\Theta_m} \mathbf{1}(\chi^m_{n-1} > 0).$$

Then,

$$\begin{aligned} \Phi_{m+1} &= \zeta_m + \left(\Phi_m + \sum_{n=1}^{\Theta_m} \xi^m_{n-1} - \sum_{n=1}^{\Theta_m} \mathbf{1}(\chi^m_{n-1} > 0)\right)^+ \\ &\geq \zeta_m + \left(\Phi_m + \sum_{n=1}^{\Theta_m} \xi^m_{n-1} - \Theta_m\right)^+ \\ &\geq \zeta_m + \left(\widetilde{\Phi}_m + \sum_{n=1}^{\Theta_m} \xi^m_{n-1} - \Theta_m\right)^+ = \widetilde{\Phi}_{m+1} \end{aligned}$$

and thus we establish the inductive step.

$\{\widetilde{\Phi}_m\}$ can be thought of as the Markov chain describing the queue length in a system with batch arrivals and batch services. The stability condition for this system is $-E\Theta + Eu + Ev < 0$ or equivalently $(1-\rho)E\Theta + \lambda EG < 0$ which is precisely (1). Thus, when the inequality in (1) is reversed, the auxiliary system is unstable (the Markov chain $\{\widetilde{\Phi}_m\}$ is transient–see Meyn and Tweedie [26]). The stochastic ordering relation between the auxiliary chain and the original system implies thus that (1) is also necessary for the positive recurrence of the Markov chain $\{\Phi_m\}$ for the complete batch policy.

10.1.2 Stability under the partial batch policy

When the partial batch policy is used,

$$\Phi_{m+1} = \Phi_m + \zeta_m + \sum_{n=1}^{L_m} (\xi_{n-1}^m - 1)$$

where $L_m = \min\Big(\Theta_m, \inf\{i \geq 0 : \Phi_m + \sum_{n=1}^{i} (\xi_{n-1}^m - 1) = 0\}\Big)$. Thus, if we define again the process $\{\widetilde{\Phi}_m\}$ by means of (81) and we assume that $\widetilde{\Phi}_m \leq \Phi_m$ then

$$\Phi_{m+1} \geq \zeta_m + \left(\Phi_m + \sum_{n=1}^{L_m} (\xi_{n-1}^m - 1)\right)^+ \geq \zeta_m + \left(\widetilde{\Phi}_m + \sum_{n=1}^{\Theta_m} (\xi_{n-1}^m - 1)\right)^+ = \widetilde{\Phi}_{m+1}.$$

This establishes inductively the stochastic ordering relationship (82) in the case where the partial batch policy is used. Thus when (1) holds with the sense of the inequality reversed $\{\widetilde{\Phi}_m\}$ is transient and the stochastic inequality just established implies that $\{\Phi_m\}$ is transient as well.

10.2 Roots within the unit disk

Here we show that equation (19) has N roots within the unit disk. Variations of this equation abound in the bulk service literature. (See for instance Chaudhry and Templeton [8] and also Coffman and Gilbert [11].) However in these treatments it is (either explicitly or tacitly) assumed that the service and vacation distributions are *light-tailed*, i.e. that the corresponding moment generating functions exist in an open interval containing the origin. We assume only the natural conditions for the existence of a stationary version of the process i.e. *the finiteness of first moments plus the stability condition*. We will use the following theorem established in Boudreau, Griffin, and Kac [6]. (See also the recent paper by Adan, van Leeuwaarden and Winands [1].)

Theorem 13. *Suppose that $\varphi(z) := \sum_{n=0}^{\infty} f_n z^n$ is the p.g.f. of f_n, $n = 0, 1, 2, \ldots$, a non-degenerate probability distribution on the non-negative integers with finite mean $\mu := \sum_{n=0}^{\infty} n f_n$ and N is a natural number. If the condition*

$$N > \mu \tag{83}$$

holds, then the equation

$$z^n - \varphi(z) = 0 \tag{84}$$

has N roots within the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. $z = 1$ is a single root of (84) while the remaining $N-1$ roots have modulus strictly smaller than 1.

