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SHORT COMMUNICATIONS

ON THE STABILITY OF A CHARACTERIZATION OF THE POISSON DISTRIBUTION

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1. Introduction

Rao and Rubin [3] derived an interesting characterization of the Poisson distribution based on partial independence between two random variables. They showed that if the non-negative, integer-valued random variables X and Y are such that

$$\mathrm{P}(Y=r|X=n)=\binom{n}{r}p^r q^{n-r}, \qquad r=0,1,\cdots,n;\ n=0,1,\cdots, \tag{1.1}$$

where $0<p<1$ with p fixed and independent of n, $q=1-p$, then the condition

$$\mathrm{P}(Y=r)=\mathrm{P}(Y=r|X=Y), \qquad r=0,1,\cdots, \tag{1.2}$$

is necessary and sufficient for X to follow a Poisson distribution.

This result has attracted a lot of interest in the literature. Since then, many variants and extensions of it have appeared, the more general of which are due to Shanbhang [5] and Panaretos [2].

In this paper we consider the stability of the above characterization. We show that if the conditional distribution of Y on X is "approximately" binomial and the Rao-Rubin condition (1.2) is "approximately" satisfied, then the distribution of X is "close" to a Poisson distribution. (For a lucid exposition of the problem of stability of characterizations the interested reader is referred to the works of Lukacs [1] and Zolotarev [7]. The stability property of a similar characterization of the exponential distribution has been studied recently by Shimizu [6].)

2. The Main Result

Suppose that for the non-negative, integer-valued random variables X and Y we know that

$$\mathrm{P}(Y=r|X=n)=\binom{n}{r}p^r q^{n-r}+\varepsilon_{n,r}, \qquad r=0,1,2,\cdots,n, \tag{2.1}$$

where $\sum_{i=0}^{n}\varepsilon_{n},\ i=0,\ n=0,1,2,\cdots$, and

$$\mathrm{P}(Y=r|X=Y)=\mathrm{P}(Y=r)+\delta_r \tag{2.2}$$

for all $n\geqq r\geqq 0$ and $\sum_{j=0}^{\infty}\delta_j=0$. Let $c=\mathrm{P}(X=Y)$ and $\alpha=-q^{-1}\log c$. Assume that

$$\sum_{n\geqq r}|\varepsilon_{n,r}|\leqq \delta\lambda^r/r! \tag{2.3}$$

and

$$|\delta_r| \leqq \delta\lambda^r/r! \tag{2.4}$$

for some $\lambda, 0<\lambda<p\alpha$. Further, let

$$\delta^* = (2c+1)\delta, \quad \beta = \lambda/p \quad (\beta<\alpha), \tag{2.5}$$

$$\theta = \delta^*(1-e^{-\varepsilon q})^{-1}, \tag{2.6}$$

and let $G(t)$ denote the probability generating function of X.

Theorem. *When the non-negative, integer-valued random variables X and Y satisfy conditions* (2.1) *and* (2.2), *then*

$$\left|\mathbf{P}(X=n) - e^{-\alpha}\frac{\alpha^n}{n!}\right| \leqq \frac{\theta}{n!}(e^{-\varepsilon}+\beta^n), \tag{2.7}$$

where $\varepsilon = \alpha-\beta$, $n = 0, 1, 2, \cdots$.

PROOF. From (2.1) and (2.2) we find

$$G(pt) = e^{-\alpha q}G(q+pt) + y(t), \qquad |t|\leqq 1, \tag{2.8}$$

where

$$y(t) = e^{-\alpha q}\sum_{r=0}^{\infty}\sum_{n=r}^{\infty} p_n\varepsilon_{n,r}t^r + e^{-\alpha q}\sum_{r=0}^{\infty}\delta_r t^r - \sum_{r=0}^{\infty} p_r\varepsilon_{r,r}t^r \tag{2.9}$$

and $p_n = \mathbf{P}(X=n)$. It is easy to see that (2.8) implies

$$G(t) = e^{-\alpha q}G(q+t) + \psi(t), \qquad -\infty<t<+\infty, \tag{2.10}$$

with $\psi(t) = y(t/p)$. Let

$$H(t) = e^{-\alpha t}G(t) \tag{2.11}$$

and

$$c(t) = \psi(t)\, e^{-\beta t}. \tag{2.12}$$

Then (2.10) can be written as

$$H(t) = H(q+t) + c(t)\, e^{-\varepsilon t}, \qquad -\infty<t<+\infty. \tag{2.13}$$

Iterating (2.13) we obtain, for $k = 0, 1, 2, \cdots$,

$$H(t) = H(t+(k+1)q) + c_k(q)\, e^{-\varepsilon t}, \tag{2.14}$$

where

$$c_\rho(t) = \sum_{i=0}^{\rho} c(t+iq)\, e^{-\varepsilon iq}. \tag{2.15}$$

It can now be shown that $\psi^{(l)}(t)$ exists for $l = 0, 1, 2, \cdots$ and that

$$|\psi^{(l)}(t)| \leqq \delta^*\beta^l e^{\beta l}, \qquad t>0, l = 0, 1, 2, \cdots. \tag{2.16}$$

