Optimal Behavior in a Reinsurance Market - An Actuarial Viewpoint

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Abstract

In this paper we examine a reinsurance market where a number of companies cooperate in order to minimize the total premium. The analysis is carried out in the context of the classical Cramér–Lundberg model of collective risk theory and it is assumed that each company determines its premium based on its attitude towards risk which is expressed via a fixed, infinite horizon ruin probability, as specified by the model. We formulate this problem as a variational problem in which ruin probabilities (as represented by the corresponding adjustment coefficients) are treated as constraints and the sum of the premium rates charged by insurer and reinsurers is minimized. Within this framework, the optimal solution is proportional reinsurance. Connections are also made with classical studies of the reinsurance market, such as Borch (1962).

Key words: Reinsurance Markets, Premium Principles, Cramér–Lundberg model, Adjustment Coefficient
IM52, IM13, G22

1 The reinsurance market

In a seminal paper, Borch (1962) examined the economics of a reinsurance market. He introduced the following model of such a market: n insurers have n portfolios, which can be though of as independent, non–negative real random variables $X_1, X_2, \ldots, X_n$ with distribution functions F_i, $i = 1, 2, \ldots, n$. Each insurer has an initial capital S_i and a utility function $v_i(\cdot)$ which represents his attitude towards risk. Thus the utility of the ith insurer in this situation is

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$V_i = \int_0^\infty v_i(S_i - x)dF_i(x) = Ev_i(S_i - X_i)$. Suppose now that these companies have concluded reinsurance agreements (*reinsurance treaties*) represented by the functions $\psi_i : \mathbb{R}_+^n \to \mathbb{R}_+$, $i = 1, 2, \ldots, n$. As a result of these treaties the new risks for the n insurers are given by $Y_i = \psi_i(X_1, X_2, \ldots, X_n)$, where of course the ψ_i's satisfy the additional requirement that

$$\sum_{i=1}^{n} \psi_i(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i \qquad \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}_+^n.$$

The new utilities under these treaties become $V_i^\psi = Ev_i(S_i - \psi(X_1, \ldots, X_n))$. In this framework one thus examines the functions ψ_i that lead to *Pareto–optimal* solutions. It has been shown in Borch (1962) (see also Borch, 1990), that Pareto–optimality implies that the reinsurance treaties are of the *pool type* i.e. that the functions ψ_i depend only on $\sum_{i=1}^n x_i$ and not on the individual x_i's and that they satisfy the condition

$$k_i u_i(\psi_i(x)) = k_j u_j(\psi_j(x_j)), \qquad i, j = 1, 2, \ldots, n,$$

where $k_i > 0$, $i = 1, 2, \ldots, n$, are arbitrary positive numbers and $u_i(x) := v_i(S_i - x)$. The above in particular implies that

$$k_i u_i(\psi_i(x)) = u'(x) \tag{1}$$

where $u'(x)$ represents aggregate marginal utility in the market. Denoting by $R_i(x) := -u_i''(x)/u_i'(x)$, $i = 1, 2, \ldots, n$, and $R(x) := -u''(x)/u'(x)$ the corresponding *absolute risk aversion* (1) becomes

$$\psi_i'(x) = \frac{\frac{1}{R_i(\psi_i(x))}}{\sum_{j=1}^n \frac{1}{R_j(\psi_j(x))}}, \qquad i = 1, 2, \ldots, n. \tag{2}$$

In general, the above formulation does not give a unique solution for the reinsurance contracts, ψ_i. It only gives a set of Pareto-optimal solutions, the so-called Pareto efficient frontier. If one desires to arrive to a unique solution, as opposed to characterizing the set of Pareto-optimal solutions, then it is necessary to introduce an overall criterion to be optimized.

Pioneers in the field of economics of risk are Allais (1953) and Arrow (1953). Parallel with Borch contributor are Gerber (1984), De Waegenaere (1994), De Waegenaere and Delbaen (1992), see also Aase (1993) and Aase (2002). Optimal properties of stop loss reinsurance are given in Borch (1969), see also Benktander (1975). The theory of premium principles and prices in the reinsurance market can be found among others in Pressacco (1979). Finally Dickson and Waters (1996) examine optimal reinsurance policies by minimizing the insurer's ruin probabilities.

2 Optimal premium with respect to the adjustment coefficient–A variational approach

We now examine the following risk-theoretic model of n insurance companies that are involved in insuring a portfolio that consists of a stream of claims that occur according to the standard Cramér-Lundberg collective risk model. We will denote the claim distribution by F and will assume it to be absolutely continuous with density f and finite mean denoted by μ. We further assume that the claim size distribution possesses exponential moments, i.e. that there exists $\epsilon > 0$ such that $\int_0^\infty e^{\epsilon x} f(x)dx < \infty$. Claim occurrence epochs are assumed to be Poisson with rate α.

