The distribution of age-of-information performance measures for message processing systems

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Abstract

The idea behind the recently introduced "age of information" performance measure of a networked message processing system is that it indicates our knowledge regarding the "freshness" of the most recent piece of information that can be used as a criterion for real-time control. In this foundational paper, we examine two such measures, one that has been extensively studied in the recent literature and a new one that could be more relevant from the point of view of the processor. Considering these measures as stochastic processes in a stationary environment (defined by the arrival processes, message processing times and admission controls in bufferless systems), we characterize their distributions using the Palm inversion formula. Under renewal assumptions we derive explicit solutions for their Laplace transforms and show some interesting decomposition properties. Previous work has mostly focused on computation of expectations in very particular cases. We argue that using bufferless or very small buffer systems is best

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and support this by simulation. We also pose some open problems including assessment of enqueueing policies that may be better in cases where one wishes to minimize more general functionals of the age of information measures.

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Contents

1 Introduction

1.1 Technological background

The Internet is now commonly used to transmit latency-sensitive information that is part of a real-time control or decision process. As an example, consider a temperature or pressure sensor which could periodically transmit a reading to a latency-critical remote control. Other examples include decision systems for an airplane, driverless vehicles, financial transactions, power systems, sensor/actuator systems or other "cyber physical" systems. In the power system case, a high temperature reading of a transmission line could indicate reduced capacity or predict near-term failure. In the sensor system case, the sensor could indicate an alarm such as a motion detector which needs to be manually reset once tripped; any alarm message would render stale any queued or in-transmission "heartbeat" message that is periodically sent to indicate no intruder is present and that the sensor is properly functioning. In the actuator system case, messages may embody commands to a remote actuator of a time-critical control system.

1.2 Two age of information measures

Systems such as the ones described above naturally depend on the *age* of the most recently received reading from a remote sensor. This is a quantity that takes into account the time since the reading was generated. In view of the speeds involved a decision must be taken upon arrival of a new information packet: to read or not read it. The choice is crucial and depends on the packet length and the frequency of information packet arrivals, quantities that may not be completely known. If $A_t^* :=$ ***arrival time of the last completely read message before time*** t

then the quantity

$$\alpha(t) := t - A_t^*$$

has been introduced in the literature and has been given the name ***"age of information (AoI)"***. This has been used as a measure of freshness and its expectation (under specific assumptions) has been studied in, e.g., [8, 6, 14, 2, 12]. From a performance point of view, we are interested not only in its expectation but also in its probability distribution. We derive fundamental results about the latter in this paper.

One can argue that the above measure may have limited usefulness for applications that ***cannot control the arrivals*** of messages. And thus, one may assert that the freshness of information should be gauged not against the current time t but against $A_t :=$ ***arrival time of last message before*** t. By definition, $A_t^* \le A_t$ with equality if and only if the message arriving at A_t is completely read. We thus introduce the measure

$$\beta(t) := A_t - A_t^*.$$

To further explain our claim that β may be more relevant than α, consider the following scenario: messages arrive rarely and randomly at times t_n and have very small duration ε, so small that $\varepsilon \ll t_{n+1} - t_n$ for all n. Then only one message will be in the system at a time and, assuming that the processor does not idle when the message is present, every message is completely read. We can easily see that, unless t is in the extremely small interval of length ε during the processing of a message, $\alpha(t)$ equals time elapsed between the last arrival before t and t, whereas $\beta(t) = 0$. (To see this, take $t_0 + \varepsilon < t < t_1$. Then $A_t = A_t^* = t_0$, so $\alpha(t) = t - t_0$, but $\beta(t) = 0$. On the other hand, if $t_1 < t < t_1 + \varepsilon$ then $A_t = t_1$, but $A_t^* = t_0$. Hence $\alpha(t) = t - t_0$, $\beta(t) = t_1 - t_0$, and the two quantities are approximately the same since ε is extremely small.)

Thus, $\alpha(t)$ and $\beta(t)$ are almost the same when t lies in a processing interval, but vastly different when t lies in an idle interval. In the latter case, $\alpha(t)$ simply tells us the age of the arrival process but $\beta(t) = 0$ meaning that the processor possesses the freshest information. Thus, in situations where the arrival process is beyond the processor's control, trying to keep the "age of information" low should not take into account the age of the arrival process. This is why we propose the new measure as a more relevant quantity. Granted, $\alpha(t) = (t - A_t) + \beta(t)$, so, insofar as expectations are desired cost functionals, there is no difference in potential optimization problems. However if the cost functional is another function $\alpha(t)$, e.g., $\mathbb{P}(\alpha(t) > u)$, then the dependence between $t - A_t$ and $\beta(t)$ justify finding the distribution of β. Since there is no terminology for this quantity, we are free to choose one: we call it ***"new age of information (NAoI)"***.

1.3 The queueing system: bufferless instead of buffered

The age of information measures can be defined for a general queueing system that could consist of a number of queues and servers, buffers of various sizes and various policies that control the acceptance of a message and its successful processing. We define some quantities used in the paper. Messages arrive at times T_n and have processing (or service) times σ_n. An arriving message may be immediately accepted or rejected. The 1/0-valued variable χ_n denotes acceptance/rejection. A message is called successful if it is processed in its entirety. The 1/0-valued variable ψ_n denotes processing success/failure. A failed message is kicked out of the system before it is read entirely. We let T'_n be the time at which the message arriving at time T_n departs from the system either because it is rejected or because it fails to be processed entirely or because it departs successfully.

Two systems that are of main concern in this policy are as follows. There is a single server and a buffer of unit size (just to accommodate the message being processed). The first system operates under the ***pushout policy*** $(\mathcal{P})$. Every arriving message immediately kicks out (one uses the word "obsoletes") the existing message, if any, and starts being processed immediately. If no message arrives while one is processed then the latter message finishes successfully. Note that this system can be thought of as a Preemptive Last In First Out system with buffer of size 1. The second system operates under the ***blocking policy*** $(\mathcal{B})$. An arriving message immediately grabs the server if the latter is available or is immediately kicked out if the server is busy. Other policies are possible; see the examples at the end of Section 2.2.

The literature so far has focused on the AI for single server queueing systems, particularly stable FIFO or preemptive LIFO disciplines [8, 2], with infinite buffers where all messages are accepted ($\chi_n = 1$ for all n) and all messages are successful ($\psi_n = 1$ for all n). Moreover, only the mean of $\alpha(t)$ for M/M/1-FIFO [8] in steady-state has been derived: see formula (81) in the last section. We do, however, question the use of infinite buffers, based on some simple, intuitive observations. The most intuitive of all is: if it is desired to keep the age of information low then storing arriving message makes no sense as they contribute nothing to either α or β.

Consider the $\mathcal{P}$ system as described above and compare it with an infinite buffer preemptive LIFO (pLIFO) system. Assume that the same sequence of arrival and processing times is fed into both systems. Then, as explained in more detail in the last section,

$$\alpha_{\text{pLIFO}}(t) = \alpha_{\mathcal{P}}(t), \quad \beta_{\text{pLIFO}}(t) = \beta_{\mathcal{P}}(t), \quad t \in \mathbb{R}. \tag{1}$$

In fact, recently, it was shown that, among all work-conserving processing disciplines, for an infinite buffer single server queue, the preemptive LIFO discipline achieves stochastically lowest AoI in steady-state in some cases; see [2]. Concerning next an infinite buffer FIFO system, the other system studied in the age of information literature [8, 10], we conjecture that another system that we call $\mathcal{P}_2$, basically a variation of $\mathcal{P}$ but with buffer size 2, has better AoI performance than the infinite buffer FIFO. It is for these reasons that we study only bufferless systems in this paper. In studying bufferless systems, the only variable is the queueing policy. Rather than studying an optimal control problem, we focus on two very specific and, in some sense, extreme policies, $\mathcal{P}$ and $\mathcal{B}$. We do so in order to obtain concrete formulas and explain the methods. However, in principle, our methods, based on Palm calculus and renewal theory, will work on any policy.

1.4 Paper organization and contributions

The paper is organized as follows. In Section 2 we present the setup and the definition of the models and all relevant stochastic processes. Section 3 is a brief outline of some of the results, pointing out, in particular, some interesting distributional stochastic decomposition results for the various stationary performance measures. Formulas for distributions and moments of both the AoI and the NAoI for the pushout system are derived in Section 4. This is done by carefully applying classical Palm theory, first in a stationary context and then by specializing to the case involving independence assumptions. The stronger the assumptions, the more explicit the results. For the queueing theorist, it is not a surprise that the formulas become quite explicit when the arrival process is Poisson. Similarly pleasing and explicit is the case when the message lengths are independent exponentially distributed random variables. If both Poissonian assumptions hold then we are in the best of all worlds. The blocking system is the subject of Section 5. The action plan is the same as in the pushout system case, but, here, all calculations are more involved. This is due to the fact that the blocking system has more complicated dynamics than the pushout system. Nevertheless, closed-form formulas are also possible. In Section 6, we discuss variations of the AoI problem to be considered in future work; in particular, we discuss other queueing policies that may have smaller (in some sense) age of information in some cases.

The contributions of this paper are as follows: Previous literature has focused only on stationary mean AoI but even that is done in rather specific cases (infinite buffer FIFO). In this paper we derive formulas for the distributions via Laplace transforms of AoI and NAoI in steady-state under renewal assumptions. In particular, we find explicit formulas for all the means in all cases, and even this appears to be novel. In addition to deriving formulas for the stationary distributions under renewal assumptions, by adopting a top-down approach based on Palm calculus we derive a methodology on how one could compute the same things (i) for arrival/service distributions with dependencies and (ii) for policies other than $\mathcal{P}$ or $\mathcal{B}$.

2 System definitions

The goal of this section is to define the two measures of the age of information for a general bufferless processing system. We are careful to include the possibility that some of the quantities below may be restricted on a lattice. We first define such a system, allowing the possibility to accept or reject messages. We then give the definitions of the age of information measures as functions of time. Lastly, we introduce stochastic assumptions which make the age of information processes random functions of time.

2.1 Notation/terminology

The set of integers is denoted by $\mathbb{Z}$. The indicator function of a set A is denoted by $\mathbf{1}_A$. The notation $\mathbb{E}[X; A]$ stands for $\mathbb{E}[X\mathbf{1}_A]$. If S is a set and $s \in S$, then δ_s denotes the Dirac measure $\delta_s(B) = \mathbf{1}_{s\in B}$, $B \subset S$. By point measure on $\mathbb{R}$ (or $\mathbb{R}^2$) we mean a measure assuming nonnegative integer values; necessarily, it is a finite or countable sum of Dirac measures. A point process is a random point measure. If X is a positive random variable with finite expectation, we say that $\overline{X}$ is the stationary version of X if it has density $\mathbb{P}(X > x)/\mathbb{E}X$:

$$\mathbb{P}(\overline{X} \in dx) = \frac{\mathbb{P}(X > x)}{\mathbb{E}X} dx.$$

We then have

$$\mathbb{E}e^{-u\overline{X}} = \frac{1 - \mathbb{E}e^{-uX}}{u\mathbb{E}X}, \quad \mathbb{E}\overline{X} = \frac{\mathbb{E}X^2}{2\mathbb{E}X}.$$

When X and Y are random variables (on, possibly, different probability spaces) $X \overset{(d)}{=} Y$ denotes equality of their laws (distributions). The symbol $\widetilde{\mathbb{P}}$ denotes the probability governing a time-stationary system, whereas $\mathbb{P}$ denotes the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the arrival process. See section 2.4 below for exact definitions. (We choose this unconventional notation because the former symbol is used less frequently than the latter.)

2.2 Bufferless message processing systems

Messages arrive in a bufferless server which can read one message at a time. Denote by T_n, $n \in \mathbb{Z}$, the message arrival times. We assume that

$$T_n < T_{n+1},\ n \in \mathbb{Z}, \qquad \sup_{n\in\mathbb{Z}} T_n = +\infty, \qquad \inf_{n\in\mathbb{Z}} T_n = -\infty.$$

We shall fix an ordering by letting T_0 be such that $T_0 \le 0 < T_1$. We denote by

$$\mathfrak{a} := \sum_{n\in\mathbb{Z}} \delta_{T_n}$$

the arrival process, considered as a point measure. We shall also let, for all $n \in \mathbb{Z}$,

$$\tau_n := T_{n+1} - T_n. \tag{2}$$

We introduce, for each $n \in \mathbb{Z}$, the *accept/reject index* χ_n, setting

$$\chi_n = \begin{cases} 1, & \text{if the message arriving at } T_n \text{ is accepted} \\ 0, & \text{otherwise.} \end{cases}$$

The χ_n is a decision variable that depends on the ***acceptance policy***. See below for some example. In this paper we shall only consider specific policies leaving optimization/control problems for future work. The length of message n (the message arriving at time T_n) is denoted by σ_n and its departure time by T'_n. The latter given by

$$T'_n := \begin{cases} T_n, & \text{if } \chi_n = 0 \\ (T_n + \sigma_n) \wedge \inf\{T_r : r > n,\ \chi_r = 1\}, & \text{if } \chi_n = 1 \end{cases}. \tag{3}$$

This means that an arriving message will either be immediately rejected (and thus depart immediately) or accepted, in which case it will either be read in its entirety or pushed out by another accepted message. Note that the sets $\{T_n, n \in \mathbb{Z}\}$ and $\{T'_n, n \in \mathbb{Z}\}$ may have common elements (e.g., if we allow all variables take values that are integer multiples of a common unit). It is easy to see from (3) that the intervals $[T_n, T'_n)$ and $[T_m, T'_m)$ are disjoint if $m \neq n$. Thus, for all t, the quantity

$$q(t) := \sum_{n\in\mathbb{Z}} \chi_n \mathbf{1}_{T_n \le t < T'_n} \tag{4}$$

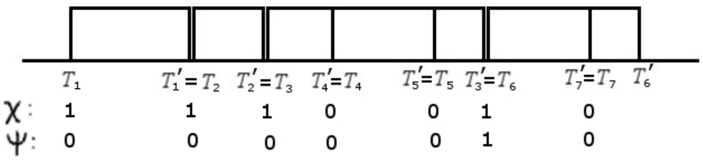


Figure 1: *A message arrives at time T_1 at an idle server and is immediately accepted. A double line indicates that a message pushes out the previous one, while a single line indicates that the message is blocked. Thus, messages 1, 2, 3 and 6 are accepted, while 4, 5 and 7 are rejected. Only message 6 is successful. The server started reading message 1 at time T_1 and finishes reading message* 6 *in its entirety at time $T'_6 = T_6 + \sigma_6$.*

is either 0 or 1. The $q(t)$ is the state of the server at time t: $q(t) = 1$ if the server is busy or 0 if not. Notice that $q(\cdot)$ is right-continuous (by choice rather than by necessity).

We call message n *successful* if it departs immediately after having being read in its entirety. The *success/failure index* is the binary variable

$$\psi_n := \mathbf{1}_{T'_n = T_n + \sigma_n}. \tag{5}$$

By definition, for all n,

$$\psi_n \le \chi_n.$$

See Figure 1 for an illustrative example of an arbitrary policy.

Consider $n \in \mathbb{Z}$ and the statement

$$\mathcal{Z}_n := \text{“}q(T_n-) = 0 \text{ or } T'_m = T_n \text{ for some } m < n\text{”}. \tag{6}$$

We can interpret $\mathcal{Z}_n$ as “the server is idle at time T_n”. If there is no possibility that a departure time coincides with the arrival time of another message then idle server simply means $q(T_n-) = 0$. But we must include the possibility that some message $m < n$ departs exactly at T_n. We shall throughout assume that the *non-idling condition*

$$\text{for all } n \in \mathbb{Z} \text{ if } \mathcal{Z}_n \text{ then } \chi_n = 1 \tag{NI}$$

holds. For those n for which $\mathcal{Z}_n$ is violated the determination of χ_n is a matter of the acceptance policy.

Here are four examples of acceptance policies. Let ℓ be a nonnegative integer.

Example 1. The *pushout* ($\mathcal{P}$) policy. All messages are accepted:

$$\chi_n = 1, \quad n \in \mathbb{Z}.$$

From (3) and (5) it is easy to see that

$$\psi_n = \mathbf{1}_{T_n+\sigma_n \le T_{n+1}} = \mathbf{1}_{\tau_n \ge \sigma_n}, \quad n \in \mathbb{Z}.$$

Example 2. The *blocking* ($\mathcal{B}$) policy. No message other than those satisfying the non-idling condition (NI) are accepted:

$$\chi_n = 1 \iff \mathcal{Z}_n \text{ holds.}$$

Note that, here, $\psi_n = \chi_n$ for all n, that is, every accepted message is successful.

Example 3. The $\mathcal{BP}(\ell)$ policy. Say a message arrives at time t at an empty system, $q(t-) = 0$. Then it starts being processed. If there are at most ℓ arrivals while the message is being processed then they are all blocked. Beyond that, the server accepts every arrival until it becomes empty again. In other words, during a reading period, the server behaves in a blocking fashion for up to ℓ arrivals and in a pushout fashion after that.

Example 4. The $\mathcal{PB}(\ell)$ policy. During a reading period, the server behaves in a pushout fashion for up to ℓ arrivals and in a blocking fashion after that.

We shall only study the first two policies in this paper, leaving the study of the others, as well as optimal policies, for future work.

