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КРАТКИЕ СООБЩЕНИЯ

ON THE STABILITY OF A CHARACTERIZATION OF THE POISSON DISTRIBUTION

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1. Introduction. Rao and Rubin [3] derived an interesting characterization of the Poisson distribution based on partial independence between two random variables. They showed that if the nonnegative, integer-valued random variables X and Y are such that

$$\mathrm{P}(Y=r \mid X=n)=\binom{n}{r} p^r q^{n-r}, \quad r=0, 1, \ldots, n; \ n=0, 1, \ldots, \tag{1.1}$$

where $0<p<1$ with p fixed and independent of n, $q=1-p$, then the condition

$$\mathrm{P}(Y=r)=\mathrm{P}(Y=r \mid X=Y), \quad r=0, 1, \ldots, \tag{1.2}$$

is necessary and sufficient for X to follow a Poisson distribution.

This result has attracted a lot of interest in the literature. Since then, many variants and extensions of it have appeared, the more general of which are due to Shanbhang [5] and Panaretos [2].

In this paper we consider the stability of the above characterization. We show that if the conditional distribution of Y on X is «approximately» binomial and the Rao—Rubin condition (1.2) is «approximately» satisfied, then the distribution of X is «close» to a Poisson distribution. (For a lucid exposition of the problem of stability of characterizations the interested reader is referred to the works of Lukacs [1] and Zolotarev [7]. The stability property of a similar characterization of the exponential distribution has been studied recently by Shimizu [6].)

2. The main result. Suppose that for the nonnegative, integer-valued random variables X and Y we have that

$$\mathrm{P}(Y=r \mid X=n)=\binom{n}{r} p^r q^{n-r}+\varepsilon_{n, r}, \quad r=0, 1, 2, \ldots, n, \tag{2.1}$$

where $\sum_{i=0}^{n} \varepsilon_{n, i}=0$, $n=0, 1, 2, \ldots$, and

$$\mathrm{P}(Y=r \mid X=Y)=\mathrm{P}(Y=r)+\delta_r \tag{2.2}$$

for all $n \geqslant r \geqslant 0$ and $\sum_{j=0}^{\infty} \delta_j=0$. Let $c=\mathrm{P}(X=Y)$ and $\alpha=-q^{-1}\log c$. Assume that

$$\sum_{n \geqslant r} |\varepsilon_{n, r}| \leqslant \delta\lambda^r/r! \tag{2.3}$$

and

$$|\delta_r| \leqslant \delta\lambda^r/r! \tag{2.4}$$

for some λ, $0<\lambda<p\alpha$. Further, let

$$\delta^*=(2c+1)\,\delta, \quad \beta=\lambda/p \ (\beta<\alpha), \tag{2.5}$$

$$\theta=\delta^*(1-e^{-\varepsilon q})^{-1}, \tag{2.6}$$

and let $G(t)$ denote the probability generating function of X.

Theorem. *When the nonnegative, integer-valued random variables X and Y satisfy conditions* (2.1) *and* (2.2) *then*

$$\left|\mathrm{P}(X=n)-e^{-\alpha}\frac{\alpha^n}{n!}\right| \leqslant \frac{\theta}{n!}(e^{-\varepsilon}+\beta^n), \tag{2.7}$$

where $\varepsilon=\alpha-\beta$, $n=0, 1, 2, \ldots$.

Proof. From (2.1) and (2.2) we find

$$G(pt) = e^{-\alpha q}\, G(q + pt) + y(t), \quad |t| \leqslant 1, \tag{2.8}$$

where

$$y(t) = e^{-\alpha q} \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} p_n \varepsilon_{n,\,r} t^r + e^{-\alpha q} \sum_{r=0}^{\infty} \delta_r t^r - \sum_{r=0}^{\infty} p_r \varepsilon_{r,\,r} t^r \tag{2.9}$$

and $p_n = \mathrm{P}(X = n)$. It is easy to see that (2.8) implies

$$G(t) = e^{-\alpha q} G(q + t) + \psi(t), \quad -\infty < t < +\infty, \tag{2.10}$$

with $\psi(t) = y(t/p)$. Let

$$H(t) = e^{-\alpha t} G(t) \tag{2.11}$$

and

$$c(t) = \psi(t)\, e^{-\beta t}. \tag{2.12}$$

Then (2.10) can be written as

$$H(t) = H(q + t) + c(t)\, e^{-\varepsilon t}, \quad -\infty < t < +\infty. \tag{2.13}$$