In particular, we have $|\psi(t)| \leqq \delta^*\varepsilon^{\beta t}$, which, in turn, implies that

$$|c_\rho(t)| \leqq \delta^*(l-e^{-\varepsilon q})^{-1} \equiv \theta. \tag{2.17}$$

The function $H(t)$, defined in (2.11) is bounded in the interval $[-1, \infty)$. We shall show that $\lim_{k\to\infty} H(t+kq)$ exists. To show this it is sufficient to prove that $H(t+kq)$ is a Cauchy sequence. Let m, s be two non-negative integers such that $m \geqq k$, $s \geqq k$. Then from (2.13) and for $t \geqq -1$ we have

$$\begin{aligned}
&|H(t+mq) - H(t+sq)| \\
&\quad \leqq |H(t+kq+(m-k)q) - H(t+kq)| + |H(t+kq+(s-k)q) - H(t+kq)| \\
&\quad = (|c_{m-k}(t+kq)| + |c_{s-k}(t+kq)|)\, e^{-\varepsilon(t+kq)} \leqq 2\theta\, e^{-\varepsilon t}\, e^{-\varepsilon kq} \to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Therefore there exists a function $\Lambda(t)$ such that

$$\lim_{k\to\infty} H(t+kq)=\Lambda(t). \tag{2.18}$$

Note that $\Lambda(t)$ is a periodic function with period q, i.e., $\Lambda(t+q)=\Lambda(t)$. Taking the limit on both sides of (2.14) as $k\to\infty$ one obtains

$$H(t)=\Lambda(t)+e^{-\varepsilon t}A(t), \tag{2.19}$$

where

$$A(t)=\lim_{k\to\infty} c_k(t)=\sum_{i=0}^{\infty} c(t+iq)\, e^{-\varepsilon iq}. \tag{2.20}$$

We may observe here that

$$|A(t)|\leqq\theta. \tag{2.21}$$

Combining (2.11) and (2.19) yields

$$G(t)=e^{\alpha t}\Lambda(t)+e^{\beta t}A(t). \tag{2.22}$$

Note that $A^{(l)}(t)$ exists and is bounded for $l=0,1,2,\cdots$. Differentiating both sides of (2.10) l times one obtains

$$G^{(l)}(t)=e^{-\alpha q}G^{(l)}(t+q)+\psi^{(l)}(t), \qquad l=0,1,2,\cdots,-\infty<t<+\infty. \tag{2.23}$$

We shall now show that (2.23) implies

$$G^{(l)}(t)=\Lambda_l(t)\, e^{\alpha t}+A_l(t)\, e^{\beta t} \tag{2.24}$$

with $\Lambda_l(t)$, $A_l(t)$ appropriately defined. Extending our earlier notation, let

$$H_l(t)=e^{-\alpha t}G^{(l)}(t) \tag{2.25}$$

and

$${}_lc(t)=\psi^{(l)}(t)\, e^{-\beta t}, \qquad l=0,1,2,\cdots,-\infty<t<+\infty; \tag{2.26}$$

(2.24), (2.25) and (2.26) give $H_l(t)=H_l(t+q)+e^{(\beta-\alpha)t}\,{}_lc(t)$, and by iteration

$$H_l(t)=H_l(t+(k+1)q)+e^{-\varepsilon iq}\,{}_lc_k(t), \qquad l,k=0,1,2,\cdots,-\infty<t<+\infty, \tag{2.27}$$

where this time

$${}_lc_p(t)=\sum_{i=0}^{p} {}_lc(t+iq)\, e^{-\varepsilon iq}. \tag{2.28}$$

Obviously

$$|{}_lc_p(t)|\leqq\delta^*\beta^l(1-e^{-\varepsilon q})^{-1}\equiv\theta\beta^l. \tag{2.29}$$

Employing the argument used to arrive at (2.18) we can show that the $\lim_{k\to\infty} H_l(t+kq)$ exists and is $\Lambda_l(t)$, say. Furthermore $\Lambda_l(t)$ is also periodic with period q. From (2.27) we have

$$H_l(t)=\Lambda_l(t)+\lim_{k\to\infty} {}_lc_k(t)\, e^{-\varepsilon t}, \tag{2.30}$$

where (from (2.28))

$$\lim_{k\to\infty} {}_lc_k(t)=\sum_{i=0}^{\infty} {}_lc(t+iq)\, e^{-\varepsilon iq}\equiv A_l(t), \tag{2.31}$$

say. Then, combining (2.30), (2.25) and (2.31) one obtains (2.24). It can be noted here that

$$|A_l(t)|\leqq\theta\beta^l. \tag{2.32}$$

If we take the l-th order derivative of both sides of (2.22), we have

$$G^{(l)}(t)=(e^{\alpha t}\Lambda(t))^{(l)}+(e^{\beta t}A(t))^{(l)}. \tag{2.33}$$