2.3 Age of information processes

To define the age of information functions (of time) we need to introduce the following functions on $\mathbb{R}$. The *last arrival epoch* before $t \in \mathbb{R}$ is defined by

$$A_t := \sup\{T_n : n \in \mathbb{Z}, T_n \le t\}.$$

The *last successful arrival epoch* before t is defined by

$$S_t := \sup\{T_n : n \in \mathbb{Z}, T_n \le t, \psi_n = 1\};$$

The *last successful departure epoch* before t is defined by

$$D_t := \sup\{T_n + \sigma_n : n \in \mathbb{Z}, T_n + \sigma_n \le t, \psi_n = 1\}.$$

Note that, under our assumptions on the sequence T_n, the sup in the definition of A_t is actually a max. Assuming further that

$$\inf\{n : \psi_n = 1\} = -\infty \tag{A1}$$

we have that the sup in S_t and D_t is replaced by a max. If, in addition,

$$\sup\{n : \psi_n = 1\} = \infty \tag{A2}$$

then $S_t, D_t < \infty$ for all t.

Definition 1. *Under assumptions (A1) and (A2), the* ***age of information (AoI)*** *function is defined by*

$$\alpha(t) := t - S_{D_t}, \quad t \in \mathbb{R}, \tag{7}$$

and the ***new age of information (NAoI) function*** *is defined by*

$$\beta(t) := A_t - S_{D_t}, \quad t \in \mathbb{R}. \tag{8}$$

Note that the functions A, S, D above are right-continuous and increasing ($s < t \Rightarrow A_s \le A_t, S_s \le S_t, D_s \le D_t$). It follows that α and β are also right-continuous. Moreover,

$$\Delta\alpha(t) := \alpha(t) - \alpha(t-) = -\Delta S_{D_t} = -\lim_{\varepsilon \downarrow 0}(S_{D_t} - S_{D_{t-\varepsilon}}) \le 0.$$

So jumps of α can only be negative. Notice that

$$\Delta\alpha(t) = S_{D_t} - S_{(D_{t-})-}.$$

On the other hand, β can have both positive and negative jumps.

We shall also use the following notations and terminology. Consider the arrival times T_n of messages arriving at a idle server:

$$\{B_k : k \in \mathbb{Z}\} := \{T_n : \ \mathcal{Z}_n \text{ holds}\}.$$

By convention, we enumerate these points as

$$\cdots < B_{-1} < B_0 \le 0 < B_1 < \cdots$$

They form the beginnings of reading intervals. An interval with endpoints B_k and B_{k+1} will be referred to as *cycle*. Define also

$$\{B'_k : k \in \mathbb{Z}\} := \{T_n + \sigma_n : n \in \mathbb{Z}, \psi_n = 1\}$$

and again assume that

$$\cdots < B'_{-1} < B'_0 \le 0 < B'_1 < \cdots$$

These are the ends of reading intervals. The two sequences, $\{B_k\}$ and $\{B'_k\}$, are interlaced: between two successive elements of one sequence there is exactly one element of the other. See Figure 2. An interval with endpoints B_k and B_{k+1} is

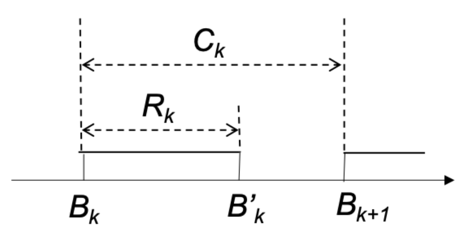


Figure 2: *The interval* $[B_k, B_{k+1})$ *is a cycle and the subinterval* $[B_k, B'_k)$ *is a reading interval.*

called a *cycle*. We set

$$\mathbf{C}_k := B_{k+1} - B_k$$

for the cycle length. The subinterval with endpoints B_k and B'_k is called a *reading interval*. We set

$$\mathbf{R}_k := B'_k - B_k$$

for the reading length.

2.4 The stationary framework and Palm probabilities

Let $(\Omega, \mathscr{F}, \widetilde{\mathbb{P}})$ be a probability space endowed with a flow, i.e., a family of invertible measurable functions $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, such that θ_t^{-1} are also measurable and such that

$$\theta_{t+s} = \theta_t \circ \theta_s, \quad s, t \in \mathbb{R}. \tag{9}$$

Assume further that the flow preserves $\widetilde{\mathbb{P}}$, that is,

$$\widetilde{\mathbb{P}} \circ \theta_t = \widetilde{\mathbb{P}}, \quad t \in \mathbb{R}.$$

Let T_n, σ_n be random variables such that the marked[1] point process $\sum_n \delta_{(T_n, \sigma_n)}$ is stationary, that is,

$$\left(\sum_n \delta_{(T_n, \sigma_n)} \right) \circ \theta_t = \sum_n \delta_{(T_n - t, \sigma_n)}, \quad t \in \mathbb{R}. \tag{10}$$

Note then that

$$A_t \circ \theta_s = A_{t+s} - s, \quad s, t \in \mathbb{R}.$$

It follows that the arrival rate

$$\lambda := \widetilde{\mathbb{E}} \sum_n \mathbf{1}_{0 \le T_n \le 1}$$

is positive and finite. Consider next a acceptance policy as specified by the acceptance random variables χ_n, $n \in \mathbb{Z}$, defined on $(\Omega, \mathscr{F})$. We say that the system is in *steady-state* if, in addition to (10),

$$\left(\sum_n \delta_{(T_n, \sigma_n, \chi_n)} \right) \circ \theta_t = \sum_n \delta_{(T_n - t, \sigma_n, \chi_n)}, \quad t \in \mathbb{R}. \tag{11}$$

[1] A point process φ on a product space $S \times M$ is called M-marked (or just marked) if $\varphi(\{s\} \times M) \in \{0, 1\}$ for all $s \in S$.

If the system is in steady-state then it follows from (11), (9) (5) and (3) that

$$\Big(\sum_n \delta_{(T_n,\sigma_n,\chi_n,\psi_n)}\Big)\circ\theta_t = \sum_n \delta_{(T_n-t,\sigma_n,\chi_n,\psi_n)}, \quad t\in\mathbb{R}, \tag{12}$$

and, for all $s,t\in\mathbb{R}$,

$$S_t\circ\theta_s = S_{t+s}-s, \quad D_t\circ\theta_s = D_{t+s}-s,$$
$$\alpha(s)\circ\theta_t = \alpha(t+s), \quad \beta(s)\circ\theta_t = \beta(t+s), \quad q(s)\circ\theta_t = q(t+s).$$

In general, it is not obvious that (11) holds. Of the four acceptance policies mentioned above, the pushout $\mathcal{P}$ immediately satisfies (11) owing to that $\chi_n = 1$ and $\psi_n = \mathbf{1}_{T_{n+1}-T_n\ge\sigma_n}$ for all n. That (11) holds is proved in [1, Section 5.3] and may require enlarging the probability space $(\Omega,\mathscr{F},\widetilde{\mathbb{P}})$.

Definition 2. *We shall denote by $\mathbb{P}$ the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the point process $\mathfrak{a} = \sum_{n\in\mathbb{Z}}\delta_{T_n}$. If (11) holds we shall denote by $\mathbb{P}^*$ the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the point process $\sum_{k\in\mathbb{Z}}\delta_{B_k}$.*

For the notion of Palm probability see, e.g., Daley and Vere-Jones [4, Chapter 13], Kallenberg [7] and Baccelli and Brémaud [1]. Formally, with $\mathscr{B}$ denoting the class of Borel sets on $\mathbb{R}$, the measure $\mathscr{B}\ni C\mapsto \widetilde{\mathbb{E}}(\mathbf{1}_A\sum_n \mathbf{1}_{T_n\in C})$ is absolutely continuous, and hence differentiable, with respect to the measure $\mathscr{B}\ni C\mapsto \widetilde{\mathbb{E}}(\sum_n \mathbf{1}_{T_n\in C})$. The value of the derivative at 0 is precisely $\mathbb{P}(A)$. The Palm probability $\mathbb{P}^*(A)$ can be obtained in exactly the same manner. However, since $\{B_k\}$ is precisely the set of T_n for which $\mathcal{Z}_n$ holds, it follows that $\mathbb{P}^*$ is obtained from $\mathbb{P}$ via elementary conditioning:

$$\mathbb{P}^* = \mathbb{P}(\cdot|\mathcal{Z}_0).$$

The hierarchy of the three measures used in the paper is

$$\widetilde{\mathbb{P}} \longrightarrow \mathbb{P} \longrightarrow \mathbb{P}^*$$

Intuitively, one thinks of $\mathbb{P}$ is obtained from $\widetilde{\mathbb{P}}$ by conditioning that a point of (T_n) is at 0 and $\mathbb{P}^*$ is obtained from $\mathbb{P}$ by conditioning on that one of this point at 0 is one of the points of (B_k). Hence $\mathbb{P}^*$ is obtained from $\widetilde{\mathbb{P}}$ as well by conditioning on both events. Hence if A is an event such that $\widetilde{\mathbb{P}}(A) = 1$ then $\mathbb{P}(A) = 1$ also and if $\mathbb{P}(A) = 1$ then $\mathbb{P}^*(A) = 1$ also. Integrals with respect to $\widetilde{\mathbb{P}}$, $\mathbb{P}$, $\mathbb{P}^*$ are denoted by $\widetilde{\mathbb{E}}$, $\mathbb{E}$, $\mathbb{E}^*$ respectively. Moreover,

$$\mathbb{P}(T_0 = 0) = 1, \quad \mathbb{P}^*(B_0 = T_0 = 0) = 1. \tag{13}$$

We denote by θ_{T_n} the map defined by $\theta_{T_n}(\omega) = \theta_{T_n(\omega)}(\omega)$. Then θ_{T_n}, $n \in \mathbb{Z}$, forms a discrete time flow that preserves $\mathbb{P}$. In other words, $\mathbb{P}$-a.s., $\theta_{T_n} \circ \theta_{T_m} = \theta_{T_{n+m}}$ for all $m, n \in \mathbb{Z}$ and $\mathbb{P} \circ \theta_{T_n} = \mathbb{P}$ for all $n \in \mathbb{Z}$. Similarly, $\mathbb{P}^*$-a.s., $\theta_{B_k} \circ \theta_{B_\ell} = \theta_{B_{k+\ell}}$ for all $k, \ell \in \mathbb{Z}$ and $\mathbb{P}^* \circ \theta_{B_k} = \mathbb{P}^*$ for all $k \in \mathbb{Z}$.

The $\mathbb{P}$-law of (τ_n, σ_n) does not depend on n. In what follows, we let (τ, σ) be a generic random element whose law is the same as the $\mathbb{P}$-law of (τ_0, σ_0). The definition of Palm probability and the fact $\lambda > 0$ implies that

$$\mathbb{E}\tau = 1/\lambda < \infty.$$

This is the minimal condition imposed by stationarity and thus it cannot be avoided. It is important to note however that we shall make no assumptions about finiteness of higher $\mathbb{P}$-moments of τ.

Referring to Figure 2, note that, under $\mathbb{P}^*$, all cycles have identical law and so do all reading intervals. We denote by $\mathbf{C}$ a *typical cycle length*, that is, a random variable whose law is the $\mathbb{P}^*$-law of the length of any cycle. Similarly, $\mathbf{R}$ denotes a *typical reading interval length*.

3 Outline of some of the results

All results concern stationary processes. Denote by $\alpha_{\mathcal{P}}$, $\alpha_{\mathcal{B}}$ the AoI processes for the pushout and blocking systems, respectively. Similarly, we let $\beta_{\mathcal{P}}$, $\beta_{\mathcal{B}}$ be the NAoI processes for the two systems.

3.1 Stochastic decomposition/representation results

These are obtained under the assumptions that, under the Palm measure $\mathbb{P}$, the (τ_i) are i.i.d. and independent of the (σ_i) which are also i.i.d. We refer to these assumptions as being the i.i.d. (or renewal) assumptions. When we say "decomposition" of (the law of) a random variable X we mean, as usual in applied probability and queueing theory, that $X \overset{(d)}{=} X_1 + X_2$ where X_1 and X_2 are independent random variables. The following are obtained in Theorems 2, 6, respectively. Under $\widetilde{\mathbb{P}}$,

$$\alpha_{\mathcal{P}}(t) \overset{(d)}{=} \overline{\tau} + \mathbf{R}_{\mathcal{P}}$$
$$\alpha_{\mathcal{B}}(t) \overset{(d)}{=} \sigma + \overline{\mathbf{C}}_{\mathcal{B}}$$

Here, $\overline{\tau}$ is a random variable whose law is the law of the stationary version of the interarrival time, $\mathbf{R}_{\mathcal{P}}$ is distributed as the typical reading interval of the pushout system, and $\overline{\mathbf{C}}_{\mathcal{B}}$ is distributed as the stationary version of the typical cycle of the blocking system. We also obtain, in Theorems 4, 7, respectively, the following representations:

$$(\beta_{\mathcal{P}}(t) | \beta_{\mathcal{P}}(t) > 0) \overset{(d)}{=} \mathbf{C}_{\mathcal{P}}$$

$$\beta_{\mathcal{B}}(t) \mathbf{1}_{\beta_{\mathcal{B}}(t) > 0} \overset{(d)}{=} \beta_{+}(t).$$

Here, $\mathbf{C}_{\mathcal{P}}$ is distributed as the typical cycle of the pushout system and $\beta_{+}(t)$ is the NAoI process for an appropriately defined variant of the fully-blocking system: remove from the system all undisturbed messages, that is, all messages that arrive at an idle system and are such that no other messages arrive while they are being processed. Moreover, we find that the NAoI always has an atom at 0. This is obvious for β_P because the it is 0 when the processor is idle, but less obvious for $\beta_{\mathcal{B}}$. The last representation result explains the appearance of an atom. For more discussion see Remark 4 of Section 5.2.

3.2 A guide to the subsequent analysis and results

We stress some points that will facilitate the reader in going through the analysis of the pushout and blocking systems, Sections 4 and 5 below. First of all, the reader should keep in mind the hierarchy of the three measures, $\widetilde{\mathbb{P}}$ (governing the stationary system), $\mathbb{P}$ (Palm with respect to arrivals), and $\mathbb{P}^*$ (palm with respect to the beginnings of cycles) should be kept in mind, as explained above.

Regarding the pushout system, the most general results are in Theorems 1 and 3:

$$\widetilde{\mathbb{E}}F'(\alpha_{\mathcal{P}}(0)) = \lambda\mathbb{E}\left[F\left(\tau_{-1} + \sum_{i=0}^{N-1}\tau_i + \sigma_N\right) - F(\sigma_N);\, \tau_{-1} > \sigma_{-1}\right],$$

$$\widetilde{\mathbb{E}}f(\beta_{\mathcal{P}}(0)) = \lambda\mathbb{E}\left[\sum_{i=0}^{N-1}\tau_i f\left(\sum_{j=-1}^{i-1}\tau_j\right) + \sigma_N f\left(\sum_{j=-1}^{N-1}\tau_j\right) + (\tau_N - \sigma_N)f(0);\, \tau_{-1} > \sigma_{-1}\right].$$

These are, in principle, expressions for the distributions of $\alpha_{\mathcal{P}}(0)$ and $\beta_{\mathcal{P}}(0)$ in steady-state because F and f are "general" functions and everything on the right-hand sides of the equations depends solely on the (joint) distribution of the infinite random sequence $(\tau_i, \sigma_i : i \in \mathbb{Z})$. In particular, N is defined as $N = \inf\{\ell \geq 0 : \tau_\ell \geq \sigma_\ell\}$ and denotes the index of the first message, among the ones numbered $0, 1, 2, \ldots$, that is successful. Note, in particular, that N has a stopping time property and this, along with the fundamental probabilist's tool, the découpage de Lévy (Lemma 2), makes, under renewal assumptions, the analysis and the obtaining of explicit formulas possible.

Regarding the blocking system, the most general results are formulas (39) and (62) of Theorems 5 and 7 below. The formulas are more complicated due to the fact that the dynamics of the system and, in particular, the construction of the unique steady-state depends on the infinite past. However, again, these formulas are again expressions for the distributions of $\alpha_{\mathcal{B}}(0)$ and $\beta_{\mathcal{B}}(0)$. We point out that the index N appearing in them is now defined as $N = \inf\{\ell \geq 1 : \tau_0 + \cdots + \tau_{\ell-1} \geq \sigma_0\}$ and is chosen so that it has the stopping time property.

Using renewal theory, we manage to turn these general formulas into explicit results for the Laplace transforms of the quantities of interest. To do so, we need to introduce several functionals of the processes which can be found by solving fixed point (renewal equations). Sometimes, the Laplace transforms can be inverted explicitly giving formulas for densities. In particular, this can be done when the random variables (τ_i) are i.i.d. exponential and the (σ_i) are also i.i.d. exponential and the two sequences are independent. This, of course, is no surprise to the queueing theorist. Finding just the expectations of the AoI and NAoI can be done either via their Laplace transforms or via the general formulas obtained via Palm calculus by choosing specific functionals. We do whatever is quicker and obtain expectation formulas that are summarized in Table 1 of the last section. To the best of our knowledge, the GI/GI formulas are new and some of the rest are consistent with [9].

4 The pushout system

The dynamics of the pushout system is quite simple: every arriving message is admitted: $\chi_n = 1$ for all $n \in \mathbb{Z}$. The message arriving at T_n is successful if and only if $T_n + \sigma_n \le T_{n+1}$. Thus

$$\psi_n = \mathbf{1}_{\tau_n \ge \sigma_n}, \quad n \in \mathbb{Z},$$

where $\tau_n = T_{n+1} - T_n$ as in (2). Since, for all n, $\chi_n = 1$ and $\psi_n = \mathbf{1}_{\tau_n \ge \sigma_n}$, it follows from (3) that the state process q of (4) is alternatively given by

$$q(t) = \begin{cases} 0, & T_n + \sigma_n \le t < T_{n+1} \text{ for some } n \\ 1, & \text{otherwise} \end{cases}.$$

If $\mathbb{P}(\tau_0 < \sigma_0) = 1$ then $\mathbb{P}(\tau_n < \sigma_n \text{ for all } n) = 1$ and so q is identically equal to 1. This is an uninteresting case resulting in infinite AoI and NAoI. We thus assume that

$$\mathbb{P}(\tau_0 \ge \sigma_0) > 0, \tag{14}$$

that is $\mathbb{P}(\psi_0 = 1) > 0$. By the Poincaré recurrence theorem [5, Theorem 7.3.4], there is a doubly-infinite subsequence ψ_{n_k}, $k \in \mathbb{Z}$, such that $\psi_{n_k} = 1$ for all k, $\mathbb{P}$-a.s. and $\widetilde{\mathbb{P}}$-a.s. In other words, $\inf\{n : \psi_n = 1\} = -\infty$, $\sup\{n : \psi_n = 1\} = +\infty$, $\mathbb{P}$-a.s., and hence $\widetilde{\mathbb{P}}$-a.s. This implies that α, β are well-defined and finitely-valued processes.

