A Note on the Sensitivity Analysis for Stationary and Ergodic Queues

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Abstract

Perturbation analysis estimators for expectations of possibly discontinuous functions of the time-stationary workload were derived in [2]. The expressions obtained however may not be valid if the customer-stationary distribution of the workload has atoms (at points other than zero). This was pointed out by Brémaud and Lasgouttes in [1]. In this note we clearly state the additional condition required for the validity of the expressions in [2]. We furthermore show how our approximation scheme can also be used to obtain the correct expressions for the right and left derivatives given in [1].

KEYWORDS: STATIONARY PROCESSES, PERFORMANCE EVALUATION AND QUEUEING, NON-MARKOVIAN PROCESSES ESTIMATION.

1 Introduction and statement of theorem

In a recently published article [2] perturbation analysis derivative estimators are constructed for expectations of functionals of stationary performance measures for a G/G/1 queue. More specifically, let $W_0(\theta)$ be the work in the system in steady-state, parametrized by a real parameter θ via its service process in a smooth way and $f : \mathbf{R}_+ \to \mathbf{R}$ be a locally bounded variation function. (For the relevant notation and assumptions we refer the reader to [2].) Then the derivative of $\phi_f(\theta) := Ef(W_0(\theta))$ is claimed to be given by formula (14) of [2], which we repeat here for convenience:

$$\phi_f'(\theta) := \frac{\partial}{\partial\theta} Ef(W_0(\theta)) = \lambda E^0 W_0'(\theta)[f(W_0(\theta)) - f(W_{t_1-}(\theta))]. \tag{1}$$

It has been recently pointed out to us [1] that this formula may fail to hold in some special cases. In fact, in [1] these cases are treated at length in a way that deals directly with a general increasing function f instead of an approximating scheme $f_n \to f$, as in [2]. The technical conditions utilized in [1] are exactly those of [2]. The purpose of this article is to correct the error in [2] and show that the approximating scheme works just as well.

We stress at the outset that the following additional condition should have been included in [2] to ensure the validity of (1):

C *Let A_f be the set of points at which f jumps. Let A_θ^+ (respectively A_θ^-) be the atoms of the distribution of $W_{0+}(\theta)$ (respectively $W_{0-}(\theta)$) under the Palm measure P^0. Suppose that, for some fixed parameter θ,*

$$A_f \cap (A_\theta^+ \cup A_\theta^-) = \emptyset \, . \tag{2}$$

In Section 2 we show the validity of the following

Theorem 1 *Suppose that conditions A1–A4 of Theorem 2 of [2] hold. Furthermore suppose that condition* **C** *introduced above holds. Then the derivative of $Ef(W_0(\theta))$ exists and is given by formula (1).*

In Section 3 we note that, even if **C** fails to hold, ϕ_f is Lipschitz continuous and therefore its derivative exists almost everywhere.

2 Proof of Theorem 1

The method that we use is essentially as in the proof of Theorem 2 of our earlier paper [2]. We shall first show that if condition **C** fails to hold then, as shown in [1], both left and right derivatives of $\phi_f(\theta)$, denoted by $D^+\phi_f(\theta)$ and $D^-\phi_f(\theta)$ respectively, exist and are given by the expressions

$$D^+\phi_f(\theta) = \lambda E^0[W_0'[f(W_0) - f(W_{t_1-})] + (W_0')^+\mu_f\{W_0\} \; - (W_{t_1-}')^-\mu_f\{W_{t_1-}\}], \tag{3}$$

$$D^-\phi_f(\theta) = \lambda E^0[W_0'[f(W_0) - f(W_{t_1-})] - (W_0')^-\mu_f\{W_0\} + (W_{t_1-}')^+\mu_f\{W_{t_1-}\}], \tag{4}$$

where the dependence on θ is omitted for readability, the workload process $W_t(\theta)$ is defined to be *right-continuous*, and $\mu_f\{x\} = \lim_{\epsilon\downarrow 0}[f(x+\epsilon) - f(x-\epsilon)]$. After (3), (4) have been established, it is apparent that, in presence of condition **C**, the terms $E^0[(W_0')^+\mu_f\{W_0\}]$ and $E^0[(W_{t_1-}')^-\mu_f\{W_{t_1-}\}]$ are both equal to zero.