2.1 The adjustment coefficient and the willingness to assume risk

We will adopt a simplified view whereby the ith insurance company sets its insurance premium rate *by deciding on a fixed ruin probability* and setting the corresponding premium rate by means of this fixed ruin probability which, in view of our assumption regarding the light–tailed nature of the claim size distribution, is given by the Cramér-Lundberg model. According to this model the infinite horizon ruin probability is asymptotically equal to

$$A_j e^{-u_j R_j} \tag{3}$$

where u_j is the initial capital of company j, R_j is the corresponding *adjustment coefficient* (or Lundberg exponent) determined by the equation

$$1 + R_j \frac{c_j}{\alpha\mu} = \int_0^\infty e^{R_j x} f(x)dx, \tag{4}$$

where c_j is the premium rate, and A_j is a constant which depends on the claim distribution and the premium rate. This is of course an approach taken in many other studies before, e.g. in Hesselager (1990) and its validity has been investigated in Dickson and Waters (1996). Here we will ignore the constant altogether and, denoting by e_j the *negative logarithm of the ruin probability*, we have the approximate equality (for large initial capital u_j)

$$e_j = u_j R_j. \tag{5}$$

However, instead of looking at the adjustment coefficient as a quantity to be minimized (as in Dickson and Waters (1996), Hesselager (1990) etc.) we will view the inverse of the adjustment coefficient,

$$R_j^{-1} = u_j / e_j, \tag{6}$$

as *a measure of the willingness of the jth company to assume risk.* It increases with the initial capital of the company and also increases with its tolerance of "ruin" as represented by e_j^{-1}.

2.2 The principle of minimizing the total premium

In order to decide on the retention level for each company we will use the following principle: *The portfolio will be split among the n companies in such a way as to minimize the total premium for the insured.* This principle is based on the assumption that the n companies cooperate fully in order to obtain the best possible position against outside competitors.

To be more specific, we will assume that a claim x must be covered from a group of n insurance companies in the following way. The part of the claim that corresponds to company j is defined to be $x\psi_j(x)$ where the following portfolio partitioning constraints must obviously hold.

$$\begin{aligned} &0 \leq \psi_j(x) \leq 1, \qquad j = 1, 2, \ldots, n \\ &\sum_{j=1}^{n} \psi_j(x) = 1, \quad \text{for every } x \in \mathbb{R}^+. \end{aligned} \tag{7}$$

Thus the functions $\psi_j : \mathbb{R}^+ \mapsto [0, 1]$ describe completely the reinsurance treaties between the n companies. For instance, if $\psi_j(x) = b_j > 0$ with $\sum_{j=1}^{n} b_j = 1$, then we have a proportional reinsurance treaty. If $\psi_j(x) = \min((x - a_{j-1})^+, a_j - a_{j-1})$, with $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = \infty$ then we have a band reinsurance treaty, and so forth.

Suppose that each insurer quotes a premium rate based on a fixed probability of ruin which in turn translates into a fixed adjustment coefficient via equation (6). *Fixing the adjustment coefficient R_j* for the jth insurer gives in turn, for each reinsurance treaty $\psi_j(x)$ a corresponding premium rate c_j from (4):

$$c_j \;=\; R_j^{-1} \int_0^{\infty} e^{R_j x \psi_j(x)} f(x) dx \;-\; R_j^{-1}.$$

Then the total premium rate charged to the insured is

$$c \;=\; \sum_{j=1}^{n} [R_j^{-1} \int_0^{\infty} e^{R_j x \psi_j(x)} f(x) dx \;-\; R_j^{-1}]. \tag{8}$$

The minimum premium rate principle we propose then consists in choosing the functions $\psi_j(x)$ so as to minimize the total premium rate charged to the insured, $c = \sum_{j=1}^{n} c_j$, subject to given attitudes toward risk for the n companies as determined through the corresponding adjustment coefficients R_j, $j = 1, 2, \ldots, n$ and under the portfolio partitioning constraints (7).

This becomes then a variational problem which turns out to have a very simple solution.