It is easy to see that, for the pushout system, the beginnings of cycles satisfy

$$\{B_k : k \in \mathbb{Z}\} = \{T_n : n \in \mathbb{Z}, \psi_{n-1} = 1\}.$$

We therefore have:

Lemma 1. *The Palm probability $\mathbb{P}^*$ of Definition 2 is the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the (stationary) point process*

$$\sum_{n\in\mathbb{Z}} \psi_{n-1}\delta_{T_n}$$

and

$$\mathbb{P}^* = \mathbb{P}(\cdot|\psi_{-1} = 1) = \mathbb{P}(\cdot|\tau_0 \geq \sigma_0). \tag{15}$$

In particular,

$$B_1 = \inf\{T_n : n \in \mathbb{Z}, T_n > 0, \psi_{n-1} = 1\}, \quad B_0 = \sup\{T_n : n \in \mathbb{Z}, T_n \leq 0, \psi_{n-1} = 1\}. \tag{16}$$

4.1 The age of information for the pushout system

To compute the law of $\alpha(0)$ we shall use the Palm inversion formula

$$\widetilde{\mathbb{E}} f(\alpha(0)) = \frac{\mathbb{E}^* \int_{B_0}^{B_1} f(\alpha(t))\, dt}{\mathbb{E}^*(B_1 - B_0)}, \tag{17}$$

where $f : \mathbb{R} \to \mathbb{R}$ is bounded and measurable or of constant sign and measurable. The denominator is easy to compute:

$$\mathbb{E}^*(B_1 - B_0) = \left(\widetilde{\mathbb{E}} \sum_n \psi_{n-1} \mathbf{1}_{0<T_n<1}\right)^{-1} = \left(\lambda \mathbb{E} \int_{\mathbb{R}} \psi_{-1} \mathbf{1}_{0<t<1}\, dt\right)^{-1} = \frac{1}{\lambda \mathbb{P}(\tau_0 \geq \sigma_0)}, \tag{18}$$

where we used Campbell's formula. By the non-triviality assumption (14), $\mathbb{E}^*(B_1 - B_0) < \infty$.

We will need the following random integer below.

$$N := \inf\{\ell \geq 0 : \tau_\ell \geq \sigma_\ell\} = \min\{\ell \geq 0 : \tau_\ell \geq \sigma_\ell\}. \tag{19}$$

Theorem 1. *Consider the pushout system under stationarity assumptions and assume that (14) holds. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a bounded absolutely continuous function with a.e. derivative F'. Then*

$$\widetilde{\mathbb{E}} F'(\alpha(0)) = \lambda \mathbb{E}\left[F\left(\tau_{-1} + \sum_{i=0}^{N-1} \tau_i + \sigma_N\right) - F(\sigma_N);\, \tau_{-1} > \sigma_{-1}\right], \tag{20}$$

where N is defined in (19).

Proof. We have $N < \infty$ because of stationarity and hence the expression in the brackets of (20) makes sense. Message N is successful ($\psi_N = 1$) and, by the first of (16) and (13),

$$B_0 = T_0 = 0,\ B'_0 = T_N + \sigma_N,\ B_1 = T_{N+1}, \quad \mathbb{P}^*\text{-a.s.}$$

To compute the integral in the numerator of (17) we take a close look at the function α restricted on the interval $[B_0, B_1) = [T_0, T_{N+1})$. Note that the only successful departures are precisely the points B'_k where reading periods end, whereas the only successful arrivals are the last arrivals on a reading period. If $T_0 \le t < T_N + \sigma_N$ then $D_t = B'_{-1}$ and so $S_{D_t} = S_{B'_1} = T_{-1}$, since $T_0 = 0$ initiates a reading period, so the last successful arrival before this is the arrival that ended the previous reading period. If $T_N + \sigma_N \le t < T_{N+1}$ then $D_t = B'_0 = T_N + \sigma_N$ and $S_{D_t} = S_{B'_0} = T_N$. Thus,

$$\alpha(t) = \begin{cases} t - T_{-1}, & T_0 \le t < T_N + \sigma_N \\ t - T_N, & T_N + \sigma_N \le t < T_{N+1} \end{cases}, \quad \mathbb{P}^*\text{-a.s.}$$

Then, $\mathbb{P}^*$-a.s., $B_0 = T_0 = 0$ (see (13)) and

$$\int_{B_0}^{B_1} f(\alpha(t))\,dt = \int_{T_0}^{T_{N+1}} f(\alpha(t))\,dt = \int_{T_0}^{T_N+\sigma_N} f(t - T_{-1})\,dt + \int_{T_N+\sigma_N}^{T_{N+1}} f(t - T_N)\,dt$$
$$= F(T_N + \sigma_N - T_{-1}) - F(T_0 - T_{-1}) + F(T_{N+1} - T_N) - F(\sigma_N),$$

and thus, since $\mathbb{E}^* F(T_0 - T_{-1}) = \mathbb{E}^* F(T_{N+1} - T_N)$,

$$\mathbb{E}^* \int_{B_0}^{B_1} f(\alpha(t))\,dt = \mathbb{E}^* \left[F\left(\tau_{-1} + \sum_{i=0}^{N-1} \tau_i + \sigma_N \right) - F(\sigma_N) \right].$$

We can rewrite (18) as $\mathbb{E}^*(B_1 - B_0) = 1/\lambda \mathbb{P}(\tau_{-1} \ge \sigma_{-1})$. Dividing the last display by this expression and using the relation (15) between $\mathbb{P}^*$ and $\mathbb{P}$ we arrive at (20). □

At this level of generality it is not possible to have a more explicit formula. However, given information about the law of the sequence (τ_n, σ_n), $n \in \mathbb{Z}$, we can proceed further. For example, assuming that the τ_n, $n \in \mathbb{Z}$, is independent of σ_n, $n \in \mathbb{Z}$, and both sequences have known laws then a further simplification is possible. If, in addition, the $\mathbb{P}$-law of one of the sequences is that of i.i.d. exponential random variables then it is possible to elaborate further and derive an almost closed-form formula.

Theorem 2. *Consider the pushout system and assume that (τ_n, σ_n), $n \in \mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0 < \infty$ and $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$. Assume further that τ_n is independent of σ_n for all n. Then, for $u > 0$,*

$$\widetilde{\mathbb{E}} e^{-u\alpha(0)} = \frac{1 - \mathbb{E}e^{-u\tau}}{u\mathbb{E}\tau} \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]} \tag{21}$$

In particular, under $\widetilde{\mathbb{P}}$, $\alpha(0)$ is the sum of two independent random variables:

$$\alpha(0) \overset{(\mathrm{d})}{=} \overline{\tau} + \mathbf{R}, \tag{22}$$

where $\overline{\tau}$ is the stationary version of τ and $\mathbf{R}$ is a typical reading interval length.

Corollary 1. *The $\widetilde{\mathbb{P}}$-distribution of $\alpha(0)$ is absolutely continuous.*

To prove Theorem 2, we shall make use of the following elementary fact, often known under the name "découpage de Lévy".

Lemma 2. *Let $X_1, X_2, \ldots$ be i.i.d. random elements in an arbitrary measurable space $(S, \mathscr{S})$ with common law μ and let $B \in \mathscr{S}$ have $\mu(B) > 0$. Let $N = \inf\{n \geq 1 : X_n \in B\}$. Then*

(i) $(X_1, \ldots, X_{N-1})$ *is independent of* X_N*;*

(ii) X_N *has law* $\mu(\cdot|B)$*;*

(iii) $\mathbb{P}(N = n) = \mu(S - B)^{n-1}\mu(B)$, $n \geq 1$.

Moreover, the distribution of $(X_1, \ldots, X_N)$ can be expressed neatly as follows. Let $X'', X'_1, X'_2, \ldots$ be independent random elements, and independent of N, such that

$$\mathbb{P}(X'' \in \cdot) = \mu(\cdot|B), \quad \mathbb{P}(X'_i \in \cdot) = \mu(\cdot|S - B), \quad i = 1, 2, \ldots$$

Then

$$(X_1, \ldots, X_N) \overset{(\mathrm{d})}{=} (X'_1, \ldots, X'_{N-1}, X''),$$

where, by definition, $(X'_1, \ldots, X'_{N-1}, X'') = X''$ if $N = 1$.

The proof is trivial and is thus omitted.

Proof of Theorem 2. For fixed $u > 0$, let $F(x) = e^{-ux}$, $x \geq 0$. Then $F'(x) = -ue^{-ux}$ and $F(x_1 + x_2) = F(x_1)F(x_2)$ for all $x_1, x_2 \geq 0$. With a view towards applying Lemma 2 to the sequence (τ_n, σ_n), $n \geq 0$, let $B := \{(t,s) \in \mathbb{R}^2 : t \geq s \geq 0\}$. For simplicity, let

$$p := \mathbb{P}(\tau \geq \sigma), \quad q = 1 - p.$$

By (20),

$$\widetilde{\mathbb{E}}F'(\alpha(0)) = \lambda p\, \mathbb{E}^* \left[F\left(\tau_{-1} + \sum_{i=0}^{N-1} \tau_i + \sigma_N\right) - F(\sigma_N) \right] = \lambda p\, \mathbb{E} \left[F\left(\tau'' + \sum_{i=0}^{N-1} \tau_i' + \sigma''\right) - F(\sigma'') \right],$$

where $N, \tau'', \tau_1', \tau_2', \ldots, \sigma''$ are independent random variables such that

$$\mathbb{P}(N = n) = q^n p, \quad \tau'' \stackrel{(d)}{=} (\tau | \tau > \sigma), \quad \sigma'' \stackrel{(d)}{=} (\sigma | \tau > \sigma), \quad \tau' \stackrel{(d)}{=} (\tau | \tau \leq \sigma). \tag{23}$$

Hence, letting $F(x) = e^{-ux}$ for some fixed $u > 0$ we have

$$\begin{aligned} \widetilde{\mathbb{E}}F'(\alpha(0)) &= \lambda p\, \mathbb{E} \left\{ F(\tau'')F(\sigma'') \prod_{i=0}^{N-1} F(\tau_i') - F(\sigma'') \right\} = \lambda p\, \mathbb{E}F(\sigma'')\ \left\{ \mathbb{E}F(\tau'')\, \mathbb{E}[(\mathbb{E}F(\tau'))^N] - 1 \right\} \\ &= \lambda p\, \mathbb{E}F(\sigma'')\ \left\{ \mathbb{E}F(\tau'')\, \frac{p}{1 - q\mathbb{E}F(\tau')} - 1 \right\} = \lambda p\, \frac{\mathbb{E}F(\sigma'')\,(\mathbb{E}F(\tau) - 1)}{1 - q\mathbb{E}F(\tau')}, \end{aligned}$$

whence, after a little algebra, we obtain (21):

$$-u\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \lambda(\mathbb{E}e^{-u\tau} - 1)\, \frac{p\mathbb{E}e^{-u\sigma''}}{1 - q\mathbb{E}e^{-u\tau'}} = \lambda(\mathbb{E}e^{-u\tau} - 1)\, \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]}.$$

To prove (22) note that the first term in (21) equals $\frac{1 - \mathbb{E}e^{-u\tau}}{u\mathbb{E}\tau}$ is equal to $\mathbb{E}e^{-u\overline{\tau}}$. So $\alpha(0) \stackrel{(d)}{=} \overline{\tau} + Y$ where Y is an independent random variable whose Laplace transform is the second term in (21):

$$\mathbb{E}e^{-uY} = \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]}. \tag{24}$$

Recalling that N is the index of the first successful arrival after the origin, we see that, again after a little algebra involving a geometric series,

$$\mathbb{E}e^{-u(T_N + \sigma_N)} = \mathbb{E} \sum_{n=0}^{\infty} e^{-u(\tau_0 + \cdots + \tau_{n-1} + \sigma_n)}\, \mathbf{1}_{\tau_0 < \sigma_0, \ldots, \tau_{n-1} < \sigma_{n-1}, \tau_n \geq \sigma_n} = \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]}. \tag{25}$$

This shows that $\mathbb{E}e^{-uY} = \mathbb{E}e^{-u(T_N+\sigma_N)}$ for all $u > 0$, and thus

$$Y \overset{(\mathrm{d})}{=} T_N + \sigma_N.$$

But, $\mathbb{P}^*$–a.s., $T_N + \sigma_N = B_0' - B_0 \overset{(\mathrm{d})}{=} \mathbf{R}$. □

***Remark* 1.** We may decompose $\alpha(0)$ in a different way. Rearranging terms in the $\widetilde{\mathbb{P}}$-Laplace transform of $\alpha(0)$ we have

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \mathbb{E}e^{-u\sigma''} \frac{\lambda p}{u} \frac{1 - \mathbb{E}e^{-u\tau}}{1 - q\mathbb{E}e^{-u\tau'}},$$

which implies that there is a second decomposition for the law of $\alpha(0)$:

$$\alpha(0) \overset{(\mathrm{d})}{=} \sigma'' + Z,$$

where σ'' and Z are independent random variables, with σ'' having the law of σ conditional on $\tau \geq \sigma$ and Z having Laplace transform $(\lambda p/u)(1 - \mathbb{E}e^{-u\tau})/(1 - q\mathbb{E}e^{-u\tau'})$.

Corollary 2. *Under the assumptions of Theorem 2, we have*

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau \wedge \sigma}{\mathbb{P}(\tau \geq \sigma)}. \tag{26}$$

Proof. Look at (22). We have $\mathbb{E}\overline{\tau} = \mathbb{E}\tau^2/2\mathbb{E}\tau$ and

$$\mathbb{E}\mathbf{R} = \mathbb{E}(T_N + \sigma_N) = \frac{\mathbb{E}\tau \wedge \sigma}{p}.$$

□

Corollary 3. *Under the assumptions of Theorem 2, and if, in addition, the variables σ_n are exponential with rate μ, then, under $\widetilde{\mathbb{P}}$,*

$$\alpha(0) \overset{(\mathrm{d})}{=} \overline{\tau} + \frac{\mathbf{e}}{\mu},$$

where $\mathbf{e}$ is a rate-1 exponential random variable, independent of $\overline{\tau}$ and so

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{1}{\mu}.$$

Proof. We use (22). We just have to show that the reading interval length $\mathbf{R}$ is exponential with rate μ. Since

$$\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma] = \mathbb{E}\int_0^\tau e^{-us}\mu e^{-\mu s} ds = \mu\mathbb{E}\int_0^\tau e^{-(u+\mu)s} ds = \frac{\mu}{u+\mu}[1 - \mathbb{E}e^{-(u+\mu)\tau}],$$

$$\mathbb{E}[e^{-u\tau}; \tau < \sigma] = \mathbb{E}e^{-u\tau}\mathbb{P}(\sigma \geq \tau|\tau) = \mathbb{E}e^{-u\tau}e^{-\mu\tau} = \mathbb{E}e^{-(u+\mu)\tau},$$

we have, from (24), that the Laplace transform of $\mathbf{R}$ is

$$\mathbb{E}e^{-u\mathbf{R}} = \frac{\mathbb{E}[e^{-u\sigma}; \tau \geq \sigma]}{1 - \mathbb{E}[e^{-u\tau}; \tau < \sigma]} = \frac{\frac{\mu}{u+\mu}[1 - \mathbb{E}e^{-(u+\mu)\tau}]}{1 - \mathbb{E}e^{-(u+\mu)\tau}} = \frac{\mu}{u+\mu}.$$

□

Corollary 4. *Under the assumptions of Theorem 2, and if, in addition, the variables τ_n are exponential with rate λ, then*

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{\lambda\mathbb{E}e^{-(\lambda+u)\sigma}}{u + \lambda\mathbb{E}e^{-(\lambda+u)\sigma}}, \quad \widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\lambda\,\mathbb{E}e^{-\lambda\sigma}}.$$

Proof. Since τ is exponential we have $\overline{\tau} \stackrel{(\mathrm{d})}{=} \tau$ and so

$$\mathbb{E}e^{-u\overline{\tau}} = \mathbb{E}e^{-u\tau} = \frac{\lambda}{u+\lambda}.$$

Using (24), we have

$$\mathbb{E}e^{-u\mathbf{R}} = \frac{(u+\lambda)\mathbb{E}e^{-(u+\lambda)\sigma}}{u + \lambda\mathbb{E}e^{-(u+\lambda)\sigma}}.$$

Equation (21) says that the Laplace transform of $\alpha(0)$ is the product of the last two displays and so this derives the first formula. Next use (26). Since

$$\mathbb{E}\tau \wedge \sigma = \frac{1}{\lambda}(1 - \mathbb{E}e^{-\lambda\sigma}), \quad \mathbb{P}(\tau > \sigma) = \mathbb{E}e^{-\lambda\sigma},$$

we have

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\lambda} + \frac{1}{\lambda}\cdot\frac{1 - \mathbb{E}e^{-\lambda\sigma}}{\mathbb{E}e^{-\lambda\sigma}} = \frac{1}{\lambda\mathbb{E}e^{-\lambda\sigma}}.$$

□

Finally, a direct consequence of either of the above corollaries is:

Corollary 5. *If the τ_n are i.i.d. exponential with rate λ, if the σ_n are i.i.d. exponential with rate μ, and if the two sequences are independent, then, under $\widetilde{\mathbb{P}}$,*

$$\alpha(0) \stackrel{(\mathrm{d})}{=} \frac{\mathbf{e}_1}{\lambda} + \frac{\mathbf{e}_2}{\mu},$$

where $\mathbf{e}_1, \mathbf{e}_2$ are two independent unit-rate exponential random variables.