Using (2.20) and (2.12) we find that

$$(e^{\beta t}A(t))^{(l)}=\sum_{i=0}^{\infty}\psi^{(l)}(t+iq)\,e^{-\alpha iq}. \tag{2.34}$$

On the other hand, taking into account (2.31) and (2.26) we see that

$$e^{\beta t}A_l(t)=\sum_{i=0}^{\infty}\psi^{(l)}(t+iq)\,e^{-\alpha iq}. \tag{2.35}$$

Comparison of (2.34) and (2.35) results in

$$(e^{\beta t}A(t))^{(l)}=e^{\beta t}A_l(t). \tag{2.36}$$

Furthermore, (2.24), (2.33) and (2.36) give

$$\Lambda_l(t)\,e^{\alpha t}=(e^{\alpha t}\Lambda(t))^{(l)}, \tag{2.37}$$

or, by setting

$$G_0(t)=\Lambda(t)\,e^{\alpha t}, \tag{2.38}$$

$$G_0^{(l)}(t)=\Lambda_l(t)\,e^{\alpha t}. \tag{2.39}$$

We shall now show that $\Lambda(t)$ is constant, say Λ. For $l=1,2,\cdots$ we have (using (2.22) and (2.24) and the fact that $A_l(t)$ is bounded and $\Lambda_l(t)$ is periodic)

$$\begin{aligned}\lim_{k\to\infty}\frac{G^{(l)}(t+kq)}{G(t+kq)}&=\lim_{k\to\infty}\frac{\Lambda_l(t+kq)\,e^{\alpha(t+kq)}+A_l(t+kq)\,e^{\beta(t+kq)}}{\Lambda(t+kq)\,e^{\alpha(t+kq)}+A(t+kq)\,e^{\beta(t+kq)}}\\&=\lim_{k\to\infty}\frac{\Lambda_l(t)\,e^{\alpha t}+A_l(t+kq)\,e^{\beta t}\,e^{-\varepsilon kq}}{\Lambda(t)\,e^{\alpha t}+A(t+kq)\,e^{\beta t}\,e^{-\varepsilon kq}}=\frac{\Lambda_l(t)}{\Lambda(t)}=\frac{G_0^{(l)}(t)}{G_0(t)}\end{aligned} \tag{2.40}$$

(because of (2.39)). Specializing (2.40) for $l=1$ and $l=2$ and using the technique of Shanbhag [4] we obtain

$$\begin{aligned}\frac{d}{dt}\left\{\frac{G_0'(t)}{G_0(t)}\right\}&=\frac{G_0''(t)}{G_0(t)}-\left\{\frac{G_0'(t)}{G_0(t)}\right\}^2=\lim_{k\to\infty}\frac{G''(t+kq)}{G(t+kq)}-\left\{\lim_{k\to\infty}\frac{G'(t+kq)}{G(t+kq)}\right\}^2\\&=\lim_{k\to\infty}\left[\frac{\sum_{j=1}^{\infty}j^2p_j(t+kq)^{j-2}}{\sum_{j=0}^{\infty}p_j(t+kq)^j}-\left\{\frac{\sum_{j=1}^{\infty}jp_j(t+kq)^{j-1}}{\sum_{j=1}^{\infty}p_j(t+kq)^j}\right\}^2\right]\geqq 0.\end{aligned} \tag{2.41}$$

This means that $\log G_0(t)$ is a convex function, i.e., (from (2.39)) $\log\Lambda(t)+\alpha(t)$ is convex. This can happen if and only if $\Lambda(t)$ is a constant function (as $\Lambda(t)$ is periodic). So

$$\Lambda(t)=\Lambda,\ \text{a constant}. \tag{2.42}$$

Then (2.37) and (2.42) give

$$\Lambda_l(t)=\alpha^l\Lambda. \tag{2.43}$$

Thus, using (2.23) and (2.43) successively, we have, for $n=0,1,2,\cdots$,

$$P(X=n)=\frac{1}{n!}G^{(n)}(0)=\frac{1}{n!}(\Lambda_n(0)+A_n(0))=\frac{1}{n!}(\alpha^n\Lambda+A_n(0)). \tag{2.44}$$

Putting $\Lambda(t)=\Lambda$ and $t=1$ we obtain from (2.22)

$$\Lambda=e^{-\alpha}-A(1)\,e^{-\varepsilon}. \tag{2.45}$$

Consequently, by taking into account (2.45), (2.21), (2.32),

$$\left|\mathrm{P}(X=n)-e^{-\alpha}\frac{\alpha^n}{n!}\right| = \left|-\frac{1}{n!}A(1)\,e^{-\varepsilon}+\frac{1}{n!}A_n(0)\right|$$

$$\leqq \frac{1}{n!}\{e^{-\varepsilon}|A(1)|+|A_n(0)|\}\leqq \frac{\theta}{n!}(e^{-\varepsilon}+\beta^n).$$

This completes the proof of the theorem.

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