We start, as in [2], with a simple function of the form $f(w) = 1(w > x)$. Using the Palm construction of [2] we have

$$\begin{aligned}\frac{1}{\delta}[P(W_0(\theta+\delta) > x) - P(W_0(\theta) > x)] &= \lambda_b^* E_b^* \int_{T_0(b)}^{T_1(b)} \frac{1}{\delta}[1(W_t(\theta+\delta) > x) - 1(W_t(\theta) > x)]dt \\ &=: \lambda_b^* E_b^* I(\delta). \end{aligned}\tag{5}$$

This is formula (10) of [2]. Omitting θ for readability whenever no confusion arises and letting

$$R_i(\delta) = \frac{W_{t_i}(\theta+\delta) - W_{t_i}(\theta)}{\delta} \, , \tag{6}$$

the quantity inside the expectation of (5) is exactly equal to

$$\begin{aligned} I(\delta) = \sum_{T_0(b)\le t_i < T_1(b)} & [R_i(\delta)\, 1\{W_{t_i} > x, W_{t_{i+1}-} < x\} \\ &+ R_i(\delta)^+ 1\{W_{t_i} = x\} - R_i(\delta)^- 1\{W_{t_{i+1}-} = x\}] \, , \end{aligned}\tag{7}$$

provided that δ (generally dependent on the sample path) is sufficiently small. From this it is clear that

$$\lim_{\delta\downarrow 0} I(\delta) = \sum_{T_0(b)\le t_i<T_1(b)} [W'_{t_i} 1\{W_{t_i} > x, W_{t_{i+1}-} < x\} + (W'_{t_i})^+ 1\{W_{t_i} = x\} - (W'_{t_i})^- 1\{W_{t_{i+1}-} = x\}], \tag{8}$$

$$\lim_{\delta\uparrow 0} I(\delta) = \sum_{T_0(b)\le t_i<T_1(b)} [W'_{t_i} 1\{W_{t_i} > x, W_{t_{i+1}-} < x\} - (W'_{t_i})^- 1\{W_{t_i} = x\} + (W'_{t_i})^+ 1\{W_{t_{i+1}-} = x\}]. \tag{9}$$

The last two terms in formulas (8) and (9), were not included in [2]. These terms correspond to the cases in which *the workload process immediately after or immediately before the arrival of a customer is exactly equal to x.*

We observe next that the Dominated Convergence Theorem still holds (the bound on $I(\delta)$ obtained in [2] remains intact) and we can interchange limit and expectation in (5). An application of the Cycle Formula yields, in each case, formulas (3) and (4) for the specific function $f(w) = 1(w > x)$. This establishes (3) and (4) for simple functions f (finite linear combinations of indicator functions).

Consider now a general locally bounded variation function $f : \mathbf{R}_+ \to \mathbf{R}$. Without loss of generality we consider f to be nonnegative and increasing. Then f can be approximated from below by an increasing sequence of increasing elementary functions f_n that converge to f uniformly over compact sets. For each f_n formulas (3) and (4) hold. The right derivative is given by the expression

$$D^+\phi_{f_n}(\theta) = \lambda E^0 Y_n(\theta), \tag{10}$$

where

$$Y_n(\theta) = W'_0[f_n(W_0) - f_n(W_{t_1-})] + (W'_0)^+ \mu_{f_n}\{W_0\} \; - (W'_{t_1-})^- \mu_{f_n}\{W_{t_1-}\}.$$

We claim that $D^+\phi_f(\theta)$ exists and is given by

$$D^+\phi_f(\theta) = \lambda E^0 Y(\theta), \tag{11}$$

where

$$Y(\theta) = W'_0[f(W_0) - f(W_{t_1-})] + (W'_0)^+ \mu_f\{W_0\} \; - (W'_{t_1-})^- \mu_f\{W_{t_1-}\}.$$

From Theorem 2 of the Appendix we realize that in order to establish (11) one needs to show that ϕ_{f_n} converges uniformly to ϕ_f and that

$$\sup_\theta E^0|Y_n(\theta) - Y(\theta)| \to 0. \tag{12}$$

For (12) it is enough to show that

$$E^0 \sup_\theta |f_n(W_0(\theta)) - f(W_0(\theta))| \to 0, \tag{13}$$

$$E^0 \sup_\theta |f_n(W_{t_1-}(\theta)) - f(W_{t_1-}(\theta))| \to 0, \tag{14}$$

$$E^0 \sup_\theta |\mu_{f_n}\{W_0(\theta)\} - \mu_f\{W_0(\theta)\}| \to 0, \tag{15}$$

$$E^0 \sup_\theta |\mu_{f_n}\{W_{t_1-}(\theta)\} - \mu_f\{W_{t_1-}(\theta)\}| \to 0. \tag{16}$$