4.2 The new age of information for the pushout system

Recall that $\beta(t) = A_t - S_{D_t}$. Under $\widetilde{\mathbb{P}}$, the law of $\beta(t)$ is independent of t.

Lemma 3. *The $\widetilde{\mathbb{P}}$-law of $\beta(t)$ has a nontrivial atom at* 0.

Proof. Indeed,

$$\widetilde{\mathbb{P}}(\beta(t) = 0) = \widetilde{\mathbb{P}}(A_t = S_{D_t}) = \widetilde{\mathbb{P}}(q(t) = 0) > 0.$$

The latter is positive because of the non-triviality assumption (14). □

Theorem 3. *Consider the pushout system under stationarity assumptions and assume that* (14) *holds. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function that is bounded or nonnegative. Then*

$$\widetilde{\mathbb{E}}f(\beta(0)) = \lambda\mathbb{E}\left[\sum_{i=0}^{N-1} \tau_i f\left(\sum_{j=-1}^{i-1} \tau_j\right) + \sigma_N f\left(\sum_{j=-1}^{N-1} \tau_j\right) + (\tau_N - \sigma_N) f(0);\, \tau_{-1} > \sigma_{-1}\right], \tag{27}$$

where N is as in Theorem 1.

Proof. We use again the Palm inversion formula

$$\widetilde{\mathbb{E}}f(\beta(0)) = \frac{\mathbb{E}^* \int_{B_0}^{B_1} f(\beta(t))\, dt}{\mathbb{E}^*(B_1 - B_0)}, \tag{28}$$

where the notation is as before. We now have

$$\beta(t) = A_t - S_{D_t} = \begin{cases} T_i - T_{-1}, & T_0 \le T_i \le t < T_{i+1} \le T_N + \sigma_N,\ i \ge 0, \\ 0, & T_N + \sigma_N \le t < T_{N+1} \end{cases}, \quad \mathbb{P}^*\text{-a.s.}$$

Hence the integral in (28) is

$$\begin{aligned} \int_{B_0}^{B_1} f(\beta(t))\, dt &= \int_{T_0}^{T_{N+1}} f(\beta(t))\, dt \\ &= \sum_{i: T_0 \le T_i < T_{i+1} \le T_N} \int_{T_i}^{T_{i+1}} f(T_i - T_{-1}) dt + \int_{T_N}^{T_N+\sigma_N} f(T_N - T_{-1}) dt + \int_{T_N+\sigma_N}^{T_{N+1}} f(0) dt \\ &= \sum_{i=0}^{N-1} \tau_i f(T_i - T_{-1}) + \sigma_N f(T_N - T_{-1}) + (\tau_N - \sigma_N) f(0). \end{aligned}$$

Substitute this into (28) and use $\mathbb{E}^*(B_1 - B_0) = 1/\lambda\mathbb{P}(\tau_{-1} \ge \sigma_{-1})$ to obtain (27). □

Corollary 6 (Continuation of Lemma 3). *The atom of $\beta(0)$ at 0 has value*

$$\widetilde{\mathbb{P}}(\beta(0)=0)=\lambda\,\mathbb{E}[(\tau_N-\sigma_N);\,\tau_{-1}>\sigma_{-1}]. \tag{29}$$

Proof. Let, in (27), $f(x):=\mathbf{1}_{x=0}$. Since all the τ_n and σ_n are nonzero with probability 1, (29) follows. □

Theorem 4. *Consider the pushout system and assume that (τ_n,σ_n), $n\in\mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0<\infty$ and $\mathbb{P}(\tau_0\geq\sigma_0)>0$. Assume further that τ_n is independent of σ_n for all n. Then the $\widetilde{\mathbb{P}}$-law of $\beta(0)$ can be described as*

$$\beta(0)\overset{(\mathrm{d})}{=}\begin{cases}0, & \textit{with probability } \dfrac{\mathbb{E}(\tau-\sigma)^+}{\mathbb{E}\tau}\\[2ex] \mathbf{C}, & \textit{with probability } \dfrac{\mathbb{E}\tau\wedge\sigma}{\mathbb{E}\tau}\end{cases}, \tag{30}$$

where $\mathbf{C}$ has the distribution of a typical cycle length;

$$\mathbb{E}e^{-u\mathbf{C}}=\mathbb{E}^*e^{-u(B_1-B_0)}=\frac{\mathbb{E}[e^{-u\tau};\tau>\sigma]}{1-\mathbb{E}[e^{-u\tau};\tau\leq\sigma]}, \tag{31}$$

In particular,

$$\widetilde{\mathbb{E}}\beta(0)=\frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{P}(\tau\geq\sigma)}. \tag{32}$$

Proof. Using (29) and independence,

$$\widetilde{\mathbb{P}}(\beta(0)=0)=\lambda\,\mathbb{E}(\tau_N-\sigma_N)\,\mathbb{P}(\tau_{-1}>\sigma_{-1}).$$

By Lemma 2 and (23), we further have

$$\begin{aligned}\widetilde{\mathbb{P}}(\beta(0)=0)&=\lambda\,\mathbb{E}(\tau''-\sigma'')\,\mathbb{P}(\tau>\sigma)\\&=\lambda\,\mathbb{E}(\tau-\sigma|\tau>\sigma)\,\mathbb{P}(\tau>\sigma)\\&=\lambda\,\mathbb{E}(\tau-\sigma)^+.\end{aligned}$$

This proves the upper part of (30). To prove the lower part notice, from (27),

$$\begin{aligned}\widetilde{\mathbb{E}}[f(\beta(0));\beta(0)>0]&=\lambda\,\mathbb{P}(\tau_{-1}>\sigma_{-1})\,\mathbb{E}\left[\sum_{i=0}^{N-1}\tau_i f\left(\sum_{j=-1}^{i-1}\tau_j\right)+\sigma_N f\left(\sum_{j=-1}^{N-1}\tau_j\right)\Bigg|\tau_{-1}>\sigma_{-1}\right]\\&=\lambda p\,\mathbb{E}\left[\sum_{i=0}^{N-1}\tau_i' f\left(\tau_{-1}''+\sum_{j=0}^{i-1}\tau_j'\right)+\sigma'' f\left(\tau_{-1}''+\sum_{j=0}^{N-1}\tau_j'\right)\right],\end{aligned}$$

where we used Lemma 2 and the definitions (23). Next, let $f(x) = e^{-ux}$ and write the above as

$$
\begin{aligned}
\widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) > 0] &= \lambda p\,(\mathbb{E} f(\tau''))\,\mathbb{E}\left[\sum_{i=0}^{N-1} (\mathbb{E}\tau')(\mathbb{E} f(\tau'))^i + (\mathbb{E}\sigma'')(\mathbb{E} f(\tau'))^N\right] \\
&= \lambda p\,(\mathbb{E} f(\tau''))\left[(\mathbb{E}\tau')\,\mathbb{E}\left(\frac{1-(\mathbb{E} f(\tau'))^N}{1-\mathbb{E} f(\tau')}\right) + (\mathbb{E}\sigma'')(\mathbb{E} f(\tau'))^N\right] \\
&= \lambda p\,(\mathbb{E} f(\tau''))\left[\frac{\mathbb{E}\tau'}{1-\mathbb{E} f(\tau')}\left(1 - \frac{p}{1-q\mathbb{E} f(\tau')}\right) + (\mathbb{E}\sigma'')\frac{p}{1-q\mathbb{E} f(\tau')}\right] \\
&= \lambda p\,(\mathbb{E} f(\tau''))\,\frac{q\mathbb{E}\tau' + p\mathbb{E}\sigma''}{1-q\mathbb{E} f(\tau')} = \lambda p\,(\mathbb{E} f(\tau''))\,\frac{\mathbb{E}\tau\wedge\sigma}{1-q\mathbb{E} f(\tau')},
\end{aligned}
$$

that is precisely the lower part of (30). The last equality in (31) is easily verified along the same lines. To finally show (32) just note that

$$
\widetilde{\mathbb{E}}\beta(0) = \frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{E}\tau}\,\mathbb{E}C = \frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{E}\tau}\,\frac{\mathbb{E}\tau}{\mathbb{P}(\tau > \sigma)}.
$$

□

***Remark* 2.** Notice that β does not suffer from the same drawback as α when τ^2 is not integrable. Indeed, here, under the condition $\mathbb{E}\tau < \infty$ we have $\widetilde{\mathbb{E}}\beta(0) \le 1$, regardless of the variance of τ.

Corollary 7. *Let the assumptions of Theorem 4 hold true.*

(i) If the variables τ_n are exponential with rate λ, then

$$
\widetilde{\mathbb{E}} e^{-u\beta(0)} = 1 - \frac{u(1-\mathbb{E}e^{-\lambda\sigma})}{u + \lambda\mathbb{E}e^{-(\lambda+u)\sigma}}, \qquad \widetilde{\mathbb{E}}\beta(0) = \frac{1}{\lambda\mathbb{E}e^{-\lambda\sigma}} - \frac{1}{\lambda}.
$$

(ii) If the variables σ_n are exponential with rate μ, then

$$
\widetilde{\mathbb{E}} e^{-u\beta(0)} = 1 - \frac{1-\mathbb{E}e^{-\mu\tau}}{\mu\mathbb{E}\tau}\,\frac{1-\mathbb{E}e^{-u\tau}}{1-\mathbb{E}e^{-(u+\mu)\tau}}, \qquad \widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu}.
$$

(iii) If the τ_n are with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$
\beta(0) \overset{(\mathrm{d})}{=} \begin{cases} 0, & \text{with probability } \frac{\mu}{\lambda+\mu} \\ \frac{\mathbf{e}_1}{\lambda} + \frac{\mathbf{e}_2}{\mu}, & \text{with probability } \frac{\lambda}{\lambda+\mu} \end{cases}, \qquad \widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu},
$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are two independent unit-rate exponential random variables.

5 The blocking system

The blocking system is defined by the requirement that only those messages for which $\mathscr{Z}_n$ holds are admitted. The remaining ones are immediately rejected (blocked). It is well-known that if

$$\mathbb{P}(\sup_{i\leq -1}(\sigma_i + T_i) \leq 0) > 0 \tag{33}$$

then the system admits a unique steady-state, see [1, Chapter 2, Section 5.2]. Under this condition, (12) holds.

We have $\psi_n = \chi_n$ for all $n \in \mathbb{Z}$ (a message is successful if and only if it is admitted) and

$$\psi_n \text{ is a measurable function of } (\tau_m, \sigma_m : m \leq n-1). \tag{34}$$

Recall that we use letters B_k, B'_k for the beginnings and ends of reading periods, respectively. In other words,

$$\{B_k : k \in \mathbb{Z}\} = \{T_n : n \in \mathbb{Z}, \psi_n = 1\},$$
$$\{B'_k : k \in \mathbb{Z}\} = \{T_n + \sigma_n : n \in \mathbb{Z}, \psi_n = 1\}.$$

Therefore the Palm probability $\mathbb{P}^*$ of $\widetilde{\mathbb{P}}$ with respect to $\{B_k\}$ admits a simpler representation:

Lemma 4. *$\mathbb{P}^*$ is the Palm probability of $\widetilde{\mathbb{P}}$ with respect to the (stationary) point process*

$$\sum_{n\in\mathbb{Z}} \psi_n \delta_{T_n}$$

and

$$\mathbb{P}^* = \mathbb{P}(\cdot|\psi_0 = 1). \tag{35}$$

Recalling that $\{B_k\}$ and $\{B'_k\}$ are interlaced sequences let us compute the quantities S_t (last successful arrival before t), D_t (last successful departure before t), and S_{D_t} (last successful arrival before D_t) depending whether t falls in a reading interval (that is, between B_k and B'_k for some k) or not (that is, between B'_k and

B_{k+1} for some k). Since $\{B_k\}$ is the totality of successful arrivals, we have that, for all $k \in \mathbb{Z}$,

$$B_k \le t < B_{k+1} \Rightarrow S_t = B_k.$$

Since $\{B'_k\}$ is the totality of successful departures, we have that, for all $k \in \mathbb{Z}$,

$$B'_k \le t < B'_{k+1} \Rightarrow D_t = B'_k.$$

It then follows that, for all $k \in \mathbb{Z}$,

$$S_{D_t} = \begin{cases} B_{k-1}, & \text{if } B_k \le t < B'_k \\ B_k, & \text{if } B'_k \le t < B_{k+1} \end{cases}. \tag{36}$$

5.1 The age of information for the blocking system

We shall use the Palm inversion formula (17) for the process $\alpha(t) = t - S_{D_t}$, $t \in \mathbb{R}$, for the blocking system. By Campbell's formula we have that the denominator of (17) is

$$\mathbb{E}^*(B_1 - B_0) = \frac{1}{\lambda \mathbb{P}(\psi_0 = 1)}, \tag{37}$$

however, unlike in the pushout system, the probability in the denominator depends on the full distribution and the dynamics of the system and so it does not admit an explicit form without further assumptions. In what follows, let

$$N := \inf\{\ell \ge 1 : \tau_0 + \cdots + \tau_{\ell-1} \ge \sigma_0\}. \tag{38}$$

Theorem 5. *Consider the blocking system under stationarity assumptions and assume that (33) holds. Let f be bounded and measurable or locally integrable and nonnegative function and let F be such that $F' = f$. Then*

$$\widetilde{\mathbb{E}} f(\alpha(0)) = \lambda\, \mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N);\, \psi_0 = 1] = \frac{\mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N) | \psi_0 = 1]}{\mathbb{E}[T_N | \psi_0 = 1]}, \tag{39}$$

where N is defined by (38).

Proof. Under $\mathbb{P}^*$, message 0 is successful (admitted) and N is the first successful (admitted) message after that. Note that $N < \infty$. Thus,

$$B_1 = T_N, \quad \mathbb{P}^*\text{-a.s.} \tag{40}$$

Note also that, with $\mathfrak{a} = \sum_{n\in\mathbb{Z}} \delta_{T_n}$,

$$N = \mathfrak{a}([0,\sigma_0]) = \sum_{n=0}^{\infty} \mathbf{1}_{T_n \le \sigma_0}, \quad \mathbb{P}\text{-a.s. and (hence) } \mathbb{P}^*\text{-a.s.} \tag{41}$$

By (36), and since $B_0' = T_0 + \sigma_0$, $\mathbb{P}^*$-a.s., the function α on $[B_0, B_1)$ is given by

$$\alpha(t) = t - S_{D_t} = \begin{cases} t - B_{-1}, & T_0 \le t < T_0 + \sigma_0 \\ t - B_0, & T_0 + \sigma_0 \le t < T_N \end{cases}, \quad \mathbb{P}^*\text{-a.s.}$$

Hence, for functions f, F as in the theorem statement, with $F' = f$,

$$\int_{B_0}^{B_1} f(\alpha(t))\,dt = \int_{T_0}^{T_N} f(\alpha(t))\,dt = \int_{T_0}^{T_0+\sigma_0} f(t - B_{-1})\,dt + \int_{T_0+\sigma_0}^{T_N} f(t - B_0)\,dt$$
$$= F(B_0 - B_{-1} + \sigma_0) - F(B_0 - B_{-1}) + F(B_1 - B_0) - F(\sigma_0), \quad \mathbb{P}^*\text{-a}$$

and thus, since $\mathbb{E}^* F(B_0 - B_{-1}) = \mathbb{E}^* F(B_1 - B_0)$,

$$\mathbb{E}^* \int_{B_0}^{B_1} f(\alpha(t))\,dt = \mathbb{E}^* F(B_0 - B_{-1} + \sigma_0) - \mathbb{E}^* F(\sigma_0)$$
$$= \mathbb{E}^* F(B_1 - B_0 + \sigma_N) - \mathbb{E}^* F(\sigma_N).$$

Here we used the fact that $\mathbb{P}^*$ is preserved by θ_{B_k} for all $k \in \mathbb{Z}$. Taking into account (17), (37) and (40), we can conclude. □

***Remark* 3.** Note that, since there is no ready-made expression for $\mathbb{P}(\psi_0 = 1)$, the second formula in (39) turns out to be more useful for further computations.