2.3 The Variational Problem

Write (8) in the form

$$c[\boldsymbol{\psi}] \;=\; \int_0^\infty L[x;\boldsymbol{\psi}]dx \;-\; \sum_{j=1}^n R_j^{-1}, \tag{9}$$

where $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$ and

$$L[x;\boldsymbol{\psi}] := f(x) \sum_{j=1}^n R_j^{-1} e^{R_j x \psi_j(x)}. \tag{10}$$

We will consider the problem of minimizing the above functional of $\boldsymbol{\psi}$ under the portfolio partitioning constraints (7) and using variational methods we will establish the following

Theorem 1 *The reinsurance treaties $\psi_j(x)$, $j = 1, 2, \ldots, n$ that minimize the total premium c are the proportional treaties given by*

$$\psi_j(x) = \frac{R_j^{-1}}{\sum_{k=1}^n R_k^{-1}}, \quad \forall x \in \mathbb{R}^+. \tag{11}$$

The corresponding minimum premium rate is then given by

$$c^* \;=\; \sum_{j=1}^n R_j^{-1} \left(M(R^*) - 1\right) \tag{12}$$

where $M(t) := \int_0^\infty e^{tx} f(x)dx$ is the moment generating function that corresponds to the claim distribution and $R^* = \left(\sum_{j=1}^n R_j^{-1}\right)^{-1}$ is the harmonic mean of the R_j.

Proof: Let us denote by $\mathbb{S}_n := \{(y_1, \ldots, y_n) : \sum_{j=1}^n y_j = 1;\; y_j \geq 0,\; j = 1, 2, \ldots, n\}$ the $(n-1)$–dimensional simplex in $\mathbb{R}^n$. We are faced with the problem of finding a function

$$\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n) : [0, \infty) \mapsto \mathbb{S}_n \tag{13}$$

so as to minimize the right hand side of (9). Ignoring for the moment the constraint imposed by the range of $\boldsymbol{\psi}$ in (13) we consider the problem of minimizing the right hand side of (9) when

$$\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n) : [0, \infty) \mapsto \mathbb{H}_n \tag{14}$$

where $\mathbb{H}_n := \{(y_1, \ldots, y_n) : \sum_{j=1}^n y_j = 1\}$ is the hyperplane that contains $\mathbb{S}_n$. The Hamiltonian of the system is given by

$$H[x; \boldsymbol{\psi}] := L[x; \boldsymbol{\psi}] - \lambda(x) G[x; \boldsymbol{\psi}], \tag{15}$$

where

$$G[x; \boldsymbol{\psi}] \ := \ \sum_{j=1}^{n} \psi_j(x) \ - 1 \tag{16}$$

and $\lambda(\cdot)$ is a piecewise continuous function. (The Lagrange multiplier function.)

The first order necessary conditions for an extremum are

$$\frac{\partial H}{\partial \psi_j} = 0, \quad j = 1, 2, \ldots, n, \tag{17}$$

$$\frac{\partial H}{\partial \lambda} = 0. \tag{18}$$

In view of (10) and (15) the first order conditions (17) above become

$$x f(x) e^{R_j x \psi_j(x)} - \lambda(x) = 0 \quad \text{for all } x \geq 0 \text{ and } j = 1, 2, \ldots, n.$$

These give for each j

$$e^{R_j x \psi_j(x)} = \frac{\lambda(x)}{x f(x)}$$

or

$$\psi_j(x) = \frac{1}{R_j x} \log \frac{\lambda(x)}{x f(x)} \qquad \text{for } j = 1, 2, \ldots, n. \tag{19}$$

In view of (15), (16), and (18) it follows from (19) that

$$\sum_{j=1}^{n} \frac{1}{R_j x} \log \frac{\lambda(x)}{x f(x)} = 1,$$

or

$$\frac{1}{x} \log \frac{\lambda(x)}{x f(x)} = \frac{1}{\sum_{j=1}^{n} \frac{1}{R_j}}.$$

Hence, according to (19), the functions ψ_j that satisfy the first order conditions for an extremum are the *constant functions* given by

$$\psi_j(x) = \frac{R_j^{-1}}{\sum_{j=1}^{n} R_j^{-1}}, \qquad x \in [0, \infty), \quad j = 1, 2, \ldots, n. \tag{20}$$

Denote by $\boldsymbol{\psi}^*$ the element of $\mathbb{H}^n$ defined by (20). Since $\boldsymbol{\psi}^*$ is the unique function that satisfies the first order conditions it is clear that it is an extremal point. Furthermore, if $\alpha \in [0,1]$, from (9) and the convexity of the exponential function on $\mathbb{R}$ we can readily check that $c[\alpha\boldsymbol{\psi}+(1-\alpha)\boldsymbol{\psi}'] \leq \alpha c[\boldsymbol{\psi}]+(1-\alpha)c[\boldsymbol{\psi}']$ for any $\boldsymbol{\psi}, \boldsymbol{\psi}' \in \mathbb{H}^n$, i.e. that $c : \mathbb{H}^n \mapsto \mathbb{R}^+$ is a convex function. Thus it follows that $\boldsymbol{\psi}^*$ minimizes c over $\mathbb{H}^n$ and hence over $\mathbb{S}^n$ since $\boldsymbol{\psi}^* \in \mathbb{S}^n \subset \mathbb{H}^n$. ∎

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