The main idea of the proof is to show that the equation $z^n - w\varphi(z) = 0$ has N roots within the unit disk when $0 < w < 1$ and then use a continuity argument to show that this remains true as $w \to 1$. In our case, $\varphi(z) = D(z) \sum_{n=1}^{n} \theta_n z^{N-n} U(z)^n$ and $\mu = \varphi'(1) = EG - (1-\rho)E\Theta + N$, thus (83) is equivalent to the stability condition for the system (1).

10.3 Determination of the constants

In this section of the Appendix we give an explicit procedure for the computation of the N constants, $F_0, \ldots, F_{N-1}$ in terms of the quantities y_i, $i = 1, 2, \ldots, N-1$, defined in (20), and C, defined in (26), in the case of the partial batch policy with finite cart capacity. These constants can be obtained from the identity (22) as follows. Let us denote by $S_k := S_k(y_1, y_2, \ldots, y_{N-1})$, $k = 1, 2, \ldots, N-1$ the *elementary symmetric functions* in $N-1$ variables defined as

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N-1} y_{i_1} y_{i_2} \cdots y_{i_k}, \qquad k = 1, 2, \ldots, N-1.$$

Then $P(y) := C(y-1) \prod_{i=1}^{N-1} (y - y_i)$ can be expressed as

$$\begin{aligned} P(y) &= Cy^n - y^{N-1}C(1+S_1) + y^{N-2}C(S_1+S_2) - y^{N-3}C(S_2+S_3) + \cdots \\ &\quad +(-1)^{N-1}yC(S_{N-2}+S_{N-1}) + (-1)^n C S_{N-1}, \end{aligned}$$

where the constant C is given in (26). On the other hand, from (21),

$$\begin{aligned} P(y) &= y^n \left(\sum_{k=0}^{N-1} F_k \sum_{j=k+1}^{n} \theta_j \right) - y^{N-1} \left(\sum_{k=0}^{N-1} F_k \theta_{k+1} \right) - \cdots - y^{N-i} \left(\sum_{k=0}^{N-i} F_k \theta_{k+i} \right) - \cdots \\ &\quad -y^2 \left(F_0\theta_{N-2} + F_1\theta_{N-1} + F_2\theta_{N-2} \right) - y \left(F_0\theta_{N-1} + F_1\theta_n \right) - F_0\theta_n. \end{aligned}$$

Equating the coefficients of y^i, $i = 0, 1, \ldots, N-1$, in the above equations we obtain the following triangular linear system which allows us to determine the constants F_k.

$$\begin{aligned} \theta_n F_0 &= (-1)^{N-1} C S_{N-1} \\ \theta_n F_1 + \theta_{N-1} F_0 &= (-1)^{N-2} C \left(S_{N-1} + S_{N-2} \right) \\ &\vdots \\ \theta_n F_i + \cdots + \theta_{N-i-1} F_1 + \theta_{N-i} F_0 &= C(-1)^{N-i-1} \left(S_{N-i} + S_{N-i-1} \right) \\ &\vdots \\ \theta_n F_{N-2} + \cdots + \theta_3 F_1 + \theta_2 F_0 &= -C \left(S_2 + S_1 \right) \\ \theta_n F_{N-1} + \theta_{N-1} F_{N-2} + \cdots + \theta_2 F_1 + \theta_1 F_0 &= C \left(S_1 + 1 \right) \end{aligned}$$

(One additional equation, namely $F_0 \left(\theta_1 + \cdots + \theta_n \right) + \cdots + F_k \left(\theta_{k+1} + \cdots + \theta_n \right) + \cdots + F_{N-1}\theta_n = C$, which is obtained by equating the coefficients of y^n, is redundant since it can be obtained by adding all the N equations above and noting that the right hand side reduces to C.)

References

[1] I.J.B.F. Adan, J.S.H. van Leeuwaarden and E.M.M. Winands (2005). On the application of Rouché's theorem in queueing theory, to appear in *Operations Research Letters*.

[2] S. Asmussen (2003). *Applied Probability and Queues*, 2nd edition, Springer Verlag, New York.

[3] F. Baccelli and P. Brémaud (2003). *Elements of Queueing,* 2nd edition, Springer Verlag, New York.

[4] D. Bertsimas and D. Nakazato (1995). The distributional Little's law and its applications, *Operations Research* **43**, 298–310.