We now introduce

$$\mathfrak{a}(t) := \inf\{\ell \ge 0 : T_\ell \ge t\}, \quad t \ge 0, \tag{42}$$

so that the variable N defined by (38) is simply the value of $\mathfrak{a}(t)$ for $t = \sigma_0$:

$$\mathfrak{a}(\sigma_0) = N.$$

Note that $\mathfrak{a}(t)$ is left-continuous at all $0 < t < \infty$ with $zarr(0) = 0$ and $\mathfrak{a}(0+) = 1$. Since $\mathfrak{a} = \sum_{n\in\mathbb{Z}} \delta_{T_n}$, we have

$$\mathfrak{a}(t) = \mathfrak{a}([0,t)) = 1 + \mathfrak{a}((0,t)), \quad t \ge 0.$$

Remembering that $\mathbb{P}$ is a Palm probability and $\mathbb{P}(T_0 = 0) = 1$, define

$$U(t) := \mathbb{E}\mathfrak{a}(t) = \sum_{n=0}^{\infty} \mathbb{P}(T_n < t), \quad t \geq 0. \tag{43}$$

If the τ_n are i.i.d., then U is known as 0-potential function (if $T_0, T_1, T_2, \ldots$ is thought of as a random walk) or renewal function (if $T_0, T_1, T_2, \ldots$ are thought of as the points of a renewal process). We have that U is left-continuous on $[0, \infty)$ with $U(0) = 0$, $U(0+) = 1$. We shall deal with the renewal case next. We will also need the definition

$$W(f, t) := \mathbb{E}f(T_{\mathfrak{a}(t)}), \quad t \geq 0, \tag{44}$$

where f is an appropriate function for which the expectation exists. In particular, with $f(x) = e^{-ux}$ for some $u > 0$, we let

$$W_u(t) = \mathbb{E}e^{-uT_{\mathfrak{a}(t)}}, \tag{45}$$

and with $f(x) = x^p$ for some $p > 0$, we let

$$M_p(t) = \mathbb{E}T^p_{\mathfrak{a}(t)}.$$

The following result gives the Laplace transform of the $\widetilde{\mathbb{P}}$-marginal of $\alpha(t)$ in terms of functions that can be computed as unique solutions to fixed-point equations.

Theorem 6. *Consider the blocking system and assume that (τ_n, σ_n), $n \in \mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0 < \infty$ and $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$. Assume further that τ_n is independent of σ_n for all n. Then, for $u > 0$,*

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \mathbb{E}e^{-u\sigma} \cdot \frac{1 - \mathbb{E}e^{-uT_N}}{u\mathbb{E}T_N} = \mathbb{E}e^{-u\sigma} \cdot \frac{1 - \mathbb{E}W_u(\sigma)}{u\,\mathbb{E}\tau\,\mathbb{E}U(\sigma)}, \tag{46}$$

where U and W_u are the unique solutions to the fixed-point equations

$$U(t) = 1 + \int_{(0,t]} U(t-x)\mathbb{P}(\tau \in dx) \tag{47}$$

$$W_u(t) = \int_{(t,\infty)} e^{-ux}\mathbb{P}(\tau \in dx) + \int_{(0,t]} W_u(t-x)e^{-ux}\mathbb{P}(\tau \in dx). \tag{48}$$

In particular, under $\widetilde{\mathbb{P}}$, $\alpha(0)$ is the sum of two independent random variables:

$$\alpha(0) \overset{\text{(d)}}{=} \sigma + \overline{T_N}, \tag{49}$$

where $\overline{T_N}$ is the stationary version of T_N.

Proof. Observe first that $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$ implies (by the ergodic theorem) (33) and hence a unique steady-state version exists. Using the fact that ψ_n is a measurable function of the variables τ_m, σ_m with $m \leq n$ [see (34)] we write (39) as

$$\widetilde{\mathbb{E}} F'(\alpha(0)) = \frac{\mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N)]}{\mathbb{E} T_N}, \tag{50}$$

with $N = \inf\{\ell \geq 1 : \tau_0 + \cdots + \tau_{\ell-1} \geq \sigma_0\}$, $\mathbb{P}$-a.s. Since $N - 1 = \inf\{i \geq 0 : \tau_0 + \cdots + \tau_i \geq \sigma_0\}$, it follows that $N-1$ is a stopping time with respect to $\mathscr{A}_i$, $i \geq 0$, where $\mathscr{A}_i$ is the σ-algebra generated by $(\sigma_0, \tau_0, \ldots, \tau_i)$. Let $F(x) = e^{-ux}$. Then

$$\mathbb{E}[F(T_N + \sigma_N) - F(\sigma_N)] = \mathbb{E}[F(T_N)F(\sigma_N) - F(\sigma_N)] = [\mathbb{E}F(T_N) - 1]\,\mathbb{E}F(\sigma_N),$$

where the last equality needs that $N-1$ is a stopping time. Noting that $\mathbb{E}F(\sigma_N) = \mathbb{E}F(\sigma)$ we obtain the first equality in (46) from which decomposition (49) follows at once.

For the last equality of (46) we have

$$\mathbb{E}\, T_N = \mathbb{E}\sum_{i=0}^{N-1} \tau_i = \mathbb{E}\sum_{i=0}^{\infty} \tau_i\, \mathbf{1}_{T_i \leq \sigma_0} = \sum_{i=0}^{\infty} (\mathbb{E}\tau_i)\mathbb{P}(T_i \leq \sigma_0) = (\mathbb{E}\tau) \sum_{i=0}^{\infty} \mathbb{P}(T_i \leq \sigma_0) = (\mathbb{E}\tau)\, \mathbb{E}U(\sigma), \tag{51}$$

and,

$$\mathbb{E}e^{-uT_N} = \mathbb{E}e^{-uT_{\mathfrak{a}(\sigma_0)}} = \mathbb{E}\,\mathbb{E}[e^{-uT_{\mathfrak{a}(\sigma_0)}} | \sigma_0] = \mathbb{E}\, W_u(\sigma_0). \tag{52}$$

Equation (47) is the renewal equation from standard renewal theory. To obtain(48) we write

$$W(f,t) = \mathbb{E}f(T_{\mathfrak{a}(t)}) = \mathbb{E}[f(T_{\mathfrak{a}(t)}); t < \tau_0] + \mathbb{E}[f(T_{\mathfrak{a}(t)}); t \geq \tau_0].$$

If $t < \tau_0$ then $\mathfrak{a}(t) = 1$, $T_{\mathfrak{a}(t)} = T_1 = \tau_0$, $\mathbb{P}$-a.s. If $t \geq \tau_0$ and $\tau_0 = x$ then $T_{\mathfrak{a}(t)} \overset{(\mathrm{d})}{=} x + T_{\mathfrak{a}(t-x)}$, under $\mathbb{P}$. Set

$$\Phi_t := T_{\mathfrak{a}(t)}.$$

If τ is independent of (Φ_t) we have

$$T_{\mathfrak{a}(t)} \overset{(\mathrm{d})}{=} \tau + \Phi_{t-\tau}$$

and so

$$f(T_{\mathfrak{a}(t)}) \mathbf{1}_{\tau_0 \le t} \overset{(\mathrm{d})}{=} f(\tau + \Phi_{t-\tau}) \mathbf{1}_{\tau \le t}.$$

Hence

$$W(f,t) = \mathbb{E}[f(\tau);\, \tau > t] + \mathbb{E}[f(\tau + \Phi_{t-\tau});\, \tau \le t]. \tag{53}$$

Letting $f(x) = e^{-ux}$ we further have $F(\tau + \Phi_{t-\tau}) = e^{-u\tau} e^{-u\Phi_{t-\tau}}$ and so

$$\mathbb{E}[e^{-u(\tau+\Phi_{t-\tau})};\, \tau \le t] = \mathbb{E}[e^{-u\tau}\mathbb{E}(e^{-u\Phi_{t-\tau}} | \tau) \mathbf{1}_{\tau \le t}] = \mathbb{E}[e^{-u\tau} W_u(t-\tau) \mathbf{1}_{\tau \le t}],$$

and this establishes (48). □

To compute the first moment of the AoI we need to know the second moment of $T_{\mathfrak{a}(t)}$. Recall that $M_p(t) = \mathbb{E}T^p_{\mathfrak{a}(t)}$ is the p^{th} moment of $T_{\mathfrak{a}(t)}$. These moments can be computed recursively, as in the lemma below, which is of independent interest.

Lemma 5. *If p is a positive integer we have*

$$M_p(t) = \mathbb{E}M_p(t-\tau) + \mathbb{E}\tau^p + \sum_{k=1}^{p-1} \binom{p}{k} \mathbb{E}[\tau^k M_{p-k}(t-\tau)] \tag{54}$$

Proof. Proceed as in the proof of Theorem 6 but let $f(x) = x^p$ in (53):

$$\begin{aligned} M_p(t) &= \mathbb{E}[\tau^p; \tau > t] + \mathbb{E}[(\tau + \Phi_{t-\tau})^p; \tau \le t] \\ &= \mathbb{E}[\tau^p; \tau > t] + \mathbb{E}\left[\sum_{k=0}^{p} \binom{p}{k} \tau^k \Phi^{p-k}_{t-\tau}; \tau \le t\right] \\ &= \mathbb{E}[\tau^p; \tau > t] + \mathbb{E}[\tau^p; \tau \le t] + \mathbb{E}[\Phi^p_{t-\tau}; \tau \le t] + \mathbb{E}\left[\sum_{k=1}^{p-1} \binom{p}{k} \tau^k \Phi^{p-k}_{t-\tau}; \tau \le t\right] \\ &= \mathbb{E}\tau^p + \mathbb{E}M_p(t-\tau) + \sum_{k=1}^{p-1} \binom{p}{k} \mathbb{E}[\tau^k M_{p-k}(t-\tau)]. \end{aligned}$$

□

Let $(U * U)(t) := \int_0^t U(t-x)\, U(dx)$.

Corollary 8. *Under the assumptions of Theorem 6,*

$$\widetilde{\mathbb{E}}\alpha(0) = \mathbb{E}\sigma + \frac{\mathbb{E}T_N^2}{2\mathbb{E}T_N} = \mathbb{E}\sigma + \frac{\mathbb{E}M_2(\sigma)}{2\mathbb{E}M_1(\sigma)} = \mathbb{E}\sigma + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}(\tau(U*U)(\sigma-\tau))}{\mathbb{E}U(\sigma)}. \quad (55)$$

Proof. The first equality in (55) follows from the decomposition (49). The second equality follows from $\mathbb{E}T_N^p = \mathbb{E}\mathbb{E}[T^p_{\mathfrak{a}(\sigma_0)}|\sigma_0] = \mathbb{E}M_p(\sigma)$. We next have

$$M_1(t) = \mathbb{E}\tau\, U(t) \quad (56)$$

and, from (54) with $p=2$,

$$M_2(t) = \mathbb{E}\tau^2 + 2\mathbb{E}[\tau\, M_1(t-\tau)] + \mathbb{E}M_2(t-\tau) = \mathbb{E}\tau^2 + 2\mathbb{E}\tau\,\mathbb{E}[\tau\, U(t-\tau)] + \mathbb{E}M_2(t-\tau).$$

With the help of (47) we can solve this explicitly and express M_2 as a function of U:

$$M_2(t) = \mathbb{E}\tau^2 \cdot U(t) + 2\,\mathbb{E}\tau\,\mathbb{E}[\tau\,(U*U)(t-\tau)]. \quad (57)$$

Using (56) and (57) in the second equality of (55) we arrive at the third one. □

The Laplace transforms of U, W_u and M_2 are easy to obtain explicitly in terms of the Laplace transform of τ:

Lemma 6.

$$\widehat{U}(\xi) := \int_0^\infty e^{-\xi t} U(t)dt = \frac{1/\xi}{1-\mathbb{E}e^{-\xi\tau}}, \quad (58)$$

$$\widehat{W}_u(\xi) := \int_0^\infty e^{-\xi t} W_u(t)dt = \frac{1}{\xi}\cdot\frac{\mathbb{E}[e^{-u\tau} - e^{-(u+\xi)\tau}]}{1-\mathbb{E}e^{-(u+\xi)\tau}}. \quad (59)$$

$$\widehat{M_2}(\xi) := \int_0^\infty e^{-\xi t} M_2(t)dt = \frac{\mathbb{E}\tau^2}{\xi(1-\mathbb{E}e^{-\xi\tau})} + 2(\mathbb{E}\tau)\frac{\mathbb{E}(\tau e^{-\xi\tau})}{\xi(1-\mathbb{E}e^{-\xi\tau})^2}. \quad (60)$$

Proof. Equation (47) then gives

$$\widehat{U}(\xi) = \frac{1}{\xi} + \widehat{U}(\xi)\,\mathbb{E}e^{-\xi\tau},$$

and hence (58) follows. Equation (48) gives

$$\begin{aligned}\widehat{W}_u(\xi) &= \int_0^\infty e^{-\xi t}\mathbb{E}[e^{-u\tau}\mathbf{1}_{\tau>t}]\,dt + \int_0^\infty e^{-\xi t}\mathbb{E}[W_u(t-\tau)e^{-u\tau}\mathbf{1}_{\tau\le t}]\,dt \\ &= \mathbb{E}\left[e^{-u\tau}\frac{1-e^{-\xi\tau}}{\xi}\right] + \mathbb{E}\left[\mathbf{e}^{-u\tau}e^{-\xi\tau}\int_\tau^\infty e^{-\xi(t-\tau)}W_u(t-\tau)dt\right] \\ &= \frac{1}{\xi}\mathbb{E}\left[e^{-u\tau}(1-e^{-\xi\tau})\right] + \mathbb{E}\left[e^{-u\tau}e^{-\xi\tau}\right]\widehat{W}_u(\xi),\end{aligned}$$

from which (59) follows. Finally, (60) follows from (57) and (58). □

Corollary 9. *Let the assumptions of Theorem 4 hold true.*

(i) If the variables τ_n are exponential with rate λ, then

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{\lambda}{1+\lambda\mathbb{E}\sigma}\cdot\frac{(u+\lambda-\lambda\mathbb{E}e^{-u\sigma})\mathbb{E}e^{-u\sigma}}{u(u+\lambda)}, \qquad \widetilde{\mathbb{E}}\alpha(0) = \mathbb{E}\sigma + \frac{1}{\lambda} + \frac{\lambda}{2}\cdot\frac{\mathbb{E}\sigma^2}{1+\lambda\mathbb{E}\sigma}.$$

(ii) If the variables σ_n are exponential with rate μ, then

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{1}{\mathbb{E}\tau}\cdot\frac{\mu}{(\mu+u)u}\cdot\frac{(1-\mathbb{E}e^{-\mu\tau})(1-\mathbb{E}e^{-u\tau})}{1-\mathbb{E}e^{-(\mu+u)\tau}} = \frac{\mu^2}{(\mu+u)^2}\cdot\frac{\mathbb{E}e^{-\mu\overline{\tau}}\,\mathbb{E}e^{-u\overline{\tau}}}{\mathbb{E}e^{-(\mu+u)\overline{\tau}}},$$
$$\widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\mu} + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}(\tau e^{-\mu\tau})}{1-\mathbb{E}e^{-\mu\tau}}.$$

(iii) If the τ_n are exponential with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \frac{\mu^2\lambda(\lambda+\mu+u)}{(\lambda+\mu)(\lambda+u)(\mu+u)^2}, \qquad \widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\mu} + \frac{1}{\lambda} + \frac{\lambda}{\mu(\lambda+\mu)}.$$

Proof. (ia) We compute the functions $U(t)$ and $W_u(t)$ that enter formula (46). Since, under $\mathbb{P}$, $\mathfrak{a} = \sum_n \delta_{T_n}$ is a Poisson process with a point at 0 we have, directly from (43), $U(t) = 1 + \lambda t$. Since τ is exponential, (59) explicitly gives the Laplace transform of W_u:

$$\widehat{W_u}(\xi) = \frac{1}{\xi}\cdot\frac{\mathbb{E}[e^{-u\tau}(1-e^{-\xi\tau})]}{1-\mathbb{E}e^{-u\tau}e^{-\xi\tau}} = \frac{1}{\xi}\cdot\frac{\frac{\lambda}{\lambda+u}-\frac{\lambda}{\lambda+u+\xi}}{1-\frac{\lambda}{\lambda+u+\xi}} = \frac{\lambda}{\lambda+u}\cdot\frac{1}{u+\xi},$$

and hence

$$W_u(t) = \frac{\lambda}{\lambda+u}e^{-ut}.$$

Substituting into (46) we obtain the announced formula for $\widetilde{\mathbb{E}}e^{-u\alpha(0)}$.

(ib) Equations (58) and (60) give

$$\widehat{M_2}(\xi) = \mathbb{E}\tau^2\,\widehat{U}(\xi) + 2(\mathbb{E}\tau)\frac{\mathbb{E}(\tau e^{-\xi\tau})}{\xi(1-\mathbb{E}e^{-\xi\tau})^2} = \mathbb{E}\tau^2\,\widehat{U}(\xi) + \frac{2\lambda\mathbb{E}\tau}{\xi^3}.$$

Hence

$$M_2(t) = \mathbb{E}\tau^2\,U(t) + \lambda(\mathbb{E}\tau)t^2.$$

Using this and $M_1(t) = \mathbb{E}\tau\,U(t)$ in (55) we obtain the announced formula for $\widetilde{\mathbb{E}}\alpha(0)$.

(iia) If σ is exponential with rate μ then $\mathbb{E}U(\sigma) = \mu\widehat{U}(\mu)$ and $\mathbb{E}W_u(\sigma) = \mu\widehat{W}_u(\mu)$. Hence (46) gives

$$\widetilde{\mathbb{E}}e^{-u\alpha(0)} = \mathbb{E}e^{-u\sigma}\,\frac{1-\mu\widehat{W}_u(\mu)}{u\,\mathbb{E}\tau\,\mu\,\widehat{U}(\mu)}$$

But the Laplace transforms $\widehat{U}$ and $\widehat{W}_u$ are known from Lemma 6. Substituting in the last display we obtain the first announced equality for $\widetilde{\mathbb{E}}e^{-u\alpha(0)}$. For the second equality, simply replace the three terms of the form $1-\mathbb{E}e^{-\xi\tau}$ by $\xi(\mathbb{E}\tau)\mathbb{E}e^{-\xi\overline{\tau}}$.

(iib) From the middle of (55) we have

$$\widetilde{\mathbb{E}}\alpha(0) = \frac{1}{\mu} + \frac{\widehat{W}_2(\mu)}{2\widehat{W}_1(\mu)}$$

and the formula follows from the previously derived formulas for $\widehat{W}_2$ and $\widehat{W}_1$.