The proofs of (13), (14) are as in [2] while the proof of (16) follows from that of (15) to which we now turn our attention. Let $Z_n := \sup_\theta |\mu_{f_n}\{W_0(\theta)\} - \mu_f\{W_0(\theta)\}|$ be the quantity inside the expectation of (15). We need to show that $E^0 Z_n \to 0$. Write

$$E^0 Z_n = E^0[Z_n 1\{W_0(b) \le K\}] + E^0[Z_n 1\{W_0(b) > K\}]. \tag{17}$$

The fact that f is nonnegative and increasing and $W_0(\theta) \le W_0(b)$ together with the triangle inequality imply that $Z_n \le 2f(W_0(b))$. Thus for K large the second term of (17) can be made arbitrarily small. As for the first term of (17), we observe that

$$\sup_x |\mu_{f_n}\{x\} - \mu_f\{x\}| \to 0. \tag{18}$$

Thus, on the event $\{W_0(b) \le K\}$, we have $Z_n \to 0$ and the Dominated Convergence Theorem can now be applied in the same manner as in [2] to show that the first term of (17) also converges to zero. Finally we note that the uniform convergence of ϕ_{f_n} to ϕ_f follows easily from (12) and Lemma 1 of the Appendix.

3 Conclusions

1. The additional condition of Theorem 2 of our earlier paper [2] needed for the existence of the derivative of $EW_0(\theta)$ is condition **C**. We showed that this is the case by using the approximating procedure of [2].

2. Even if **C** fails to hold, the function $EW_0(\theta)$ is Lipschitz continuous in θ. This is due to the argument given in Lemma 1 of the Appendix. Therefore $EW_0(\theta)$ is absolutely continuous and its derivative exists almost everywhere.

4 Appendix

Let ϕ_f, ϕ_{f_n} be as in Section 2.

Lemma 1 *The function ϕ_{f_n} is Lipschitz continuous, i.e., $|\phi_{f_n}(x) - \phi_{f_n}(y)| \le L|x-y|$ where L does not depend on n. In particular, ϕ_{f_n} is absolutely continuous.*

Proof. (10) together with some obvious inequalities and the fact that $f_n \le f$ and f is increasing indicate that Y_n is bounded above by $2f(W_0(b)) \sup_\theta |W_0'|$ almost surely. Cauchy-Schwartz gives $|D^+\phi_{f_n}(\theta)| \le 2\lambda(E^0 f(W_0(b))^2)^{1/2}\ (E^0(\sup_\theta |W_0'|)^2)^{1/2}$, the right-hand-side being finite as a result of the assumptions A1–A4 in [2]. The same bound works for the left derivative as well. These inequalities establish the Lipschitz property of ϕ_{f_n}.

The following is a right-derivative version of a rather standard theorem (see, for instance, [3, p.152-153]) adapted to suit our purposes.

Theorem 2 *Suppose χ_n is a sequence of absolutely continuous functions converging to χ uniformly on $[a,b]$. Suppose that the right derivatives $D^+\chi_n$ converge uniformly to some function ψ. Then $D^+\chi$ exists and $D^+\chi = \psi$.*

Proof. Fix θ and for $\delta > 0$ define

$$\gamma_n(\delta) = \frac{\chi_n(\theta + \delta) - \chi_n(\theta)}{\delta}. \tag{19}$$

From the absolute continuity follows that $\gamma_n(\delta) = \frac{1}{\delta}\int_\theta^{\theta+\delta} D^+\chi_n(u)du$. Choose n_0 such that $|D^+\chi_m(u) - D^+\chi_n(u)| < \epsilon$ for $n, m \geq n_0$ and all $u \in [a, b]$. Then, from (19) we have $|\gamma_m(\delta) - \gamma_n(\delta)| \leq \epsilon$. Hence $\{\gamma_n(\delta)\}$ converges uniformly on $(0, b - \theta]$ and thus

$$\lim_{n\to\infty} D^+\chi_n(\theta) \;=\; \lim_{n\to\infty}\lim_{\delta\downarrow 0}\gamma_n(\delta) \;=\; \lim_{\delta\downarrow 0}\lim_{n\to\infty}\gamma_n(\delta) \;=\; D^+\chi(\theta),$$

the interchange of the limits in the middle being justified by uniform convergence (via a 3ϵ argument).

The same theorem holds for left derivatives.

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