[5] W. Bischof (2001). Analysis of $M/G/1$ queues with setup times and vacations under six different service disciplines, *Queueing Systems,* **39,** 265-301.

[6] P.E. Boudreau, J.S. Griffin, Jr., and M. Kac (1962). An elementary queueing problem, *American Mathematical Monthly* **69**, 8, 713-724.

[7] E. Çinlar (1975). *An Introduction to the Theory of Stochastic Processes*, Prentice Hall.

[8] M.L. Chaudhry and J.G.C. Templeton (1983). *A First Course in Bulk Queues,* John Wiley & Sons, New York.

[9] J.W. Cohen (1969). *The Single Server Queue,* North-Holland.

[10] E.T. Copson (1935). *An Introduction to the Theory of Functions of a Complex Variable,* Oxford University Press.

[11] E.G. Coffman and E.N. Gilbert (1992). Service with a queue and a cart, *Management Science* **38**, 867–883.

[12] B.T. Doshi (1986). Queueing system with vacations, *Queueing Systems* **1**, 29-66.

[13] A. Federgruen and L. Green (1986). Queueing systems with service interruptions, *Operations Research,* **34,** 5, 752-768.

[14] F.G. Foster and K.M. Nyunt (1961). Queues with batch departures I, *The Annals of Mathematical Statistics,* **32**, 4, 1324-1332.

[15] F.G. Foster and A.G.A.D. Perrera (1964). Queues with batch departures II, *The Annals of Mathematical Statistics*, **35**, 3, 1147-1156.

[16] S.W. Fuhrmann (1984). A note on the $M/G/1$ queue with server vacations, *Operations Research* **31**, 1368-1373.

[17] S.W. Fuhrmann and R.B. Cooper (1985). Stochastic decompositions in the $M/G/1$ queue with generalized vacations, *Operations Research,* **33,** 1117-1129.

[18] R. Haji and G.F. Newell (1971). A relation between stationary queue and waiting–time distributions, *Journal of Applied Probability* **8**, 617–620.

[19] J. Keilson (1962). The general bulk queue as a Hilbert problem. *Journal of the Royal Statistical Society, Series B,* **24,** 2, 344-358.

[20] J. Keilson (1964). Corrigenda to the above paper, *Journal of the Royal Statistical Society, Series B,* **26,** 3, 487.

[21] J. Keilson and L.D. Servi (1986). Oscillating random walk models for $GI/G/1$ vacation systems with Bernoulli schedules, *Journal of Applied Probability,* **23**, 790–802.

[22] J. Keilson and L.D. Servi (1988). A distributional form of Little's law, *Operations Research Letters* **7**, 223–227.

[23] J. Keilson and L.D. Servi (1990). The distributional form of Little's law and the Fuhrmann–Cooper decomposition, *Operations Research Letters* **9**, 223-227.

[24] J.H.B. Kemperman (1961). *The Passage Problem for a Stationary Markov Chain,* The University of Chicago Press.

[25] ee, T. T. 1989. M/G/l/N queue with vacation time and limited service discipline. Performance Evaluation, Vol.9, No.3 (June), pp. 181-190.

[26] Meyn, S.P. and R.L. Tweedie (1993). *Markov Chains and Stochastic Stability,* Springer Verlag.

[27] T.J. Ott (1987). The single-server queue with independent GI/G and M/G input streams, *Advances in Applied Probability,* **19**, 266-286.

[28] N.U. Prabhu (1965). *Queues and Inventories: A Study of Their Basic Stochastic Processes,* John Wiley.

[29] B. Sengupta (1990). A queue with service interruptions in an alternating random environment, *Operations Research* **38**, 2, 308–318.

[30] L. Takács (1962). *Introduction to the Theory of Queues,* Oxford University Press.

[31] T. Takine and B. Sengupta (1997). A single server queue with service interruptions, *Queueing Systems*, **26**, 285-300.

[32] Takagi, H. (1991). *Queueing Analysis. Vol. 1: Vacation and Priority Systems,* North Holland.

[33] Zazanis, M.A. (1998). Functional versions of Little's law, *Stochastic Processes and their Applications,* **74,** 2, 195-201.