(iiia) Consider the second equality in (ii). Since $\overline{\tau} \overset{\text{(d)}}{=} \tau$ we have $\mathbb{E}e^{-\xi\overline{\tau}} = \lambda/(\xi+\lambda)$. Replacing the three terms in the second equality in (ii) by such ratios we arrive at the announced formula. Alternatively, letting $\mathbb{E}e^{-\mu\sigma} = \mu/(\mu+u)$ and $\mathbb{E}\sigma = 1/\mu$ in (i) we arrive at the same formula.

(iiib) Set $\mathbb{E}\sigma = 1/\mu$, $\mathbb{E}\sigma^2 = 2/\mu^2$ in the last formula of (i). □

5.2 The new age of information for the blocking system

Recall that the NAoI process is given by $\beta(t) = A_t - S_{D_t}$, where A_t is the last arrival (accepted or not) before t and S_{D_t} is the last successful arrival before the last successful departure before t; this quantity is given by (36).

Theorem 7. *Consider the blocking system under stationarity assumptions and assume that (33) holds. Then the $\widetilde{\mathbb{P}}$-law of $\beta(0)$ has an atom at 0 satisfying*

$$\widetilde{\mathbb{P}}(\beta(0)=0) = \frac{\mathbb{E}^*(\tau_0-\sigma_0)^+}{\mathbb{E}^*T_N}, \tag{61}$$

while, for f bounded and measurable function,

$$\widetilde{\mathbb{E}}[f(\beta(0));\beta(0)>0] = \frac{1}{\mathbb{E}^*T_N}\mathbb{E}^*\left\{\sum_{i=0}^{N-1}\tau_i f(T_i-T_M) - (T_N-\sigma_0)\,f(T_{N-1}-T_M)\right\}$$
$$+\frac{1}{\mathbb{E}^*T_N}\mathbb{E}^*\left\{(T_N-\sigma_0)\,f(T_{N-1})\mathbf{1}_{T_{N-1}>0}\right\}, \tag{62}$$

where $N = \inf\{\ell \geq 1 : \psi_\ell = 1\}$ and $M = \sup\{\ell \leq -1 : \psi_\ell = 1\}$.

Proof. Notice that $N = \inf\{\ell \geq 1 : T_\ell \geq \sigma_0\}$, $\mathbb{P}^*$-a.s. We use the Palm inversion formula:

$$\widetilde{\mathbb{E}} f(\beta(0)) = \frac{\mathbb{E}^* \int_{B_0}^{B_1} f(\beta(t))\, dt}{\mathbb{E}^*(B_1 - B_0)}. \tag{63}$$

Since M, N are the indices of the admitted messages nearest to 0,

$$B_{-1} = T_M \leq T_{-1} < T_0 = B_0 = 0 < T_1 < \cdots < T_{N-1} < \sigma_0 \leq T_N = B_1, \quad \mathbb{P}^*\text{-a.s.}$$

In particular, $B_1 - B_0 = T_N$, $\mathbb{P}^*$-a.s. Since $\beta(t) = A_t - S_{D_t}$, using (36) we have

$$\beta(t) = \begin{cases} T_i - T_M, & \text{if } T_0 \leq T_i \leq t < T_{i+1} \leq T_{N-1} \\ T_{N-1} - T_M, & \text{if } T_{N-1} \leq t < T_0 + \sigma_0 \\ T_{N-1} - T_0, & \text{if } T_0 + \sigma_0 \leq t < T_N \end{cases}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and measurable. We write the integral in the numerator of (63) as:

$$\begin{aligned} \int_{T_0}^{T_N} f(\beta(t))\, dt &= \int_{T_0}^{T_{N-1}} f(\beta(t))\, dt + \int_{T_{N-1}}^{T_0+\sigma_0} f(\beta(t))\, dt + \int_{T_0+\sigma_0}^{T_N} f(\beta(t))\, dt \\ &= \sum_{i=0}^{N-2} \int_{T_i}^{T_{i+1}} f(T_i - T_M) dt + \int_{T_{N-1}}^{T_0+\sigma_0} f(T_{N-1} - T_M) dt + \int_{T_0+\sigma_0}^{T_N} f(T_{N-1} - T_0) dt \\ &= \sum_{i=0}^{N-2} \tau_i f(T_i - T_M) + (\sigma_0 - T_{N-1})\, f(T_{N-1} - T_M) + (T_N - \sigma_0)\, f(T_{N-1}). \end{aligned} \tag{64}$$

Add and subtract the term corresponding to $i = N-1$ to write the last line as

$$\begin{aligned} &= \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - \tau_{N-1} f(T_{N-1} - T_M) + (\sigma_0 - T_{N-1})\, f(T_{N-1} - T_M) + (T_N - \sigma_0)\, f(T_{N-1}) \\ &= \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) + (\sigma_0 - T_{N-1} - \tau_{N-1})\, f(T_{N-1} - T_M) + (T_N - \sigma_0)\, f(T_{N-1}) \\ &= \left\{ \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - (T_N - \sigma_0)\, f(T_{N-1} - T_M) \right\} + (T_N - \sigma_0)\, f(T_{N-1}). \end{aligned} \tag{65}$$

(For $f \geq 0$, the term in the bracket is positive because the last term of the sum is $\tau_{N-1} f(T_{N-1} - T_M)$ is bigger than $(T_N - \sigma_0)\, f(T_{N-1} - T_M)$ and this is because $\tau_{N-1} - (T_N - \sigma_0) = \sigma_0 - T_{N-1} > 0$.) By (63),

$$\mathbb{E}^* T_N\, \widetilde{\mathbb{E}} f(\beta(0)) = \mathbb{E}^* \left\{ \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - (T_N - \sigma_0)\, f(T_{N-1} - T_M) \right\} + \mathbb{E}^* \left\{ (T_N - \sigma_0)\, f(T_{N-1}) \right\}. \tag{66}$$

To reveal the atom of the $\widetilde{\mathbb{P}}$-law of $\beta(0)$ at 0, let

$$f(x) = \mathbf{1}_{x=0}.$$

Then $f(T_i - T_M) = 0$ because $T_M < 0$. Also, $f(T_{N-1}) = \mathbf{1}_{T_{N-1}=0} = \mathbf{1}_{N=1} = \mathbf{1}_{\tau_0 \geq \sigma_0}$. Hence

$$\mathbb{E}^* T_N \, \widetilde{\mathbb{P}}(\beta(0) = 0) = \mathbb{E}^* \left\{ (T_N - \sigma_0) \, \mathbf{1}_{N=1} \right\} = \mathbb{E}^* \left\{ (\tau_0 - \sigma_0) \, \mathbf{1}_{\tau_0 \geq \sigma_0} \right\} = \mathbb{E}^* (\tau_0 - \sigma_0)^+.$$

On the other hand,

$$\begin{aligned}
\widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) > 0] &= \widetilde{\mathbb{E}} f(\beta(0)) - \widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) = 0] \\
&= \widetilde{\mathbb{E}} f(\beta(0)) - f(0) \widetilde{\mathbb{P}}(\beta(0) = 0) \\
&= \widetilde{\mathbb{E}}[f(\beta(0)) - f(0) \mathbf{1}_{\beta(0)=0}] \equiv \widetilde{\mathbb{E}} g(\beta(0)),
\end{aligned}$$

where

$$g(x) = f(x) - f(0) \mathbf{1}_{x=0}.$$

We use g in place of f in (66) after noting that $g(T_i - T_M) = f(T_i - T_M) - f(0)\mathbf{1}(T_i = T_M) = f(T_i - T_M)$ for $i \geq 0$, and $g(T_{N-1}) = f(T_{N-1}) - f(0)\mathbf{1}_{T_{N-1}=0} = f(T_{N-1}) - f(0)\mathbf{1}_{N=1}$. So

$$\begin{aligned}
&\mathbb{E}^* T_N \, \widetilde{\mathbb{E}}[f(\beta(0)); \beta(0) > 0] = \\
&= \mathbb{E}^* \left\{ \sum_{i=0}^{N-1} \tau_i f(T_i - T_M) - (T_N - \sigma_0) \, f(T_{N-1} - T_M) \right\} + \mathbb{E}^* \left\{ (T_N - \sigma_0) \left(f(T_{N-1}) - f(0) \mathbf{1}_{N=1} \right) \right\}.
\end{aligned}$$

Notice that

$$f(T_{N-1}) - f(0)\mathbf{1}_{N=1} = f(T_{N-1}) - f(T_{N-1})\mathbf{1}_{N=1} = f(T_{N-1}) \, \mathbf{1}_{N>1} = f(T_{N-1}) \, \mathbf{1}_{T_{N-1}>0}$$

and substitute into the last display to obtain the announced formula. □

By Palm theory and stationarity, we have that $|M|$ and N have the same $\mathbb{P}^*$-law and so do $|T_M|$ and T_N. This simple fact is stated as an stand-alone lemma because it holds only under stationary assumptions and because it is needed when we explicitly compute distributions under independence assumptions.

***Remark* 4.** We now give a physical meaning to the $\widetilde{\mathbb{P}}$-law of $\beta(0)$ conditional on $\beta(0) > 0$. Say that the message arriving at time T_n is *undisturbed* if it is admitted (and hence successful) and no other messages arrive during the time it is being processed; i.e., $\psi_n = 1$ and $T_n + \sigma_n \le T_{n+1}$. Therefore, for $T_n + \sigma_n \le t < T_{n+1}$ we have $\beta(t) = 0$: undisturbed messages provided the freshest possible information; this is what contributes to the atom at 0 for $\beta(0)$. Define then an auxiliary system, pathwise, by removing all undisturbed messages. If $\beta_+(t)$ denotes the NAoI process for the auxiliary system then we have that, under $\widetilde{\mathbb{P}}$, $\beta(0)$ equals 0 with probability $\frac{\mathbb{E}^*(\tau_0-\sigma_0)^+}{\mathbb{E}^* T_N}$ or $\beta_+(0)$ with the remaining probability. In particular,

$$\widetilde{\mathbb{E}}[f(\beta(0));\, \beta(0) > 0] = \widetilde{\mathbb{E}} f(\beta_+(0)).$$

Lemma 7. *Assume that (τ_n, σ_n), $n \in \mathbb{Z}$, is stationary under $\mathbb{P}$. Let $N = \inf\{\ell \ge 1 :\, \psi_\ell = 1\}$ and $M = \sup\{\ell \le -1 :\, \psi_\ell = 1\}$. Then*

$$\mathbb{E}(g(-T_M)|\psi_0 = 1) = \mathbb{E}(g(T_N)|\psi_0 = 1),$$

for any bounded and measurable function g.

Proof. The point process $\sum_n \psi_n \delta_n$ is stationary under $\widetilde{\mathbb{P}}$ and the Palm probability of the latter with respect to this point process is denoted by $\mathbb{P}^*$. If $\cdots < T^*_{-1} < T^*_0 \le 0 < T^*_1 < T^*_2 < \cdots$ is an enumeration of the points of $\sum_n \psi_n \delta_n$ in their natural order then $\mathbb{E}^* g(-T^*_{-1}) = \mathbb{E}^* g(T^*_1)$ for any bounded measurable function g. But $T^*_1 = T_N$ and $T^*_{-1} = T_M$ and $\mathbb{P} = \mathbb{P}^*(\cdot|\psi_0 = 1)$. □

Under i.i.d. assumptions, and because the decision on whether to admit a message or not is past-dependent, the ensued regeneration results into further simplification and the vanishing of the M from the formula. We explain this below. First fix $u \ge 0$ and consider the function $W_u(t)$ introduced in (45) as well as

$$V_u(t) := \mathbb{E} \sum_{i=0}^{\mathfrak{a}(t)-1} e^{-uT_i}, \quad t \ge 0 \tag{67}$$

$$Q_u(t) := \mathbb{E}\big\{(T_{\mathfrak{a}(t)} - t)\, e^{-uT_{\mathfrak{a}(t)-1}}\big\}, \quad t \ge 0. \tag{68}$$

Theorem 8. *Consider the blocking system and assume that* (τ_n, σ_n), $n \in \mathbb{Z}$, *is i.i.d. under* $\mathbb{P}$ *and such that* $\mathbb{E}\tau_0 < \infty$ *and* $\mathbb{P}(\tau_0 \geq \sigma_0) > 0$. *Assume further that* τ_n *is independent of* σ_n *for all* n. *Then* $\widetilde{\mathbb{P}}(\beta(0) = 0) = \frac{\mathbb{E}(\tau_0-\sigma_0)^+}{\mathbb{E}T_N}$ *and*

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)}; \beta(0) > 0] = \frac{1}{\mathbb{E}T_N}\mathbb{E}e^{-uT_N} \left\{\mathbb{E}\tau \, \mathbb{E}\sum_{i=0}^{N-1} e^{-uT_i} - \mathbb{E}(T_N - \sigma_0)\, e^{-uT_{N-1}}\right\}$$

$$+ \frac{1}{\mathbb{E}T_N}\mathbb{E}\left\{(T_N - \sigma_0)\, e^{-uT_{N-1}}\mathbf{1}_{T_{N-1}>0}\right\}$$

$$= \frac{\mathbb{E}W_u(\sigma)\,[\mathbb{E}\tau \, \mathbb{E}V_u(\sigma) - \mathbb{E}Q_u(\sigma)] + \mathbb{E}[e^{-u\tau}Q_u(\sigma - \tau)]}{\mathbb{E}\tau \, \mathbb{E}U(\sigma)}, \quad (69)$$

where U, W_u *are unique solutions to the fixed point equations* (47), (48), *respectively, while* V_u, Q_u *are unique solutions to*

$$V_u(t) = 1 + \int_{(0,t]} V_u(t - x)\, e^{-ux}\, \mathbb{P}(\tau \in dx), \quad (70)$$

$$Q_u(t) = \mathbb{E}(\tau - t)^+ + \int_{(0,t]} Q_u(t - x)e^{-ux}\mathbb{P}(\tau \in dx). \quad (71)$$

In particular, under $\widetilde{\mathbb{P}}$, *and conditional on* $\beta(0) > 0$, *the random variable* $\beta(0)$ *is absolutely continuous.*

***Remark* 5.** The term $\mathbb{E}\tau \, \mathbb{E}V_u(\sigma) - \mathbb{E}Q_u(\sigma)$ in (69) is nonnegative and this is due to the remark made below (65) about the nonnegativity of the bracketed term in (69).

Proof. The value of $\widetilde{\mathbb{P}}(\beta(0) = 0)$ follows from (61) and (34) that allows us to replace $\mathbb{E}^*$ by $\mathbb{E}$. To show the rest, we look at the various terms in (62) with $f(x) = e^{-ux}$. Using (34) we obtain

$$\mathbb{E}^* \sum_{i=0}^{N-1} \tau_i e^{-u(T_i - T_M)} = \mathbb{E}(e^{uT_M}|\psi_0 = 1)\, \mathbb{E}\sum_{i=0}^{N-1} \tau_i e^{-uT_i} \quad (72)$$

Due to Lemma 7, the first term of the product is further written as:

$$\mathbb{E}(e^{uT_M}|\psi_0 = 1) = \mathbb{E}(e^{-uT_N}|\psi_0 = 1) = \mathbb{E}e^{-uT_N}.$$

The second term in the last product of (72) is computed as follows.

$$\mathbb{E}\sum_{i=0}^{N-1}\tau_i e^{-uT_i} = \mathbb{E}\sum_{i=0}^{\infty}\tau_i e^{-uT_i}\mathbf{1}_{T_i<\sigma_0} = \sum_{i=0}^{\infty}\mathbb{E}\{\mathbb{E}[\tau_i e^{-uT_i}\mathbf{1}_{T_i<\sigma_0}|\sigma_0,\tau_0,\dots,\tau_{i-1}]\}$$
$$= \sum_{i=0}^{\infty}\mathbb{E}\{e^{-uT_i}\mathbf{1}_{T_i<\sigma_0}\mathbb{E}[\tau_i|\sigma_0,\tau_0,\dots,\tau_{i-1}]\} = (\mathbb{E}\tau)\,\mathbb{E}\sum_{i=0}^{N-1}e^{-uT_i}. \quad (73)$$

Using the same logic,

$$\mathbb{E}^*(T_N-\sigma_0)e^{-u(T_{N-1}-T_M)} = \mathbb{E}^* e^{uT_M}\,\mathbb{E}(T_N-\sigma_0)\,e^{-uT_{N-1}} = \mathbb{E}e^{uT_M}\,\mathbb{E}(T_N-\sigma_0)\,e^{-uT_{N-1}} \quad (74)$$

$$\mathbb{E}^*\left\{(T_N-\sigma_0)\,f(T_{N-1})\mathbf{1}_{T_{N-1}>0}\right\} = \mathbb{E}\left\{(T_N-\sigma_0)\,f(T_{N-1})\mathbf{1}_{T_{N-1}>0}\right\} \quad (75)$$

Substituting (73) into (72) and then this, together with (74) and (75), into (62) we arrive at the first equality for (69). For the second equality, use (51), (52) and (45), (67), (68) and observe that

$$\mathbb{E}\big\{(T_{\mathfrak{a}(t)}-t)\,e^{-uT_{\mathfrak{a}(t)-1}}\mathbf{1}_{T_{\mathfrak{a}(t)}-1>0}\big\} = \mathbb{E}[e^{-u\tau}\,Q_u(t-\tau)].$$

To see that V_u satisfies (70), notice that

$$V_u(t) = \mathbb{E}\left[\sum_{i=0}^{\mathfrak{a}(t)-1}e^{-uT_i};\,T_1>t\right] + \mathbb{E}\left[\sum_{i=0}^{\mathfrak{a}(t)-1}e^{-uT_i};\,T_1\le t\right]$$
$$= \mathbb{E}[e^{-uT_0};\,T_1>t] + \mathbb{E}\left[e^{-uT_0}+e^{-uT_1}V_u(t-T_1);\,t-T_1\ge 0\right]$$
$$= e^{-u0} + \mathbb{E}[e^{-u\tau}V_u(t-\tau)\mathbf{1}_{\tau\le t}].$$

To see that Q_u satisfies (71), notice that

$$Q_u(t) = \mathbb{E}\left[(T_{\mathfrak{a}(t)}-t)\,e^{-uT_{\mathfrak{a}(t)-1}};\,T_1>t\right] + \mathbb{E}\left[(T_{\mathfrak{a}(t)}-t)\,e^{-uT_{\mathfrak{a}(t)-1}};\,T_1\le t\right]$$
$$= \mathbb{E}\left[(T_1-t)\,e^{-uT_0};\,T_1>t\right] + \int \mathbb{E}\left[(x+T_{\mathfrak{a}(t-x)}-t)\,e^{-u(x+T_{\mathfrak{a}(t-x)-1})}\right]\mathbf{1}_{x\le t}\,\mathbb{P}(T_1\in dx)$$
$$= \mathbb{E}\left[(\tau-t)\,e^{-u0};\,\tau>t\right] + \int_{(0,t]} e^{-ux}\,\mathbb{E}\left[(T_{\mathfrak{a}(t-x)}-(t-x))\,e^{-u(T_{\mathfrak{a}(t-x)-1})}\right]\mathbb{P}(\tau\in dx)$$
$$= \mathbb{E}(\tau-t)^+ + \int_{(0,t]} e^{-ux}\,Q_u(t-x)\,\mathbb{P}(\tau\in dx).$$

□

Continuing in the same manner as Lemma 6, we obtain the Laplace transforms of V_u and Q_u.

Lemma 8.

$$\widehat{V}_u(\xi) = \frac{1/\xi}{1 - \mathbb{E}e^{-(u+\xi)\tau}} \tag{76}$$

$$\widehat{Q}_u(\xi) = \frac{1}{\xi^2}\frac{\xi\mathbb{E}\tau - 1 + \mathbb{E}e^{-\xi\tau}}{1 - \mathbb{E}e^{-(u+\xi)\tau}} \tag{77}$$

Proof. Directly from (70) and (71). □

Corollary 10. *Let the assumptions of Theorem 4 hold true.*

(i) If the variables τ_n are exponential with rate λ, then

$$\widetilde{\mathbb{P}}(\beta(0) = 0) = \frac{\mathbb{E}e^{-\lambda\sigma}}{1 + \lambda\mathbb{E}\sigma}$$

and, with $L_\sigma(u) = \mathbb{E}e^{-u\sigma}$,

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)}; \beta(0) > 0] = \frac{\lambda}{1 + \lambda\mathbb{E}\sigma}\left[\frac{L_\sigma(u)}{\lambda + u}\,\frac{\lambda^2(1 - L_\sigma(u)) - u^2(1 - L_\sigma(\lambda))}{u(\lambda - u)} + \frac{L_\sigma(u) - L_\sigma(\lambda)}{\lambda - u}\right]$$

(ii) If the variables σ_n are exponential with rate μ, then, with $L_\tau(u) = \mathbb{E}e^{-u\tau}$,

$$\widetilde{\mathbb{P}}(\beta(0) = 0) = \frac{1}{\mu\mathbb{E}\tau}(1 - \mathbb{E}e^{-\mu\tau})(\mu\mathbb{E}\tau - 1 - \mathbb{E}e^{-\mu\tau}),$$

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)}; \beta(0) > 0] = \frac{1 - L_\tau(\mu)}{\mu\mathbb{E}\tau(1 - L_\tau(u + \mu))}\left[\frac{L_\tau(u) - L_\tau(u + \mu)}{1 - L_\tau(u + \mu)}(1 - L_\tau(\mu)) + L_\tau(u + \mu)\,(\mu\mathbb{E}\tau - 1 - L_\tau(\mu))\right.$$

(iii) If the τ_n are exponential with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$\beta(0) \overset{(d)}{=} \begin{cases} 0, & \textit{with probability } \frac{\mu^2}{(\lambda+\mu)^2} \\ \zeta, & \textit{with probability } \frac{\lambda(\lambda+2\mu)}{(\lambda+\mu)^2} \end{cases},$$

where ζ is an absolutely continuous random variable with

$$\mathbb{E}e^{-u\zeta} = \frac{\mu^2}{\lambda + 2\mu}\,\frac{u^2 + (2\lambda + \mu)u + \lambda(\lambda + 2\mu)}{(u + \lambda)(u + \mu)^2}.$$

Proof. From Theorem (8), we have $\widetilde{\mathbb{P}}(\beta(0)=0)=\lambda\mathbb{E}(\tau-\sigma)^+/\mathbb{E}U(\sigma)$ and the expressions of this are obtained by elementary integrals in all cases. We rewrite (69) as

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)};\beta(0)>0]=\frac{\mathbb{E}W_u(\sigma)\,\mathbb{E}H_u(\sigma)+\mathbb{E}Q_u^+(\sigma)}{\mathbb{E}\tau\,\mathbb{E}U(\sigma)}, \tag{78}$$

where

$$H_u(t)=\mathbb{E}\tau\,V_u(t)-Q_u(t),\quad Q_u^+(t)=\mathbb{E}[e^{-u\tau}\,Q_u(t-\tau)].$$

We thus know the Laplace transforms of all functions entering in (78) in terms of $L_\tau(\xi):=\mathbb{E}e^{-\xi\tau}$:

$$\widehat{U}(\xi)=\frac{1/\xi}{1-L_\tau(\xi)},\quad \widehat{W}_u(\xi)=\frac{1}{\xi}\frac{L_\tau(u)-L_\tau(u+\xi)}{1-L_\tau(u+\xi)},$$
$$\widehat{H}_u(\xi)=\frac{1}{\xi^2}\frac{1-L_\tau(\xi)}{1-L_\tau(u+\xi)},\quad \widehat{Q}_u^+(\xi)=\frac{L_\tau(u+\xi)}{\xi^2}\,\frac{\xi\mathbb{E}\tau-1+L_\tau(\xi)}{1-L_\tau(u+\xi)}.$$

(i) When τ is exponential, we already know that $U(t)=1+\lambda t$ and that $W_u(t)=\lambda e^{-ut}/(\lambda+u)$ and, with $L_\tau(u)=\lambda/(\lambda+u)$, we obtain

$$\widehat{H}_u(\xi)=\frac{\lambda+u+\xi}{\xi(\lambda+\xi)(u+\xi)},\quad \widehat{Q}_u^+(\xi)=\frac{1}{(u+\xi)(\lambda+\xi)},$$

that can easily be inverted to the nonnegative functions

$$H_u(t)=\frac{\lambda^2(1-e^{-ut})-u^2(1-e^{-\lambda t})}{\lambda u(\lambda-u)},\quad Q_u^+(t)=\frac{e^{-ut}-e^{-\lambda t}}{\lambda-u}.$$

The values of H_u and Q_u^+ at $u=\lambda$ should be interpreted as limits when $u\to\lambda$. Thus, $H_\lambda(t)=\lambda^{-1}[2-(\lambda t+2)]e^{-\lambda t}$, $Q_\lambda^+(t)=te^{-\lambda t}$. Substitute these functions in (78) to obtain the announced formula.

(ii) When σ is exponential with rate μ, all functions in (78) are essentially Laplace transforms of σ, for example, $\mathbb{E}W_u(\sigma)=\mu\widehat{W}_u(\mu)$. Hence

$$\widetilde{\mathbb{E}}[e^{-u\beta(0)};\beta(0)>0]=\frac{\mu\widehat{W}_u(\mu)\,\mu\widehat{H}_u(\mu)+\mu\widehat{Q}_u^+(\mu)}{\mathbb{E}\tau\,\mu\widehat{U}(\mu)},$$

and the formula is obtained because we know all Laplace transforms.

(iii) The formula readily follows from either (i) or (ii). □

Let us take a closer look at the law of the random variable ζ of Corollary 10(iii). Letting $\rho=\lambda\mu$ we have

$$\mathbb{E}e^{-u\mu\zeta}=\frac{1}{\rho+2}\,\frac{u^2+(2\rho+1)u+\rho(\rho+2)}{(u+\rho)(u+1)^2}.$$

Inverting this Laplace transform, we find that $\mu\zeta$ has density

$$g_\rho(t) = \frac{1}{(\rho+2)(\rho-1)^2}\left[\rho e^{-\rho t} + (\rho^3 - 3\rho + 1 + \rho^2(\rho-1)t)e^{-t}\right],$$

for all values of $\rho \neq 1$ and, for $\rho = 1$, the density corresponds to the limit of this expression when $\rho \to 1$:

$$g_1(t) = \frac{1}{6}(t^2 + 2t + 2)e^{-t}.$$

We now pass on to computing first moments.

Lemma 9. *Consider the blocking system under stationarity assumptions. Then*

$$\widetilde{\mathbb{E}}\beta(0) = \frac{\mathbb{E}^*\left[\sum_{i=0}^{N-1} \tau_i T_i - \sigma_0 T_M\right]}{\mathbb{E}^* T_N} \tag{79}$$

Proof. Take $f(x) = x$ in (64) and regroup the terms there to obtain

$$\int_{T_0}^{T_N} \beta(t)dt = \sum_{i=0}^{N-1} \tau_i(T_i - T_M) + (T_N - \sigma_0)\, T_M = \sum_{i=0}^{N-1} \tau_i T_i - \sigma_0 T_M$$

and then use the Palm inversion formula. □

Next define

$$Z(t) = \mathbb{E} \sum_{i=0}^{\mathfrak{a}(t)-1} T_i, \quad t \geq 0. \tag{80}$$

Lemma 10. *Consider the blocking system and assume that (τ_n, σ_n), $n \in \mathbb{Z}$, is i.i.d. under $\mathbb{P}$ and such that $\mathbb{E}\tau_0 < \infty$. Assume further that τ_n is independent of σ_n for all n. Then*

$$\widetilde{\mathbb{E}}\beta(0) = \mathbb{E}\sigma + \frac{\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right]}{\mathbb{E}N} = \mathbb{E}\sigma + \frac{\mathbb{E}Z(\sigma)}{\mathbb{E}U(\sigma)},$$

where Z is the unique solution to the fixed-point equation

$$Z(t) = \mathbb{E}[Z(t-\tau)] + \mathbb{E}[\tau U(t-\tau)]$$

and has Laplace transform

$$\widehat{Z}(\xi) = \frac{\mathbb{E}\tau e^{-\xi\tau}}{\xi(1 - \mathbb{E}e^{-\xi\tau})^2}.$$

Proof. The numerator of (79) is written as

$$\begin{aligned}\mathbb{E}^*\left[\sum_{i=0}^{N-1}\tau_i T_i - \sigma_0 T_M\right] &= \mathbb{E}^*\sum_{i=0}^{N-1}\tau_i T_i + \mathbb{E}^*\sigma_0(-T_M)\\ &= \mathbb{E}\tau\,\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right] + \mathbb{E}(-T_M|\psi_0 = 1)\,\mathbb{E}\sigma\\ &= \mathbb{E}\tau\,\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right] + \mathbb{E}T_N\,\mathbb{E}\sigma.\end{aligned}$$

Dividing this by $\mathbb{E}T_N = \mathbb{E}\tau\,\mathbb{E}N$ results in the first equality. Next use the function (80) to write $\mathbb{E}\left[\sum_{i=0}^{N-1} T_i\right] = \mathbb{E}Z(\sigma)$. The fixed point equation is obtained from first principles or by differentiating both sides of (70) with respect to u and letting $u \to 0$. The Laplace transform is obtained by taking the Laplace transform of both sides of the fixed-point equation. □

Corollary 11. *Let the assumptions of Theorem 4 hold true.*

(i) If the variables τ_n are exponential with rate λ, then

$$\widetilde{\mathbb{E}}\beta(0) = \mathbb{E}\sigma + \frac{\lambda}{2}\frac{\mathbb{E}\sigma^2}{1+\lambda\mathbb{E}\sigma}.$$

(ii) If the variables σ_n are exponential with rate μ, then, with $L_u = \mathbb{E}e^{-u\tau}$,

$$\widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu} + \frac{\mathbb{E}\tau e^{-\mu\tau}}{1-\mathbb{E}e^{-\mu\tau}}$$

(iii) If the τ_n are exponential with rate λ, and the σ_n are exponential with rate μ then, under $\widetilde{\mathbb{P}}$,

$$\widetilde{\mathbb{E}}\beta(0) = \frac{1}{\mu} + \frac{\lambda}{\mu(\lambda+\mu)}.$$

6 Concluding discussion and open problems

Summary. In summary, the contributions of this paper are: A new age of information measure (NAoI) definition was introduced and motivated. The utility of Palm calculus was demonstrated in deriving the distribution of AoI and NAoI for stationary bufferless systems under pushout and blocking policies. All formulas obtained for bufferless $\mathcal{P}$ are also valid for inifinite buffer queues under preemptive

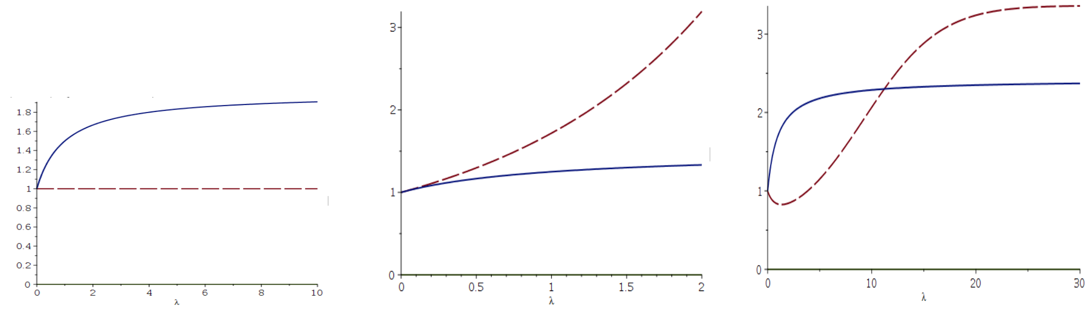
LIFO policy; see Section 1.3. In particular, the expectations of these quantities, under renewal assumptions are summarized in Table 1. Under the same assumptions, some interesting stochastic decomposition and representation results were also obtained; see Section 3 for a summary of these results.

model	pushout ($\mathcal{P}$) AoI ($\widetilde{\mathbb{E}}\alpha_{\mathcal{P}}(0)$)	pushout ($\mathcal{P}$) NAoI ($\widetilde{\mathbb{E}}\beta_{\mathcal{P}}(0)$)	blocking ($\mathcal{B}$) AoI ($\widetilde{\mathbb{E}}\alpha_{\mathcal{B}}(0)$)	blocking ($\mathcal{B}$) NAoI ($\widetilde{\mathbb{E}}\beta_{\mathcal{B}}(0)$)
GI/GI	$\frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{P}(\tau\geq\sigma)}$	$\frac{\mathbb{E}\tau\wedge\sigma}{\mathbb{P}(\tau\geq\sigma)}$	$\frac{1}{\mu} + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau(U*U)(\sigma-\tau)}{\mathbb{E}U(\tau)}$	$\frac{1}{\mu} + \frac{\mathbb{E}Z(\sigma)}{EU(\sigma)}$
M/GI	$\frac{1}{\lambda\mathbb{E}e^{-\lambda\sigma}}$	$\frac{1}{\lambda\mathbb{E}e^{-\lambda\sigma}} - \frac{1}{\lambda}$	$\frac{1}{\mu} + \frac{1}{\lambda} + \frac{\lambda}{2}\frac{\mathbb{E}\sigma^2}{1+\lambda\mathbb{E}\sigma}$	$\frac{1}{\mu} + \frac{\lambda}{2}\frac{\mathbb{E}\sigma^2}{1+\lambda/\mu}$
GI/M	$\frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu} + \frac{\mathbb{E}\tau^2}{2\mathbb{E}\tau} + \frac{\mathbb{E}\tau e^{-\mu\tau}}{1-\mathbb{E}e^{-\mu\tau}}$	$\frac{1}{\mu} + \frac{\mathbb{E}\tau e^{-\mu\tau}}{1-\mathbb{E}e^{-\mu\tau}}$
M/M	$\frac{1}{\lambda} + \frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu} + \frac{1}{\lambda} + \frac{\lambda}{\mu(\lambda+\mu)}$	$\frac{1}{\mu} + \frac{\lambda}{\mu(\lambda+\mu)}$

Table 1: Mean AoI and NAoI for different models of interarrival times (with $\mathbb{E}\tau = 1/\lambda$) and service times (with $\mathbb{E}\sigma = 1/\mu$) in the renewal case.

Using Laplace inversion, we obtained, in certain cases, the density of AoI and the density of the NAoI conditional that it be positive. We may alternately obtain expressions for the probability densities by using *level-crossing arguments* as in, e.g., [3]. We should also point out the generality of the formulas obtained in Theorems 1, 3, 5 and 7: they remain true even under general stationarity assumptions. Therefore, we can, for example, incorporate situations where messages arrive according to processes that are more general than renewal ones, e.g., Markov renewal.

What is best for a bufferless system? Let us now take a look at the issue of choosing the "best" policy for bufferless system. The choice depends not only on the arrival/processing rates but on the way that arrivals and processing times are distributed. It also depends on what we mean by "best". If "good" means low expectation and if renewal assumptions are made, then sometimes $\mathcal{P}$ always



(a) M/M (b) M/D (c) M/GI

Figure 3: *Mean NAoI as a function of the arrival rate* λ *for pushout (broken line) and blocking (solid line) policies in three cases. Here,* $\mu = 1$ *in all cases.*

outperforms $\mathcal{B}$, sometimes $\mathcal{B}$ outperforms $\mathcal{P}$ and sometimes the answer depends on how loaded the system is. Suppose $\mu = 1$.

(a) In the M/M case we have, for all λ,

$$\widetilde{\mathbb{E}}\beta_{\mathcal{P}} = 1 < \widetilde{\mathbb{E}}\beta_{\mathcal{B}} = 1 + \frac{\lambda}{\lambda + 1}.$$

(b) In the M/D case (where D stands for deterministic) we have, for all λ,

$$\widetilde{\mathbb{E}}\beta_{\mathcal{P}} = \frac{e^{\lambda} - 1}{\lambda} > \widetilde{\mathbb{E}}\beta_{\mathcal{B}} = 1 + \frac{1}{2} \cdot \frac{\lambda}{1 + \lambda}.$$

This inequality is implied by the inequality $e^x > 1 + x + x^2/2$ which is true for all $x > 0$.

(c) In the M/GI case we have a freedom to choose the law of σ. We take a mixture: σ is either equal to the constant $1/3$, or is exponentially distributed with rate $3/5$, with equal probability for each case (the parameters are chosen so that $\mathbb{E}\sigma = 1$) we have that $\mathcal{P}$ outperforms $\mathcal{B}$ for high arrival rates (roughly for $\lambda > 11.2$) but the opposite is true for smaller rates. The exact expressions are obtained from the second row of Table 1 and are plotted in Figure 3(c). A better policy, insofar as expectations are concerned, can be found by considering a $\mathcal{PB}(\ell)$ policy or a $\mathcal{BP}(\ell)$ policy for appropriate ℓ. Changing the optimality criterion changes the story completely.

Large buffers make no sense. Most of research in the AoI area so far has focused on systems with infinite storage capacity have been studied. Among these systems, FIFO seems to be worst from the point of view of AoI. Indeed, it makes no sense to store an accepted message if we are only interested in the age of information. We should process it as soon as possible, perhaps even by preempting the message that is currently being processed. It is therefore intuitive that the buffer should have capacity of at most 2, including the packet currently being processed. It seems that adding additional buffer space beyond 2 works against us. See [12] regarding this point.

Let us define a system, that we call $\mathcal{P}_2$. The buffer has size 2. An arriving message, say message 1, to an empty buffer starts being processed immediately. If a second message, say 2, arrives while 1 is being processed it is stored. If message 3 arrives while 1 is still being processed and 2 stored, it pushes 2 out and replaces it. If no message arrives for a while, then 1 finishes and 3 starts being processed, leaving one unit available to accommodate the next arriving message, if any. The point is that while a message is being processed, it is never disturbed by an arriving message. An arriving message will only disturb the stored message, if any. Could, then, adding an extra unit buffer improve the system from the point of view of AoI or NAoI? The answer seems to be no. For evidence via simulations, look, for example at the D/M case (deterministic periodic arrivals, i.i.d. exponential service times). Figure 4 compares the three policies from the point of view of mean NAoI in steady state as a function of the arrival rate λ.

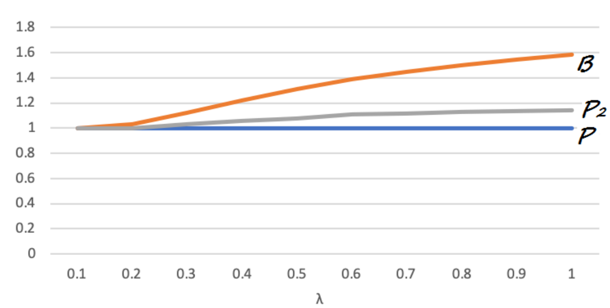


Figure 4: *The mean NAoI as a function of* λ *for D/M systems with deterministic interarrival times and i.i.d. exponential service times (with mean* 1*) in three cases.*

We next take a look at the most commonly studied system, an infinite buffer

FIFO system. But it performs even more poorly. Consider the M/M case. Here there is an explicit formula:

$$\text{Mean AoI for M/M/1/}\infty\text{/FIFO } = \frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda^2}{\mu^2} \cdot \frac{1}{\mu - \lambda}, \tag{81}$$

see [8, eq. (17)] and compare it with the mean AoI for $\mathcal{P}$ and $\mathcal{B}$, formulas as in Table 1:

$$\widetilde{\mathbb{E}}\alpha_{\mathcal{P}}(0) = \frac{1}{\lambda} + \frac{1}{\mu}, \quad \widetilde{\mathbb{E}}\alpha_{\mathcal{B}}(0) = \frac{1}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{\mu} \cdot \frac{1}{\lambda + \mu}.$$

Clearly, $\widetilde{\mathbb{E}}\alpha_{\mathcal{P}}(0)$ is the smallest of all. The FIFO mean is larger than $\widetilde{\mathbb{E}}\alpha_{\mathcal{B}}(0)$ when $\lambda > \sqrt{2}-1$. But even when $\lambda < \sqrt{2}-1$, the mean NAoI under FIFO is only 0.94% better than $\widetilde{\mathbb{E}}\alpha_{\mathcal{B}}(0)$. See Figure 5; here, the curve for $\mathcal{P}_2$ has been obtained by stochastic simulation. All curves tend to ∞ as $\lambda \to 0$ because of the fact that we plot AoI and not NAoI, and AoI also measures the time until the previous arrival. Also, the FIFO curve tends to ∞ as $\lambda \to \mu = 1$, and that is because the FIFO system becomes unstable.

The following observations provide additional evidence regarding the claim that small buffer systems perform at least as well as the well-studied infinite buffer LIFO and FIFO systems. Consider a sequence of arrival times and processing times fed to four systems, preemptive infinite LIFO buffer (pLIFO), infinite buffer FIFO, the $\mathcal{P}$ system, and the $\mathcal{P}_2$ system. Let α_{pLIFO}, β_{LIFO}, etc., be the AoIs and NAoIs for the four systems.

Observation 1. For all times t, $\alpha_{\text{pLIFO}}(t) = \alpha_{\mathcal{P}}(t)$ and $\beta_{\text{pLIFO}}(t) = \beta_{\mathcal{P}}(t)$.

To see this, recall that we use the same arrival times and same processing times for both pLIFO and $\mathcal{P}$ systems. Consider a trajectory of the pLIFO system. We will show how to construct the trajectory of the $\mathcal{P}$ system deterministically from that of the pLIFO system. Consider the arrival of a message at an empty pLIFO system, call this message 1, letting $2, 3, \ldots$ be the indices of subsequent messages. Observe the pLIFO system until the end of the busy period started with message 1. Necessarily, this busy period also ends with 1. Let j_1 be the index of the first message within this busy period that will not be disturbed by any arriving message: message j_1 is processed without interruption. Similarly, let $j_2 > j_1$ be

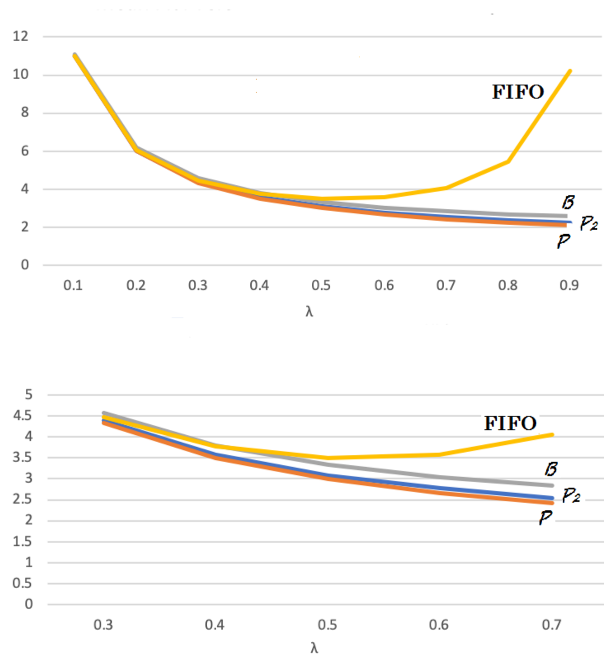


Figure 5: *The mean AoI as a function of λ for M/M systems with i.i.d. exponential interarrival times, i.i.d. exponential service times (with mean 1) in four cases: the bufferless $\mathcal{P}$ and $\mathcal{B}$ systems, the $\mathcal{P}_2$ system, and the infinite capacity system under FIFO. The right figure is a detail of the left. Note that $\mathcal{B}$ is slightly worse than FIFO for small λ. However, $\mathcal{P}_2$ is better than FIFO and $\mathcal{P}$ is better than $\mathcal{P}_2$ for all λ.*

the next uninterrupted message, and so on. We can construct the trajectory of $\mathcal{P}$ by observing that the messages $1, \ldots, j_1 - 1$ are unsuccessful and j_1 is the first successful message. Similarly, $j_1 + 1, \ldots, j_2 - 1$ are unsuccessful and j_2 is successful. The process A_t^* changes only at the departure times of $j_1, j_2, \ldots$. It is thus the same for both pLIFO and $\mathcal{P}$. Hence $\alpha_{\text{pLIFO}}(t) = \alpha_{\mathcal{P}}(t)$ for all t. Since A_t, the last arrival before t, is also the same for both systems (arrivals are coupled), we also have $\beta_{\text{pLIFO}}(t) = \beta_{\mathcal{P}}(t)$.

Observation 2. For times t that are successful departures from the $\mathcal{P}_2$ system, $\alpha_{\text{FIFO}}(t) \geq \alpha_{\mathcal{P}_2}(t)$ and $\beta_{\text{FIFO}}(t) \geq \beta_{\mathcal{P}_2}(t)$.

Assume both systems empty at time 0 with the first message indexed 1 arriving at time $T_1 \geq 0$. Under system $x \in \{\text{FIFO}, \mathcal{P}_2\}$, let $T'_{x,k}$ be departure time of message k, $\alpha_x(t)$ be the AoI at time t, and $W_x(t)$ be the work-to-be-done at time t (including the service time of an arrival at time t). For the $\mathcal{P}_2$ system, let χ_k indicate whether the k^{th} message is successfully served and let (k) be the index of the k^{th} successfully served message. Thus, $\forall k$, $\chi_{(k)} \equiv 1$. We argue inductively that

$$\forall k \geq 1, \quad W_{\text{FIFO}}(T_{(k)}) \;\geq\; W_{\mathcal{P}_2}(T_{(k)}). \tag{82}$$

Clearly (82) is true with equality ($= \sigma_1$) at $k = 1$. Assume (82) for arbitrary $k \geq 1$. This leads to the following inequality:

$$\begin{aligned}
&W_{\text{FIFO}}(T_{(k+1)}) \\
&= \max\left\{W_{\text{FIFO}}(T_{(k)}) + \sum_{i=(k)+1}^{(k+1)} \sigma_i - (T_{(k+1)} - T_{(k)}),\ \max_{(k)+1\leq \ell \leq (k+1)} \sum_{i=\ell}^{(k+1)} \sigma_i - (T_{(k+1)} - T_\ell)\right\} \\
&\geq \max\left\{W_{\mathcal{P}_2}(T_{(k)}) + \sum_{i=(k)+1}^{(k+1)} \sigma_i\chi_i - (T_{(k+1)} - T_{(k)}),\ \max_{(k)+1\leq \ell \leq (k+1)} \sum_{i=\ell}^{(k+1)} \sigma_i\chi_i - (T_{(k+1)} - T_\ell)\right\} \\
&= \max\left\{W_{\mathcal{P}_2}(T_{(k)}) + \sigma_{(k+1)} - (T_{(k+1)} - T_{(k)}),\ \sigma_{(k+1)}\right\} \\
&= W_{\mathcal{P}_2}(T_{(k+1)})
\end{aligned}$$

Thus, for all k, the departure time of (k) under FIFO,

$$T'_{\text{FIFO,(k)}} = T_{(k)} + W_{\text{FIFO}}(T_{(k)}) \;\;\geq\;\; T_{(k)} + W_{\mathcal{P}_2}(T_{(k)}) = T'_{\mathcal{P}_2,(k)}.$$

So, at time $T'_{\mathcal{P}_2,(k)}$, the index of the most recent completely served message under FIFO is $k' \leq (k)$. Therefore,

$$\alpha_{\mathcal{P}_2}(T'_{\mathcal{P}_2,(k)}) = T'_{\mathcal{P}_2,(k)} - T_{(k)} \leq T'_{\mathcal{P}_2,(k)} - T_{k'} = \alpha_{\text{FIFO}}(T'_{\mathcal{P}_2,(k)})$$

These arguments also hold for NAoI β because arrival times are coupled.

So it is unclear and rather puzzling why infinite capacity systems have been considered. As mentioned in Section 1.3, [2] showed that, among all infinite capacity systems, and under specific distributional assumptions, pLIFO is best. But pLIFO with preemption has the same AoI and NAoI as $\mathcal{P}$. So all the results obtained in this paper for $\mathcal{P}$ also hold for pLIFO.

For the M/M/1/∞-FIFO system, [8] observes that, when the service rate μ is fixed the mean AoI is minimized at $\lambda \approx 0.53\mu$. This is trivial: just minimize the expression (81) over $0 < \lambda < \mu$. To accomplish this may require that the arrival rate λ can be controlled. It is unclear how this control avoids dropping arriving (freshest) messages, or not generating them in the first place. Obviously, given λ, one can generally reduce the mean AoI by increasing μ. The assumption of a FIFO queueing discipline has been justified by its existing deployment in many practical scenarios (e.g., message transmission buffers of sensors). But practical scenarios also involve *finite* message buffers, and an arriving (freshest) message to a full buffer is dropped (unless pushout is available, in which case the queue could be operated as a bufferless system with pushout). This will be particularly problematic under heavy traffic. Generally, the AoI concept is not very interesting under light traffic.

Alternative definitions of age of information. Alternative definitions of age of information are possible and may be desirable. For example, a measure of freshness of information may involve message streams where the most recent message does *not* obsolete all previous ones. More specifically, assume that, upon arrival of a new message (with normalized "importance" 1), the importance of all prior messages can be diminished by a positive factor $\xi < 1$, and the objective could be to minimize the sum of the importance of all transmitted messages. That is, it may be desirable at the receiver to accurately interpolate between the freshest messages. This case may require a large message buffer under LIFO.

Open problems.

1. Compute the distributions and/or expectations of AoI and NAoI under the $\mathcal{PB}(\ell)$ and $\mathcal{BP}(\ell)$ policies, especially under renewal assumptions. Choose the ℓ that minimizes a given performance measure, e.g., $\widetilde{\mathbb{E}}\beta(0)$ or $\widetilde{\mathbb{P}}(\beta(0) > x)$ as a function of the interarrival and processing time distributions.

2. Formulate and solve a dynamic optimization problem. That is, decide the policy that accepts/rejects incoming messages and also decides which of them

will be successful or not. Even under renewal assumptions, this is not an easy problem. Life can possibly be made easier under specific distributional assumptions, e.g., in the good old M/M case.

3. Analyze the $\mathcal{P}_2$ system. That is, compute distributions and/or expectations for AoI and NAoI.

4. Conjecture: In steady state, $\beta_{\mathcal{P}_2}(0)$ is stochastically smaller than $\beta_{\text{FIFO}}(0)$. Evidence for this is Observation 2 above.

5. Conjecture: In steady state, $\beta_{\mathcal{P}_2}(0)$ is stochastically smaller than $\beta_{\text{npLIFO}}(0)$, where the latter is LIFO with *non*-preemptive service policy.

6. Take into account the technological constraints and see if alternative measures of the age of information can justify large buffers.

7. Better explain how these measures help real-time systems in real-life situations.

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A List of symbols

δ_x	delta measure at the point x
$\overline{X}$	a random variable with density $\mathbb{P}(X > x)/\mathbb{E}X$
T_n	arrival time of a message
χ_n	accept/reject index
ψ_n	success/failure index
$\mathcal{Z}_n$	informally, the event that the server is idle just before T_n
$\widetilde{\mathbb{P}}$	informally, probability measure governing the stationary system
$\mathbb{P}$	Palm probability of $\widetilde{\mathbb{P}}$ with respect to the arrival process
$\mathbb{P}^*$	Palm probability of $\widetilde{\mathbb{P}}$ with respect to reading intervals beginnings $= \mathbb{P}(\cdot\|\mathcal{Z}_0)$
$\mathfrak{a}$	$= \sum_n \delta_{T_n}$, arrival process as a point process
$\mathfrak{a}(t)$	$= \mathfrak{a}([0,t))$
$U(t)$	$= \mathbb{E}\mathfrak{a}(t)$
$Z(t)$	$= \mathbb{E}\sum_{i=0}^{\mathfrak{a}(t)-1} T_i$
$W_u(t)$	$= \mathbb{E}e^{-uT_{\mathfrak{a}(t)}}$
$V_u(t)$	$= \mathbb{E}\sum_{i=0}^{\mathfrak{a}(t)-1} e^{-uT_i}$
$Q_u(t)$	$= \mathbb{E}\big\{(T_{\mathfrak{a}(t)} - t)\, e^{-uT_{\mathfrak{a}(t)-1}}\big\}$
σ_n	processing time of a message
T'_n	departure time of a message either due to successful reading or not
A_t	last arrival epoch before t
S_t	last arrival epoch before t of a successful message
D_t	last departure epoch before t of a successful message
A^*_t	$= S_{D_t}$
$\Delta f(t)$	$= f(t+) - f(t-)$
τ_n	$= T_{n+1} - T_n$
B_k	beginning of a reading interval
$\mathbf{R}_k$	duration of a reading interval
B'_k	$= B_k + \mathbf{R}_k$
$\mathbf{C}_k$	$= B_{k+1} - _k$, cycle length
λ	arrival rate $= 1/\mathbb{E}\tau = \sum_n \widetilde{\mathbb{P}}(0 < T_n < 1)$

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