Iterating (2.13) we obtain, for $k = 0, 1, 2, \ldots$,

$$H(t) = H(t + (k + 1)\, q) + c_k(q) e^{-\varepsilon t}, \tag{2.14}$$

where

$$c_\rho(t) = \sum_{i=0}^{\rho} c(t + iq)\, e^{-\varepsilon iq}. \tag{2.15}$$

It can now be shown that $\psi^{(l)}(t)$ exists for $l = 0, 1, 2, \ldots$ and that

$$|\psi^{(l)}(t)| \leqslant \delta^* \beta^l e^{\beta t}, \quad t > 0, \ l = 0, 1, 2, \ldots. \tag{2.16}$$

In particular, we have $|\psi(t)| \leqslant \delta^* e^{\beta t}$, which, in turn, implies that

$$|c_\rho(t)| \leqslant \delta^* (l - e^{-\varepsilon q})^{-1} \equiv \theta. \tag{2.17}$$

The function $H(t)$, defined in (2.11) is bounded in the interval $[-1, \infty)$. We will show that $\lim\limits_{k\to\infty} H(t + kq)$ exists. To show this it is sufficient to prove that $H(t + kq)$ is a Cauchy sequence. Let m, s be two nonnegative integers such that $m \geqslant k$, $s \geqslant k$. Then from (2.13) and for $t \geqslant -1$ we have

$$\begin{aligned} &|H(t + mq) - H(t + sq)| \leqslant \\ &\leqslant |H(t + kq + (m - k)\, q) - H(t + kq)| + |H(t + kq + (s - k)\, q) - H(t + kq)| = \\ &= (|c_{m-k}(t + kq)| + |c_{s-k}(t + kq)|)\, e^{-\varepsilon(t+kq)} \leqslant 2\theta e^{-\varepsilon t} e^{-\varepsilon kq} \to 0 \ \text{as} \ k \to \infty. \end{aligned}$$

Therefore there exists a function $\Lambda(t)$ such that

$$\lim_{k\to\infty} H(t + kq) = \Lambda(t). \tag{2.18}$$

Note that $\Lambda(t)$ is a periodic function with period q, i. e., $\Lambda(t + q) = \Lambda(t)$. Taking the limit on both sides of (2.14) as $k \to \infty$ gives

$$H(t) = \Lambda(t) + e^{-\varepsilon t} A(t), \tag{2.19}$$

where

$$A(t) = \lim_{k\to\infty} c_k(t) = \sum_{i=0}^{\infty} c(t + iq)\, e^{-\varepsilon iq}. \tag{2.20}$$

We may observe here that

$$|A(t)| \leqslant \theta. \tag{2.21}$$

Combining (2.11) and (2.19) yields

$$G(t) = e^{\alpha t} \Lambda(t) + e^{\beta t} A(t). \tag{2.22}$$

Note that $A^{(l)}(t)$ exists and is bounded for $l = 0, 1, 2, \ldots$. Differentiating both sides of (2.10) l times yields

$$G^{(l)}(t) = e^{-\alpha q} G^{(l)}(t + q) + \psi^{(l)}(t), \quad l = 0, 1, 2, \ldots, \ -\infty < t < +\infty. \tag{2.23}$$

We will now show that (2.23) implies

$$G^{(l)}(t) = \Lambda_l(t)\, e^{\alpha t} + A_l(t)\, e^{\beta t} \tag{2.24}$$

with $\Lambda_l(t)$, $A_l(t)$ appropriately defined. Extending our earlier notation, let

$$H_l(t) = e^{-\alpha t} G^{(l)}(t) \tag{2.25}$$

and

$$_lc(t) = \psi^{(l)}(t)\, e^{-\beta t}, \quad l = 0, 1, 2, \ldots, -\infty < t < +\infty; \tag{2.26}$$

(2.24), (2.25) and (2.26) give $H_l(t) = H_l(t+q) + e^{(\beta-\alpha)t}{}_lc(t)$, and by iteration

$$H_l(t) = H_l(t + (k+1)q) + e^{-\varepsilon iq}{}_lc_k(t), \; l,\; k = 0, 1, 2, \ldots, -\infty < t < +\infty, \tag{2.27}$$

where this time

$$_lc_\rho(t) = \sum_{i=0}^{\rho} {}_lc(t+iq)\, e^{-\varepsilon iq}. \tag{2.28}$$

Obviously

$$|\, _lc_\rho(t)\,| \leqslant \delta^* \beta^l (1 - e^{-\varepsilon q})^{-1} \equiv \theta\beta^l. \tag{2.29}$$

Employing the argument used to arrive at (2.18) we can show that the $\lim\limits_{k\to\infty} H_l(t+kq)$ exists and is $\Lambda_l(t)$, say. Furthermore $\Lambda_l(t)$ is also periodic with period q. From (2.27) we have

$$H_l(t) = \Lambda_l(t) + \lim_{k\to\infty} {}_lc_k(t)\, e^{-\varepsilon t}, \tag{2.30}$$

where (from (2.28))

$$\lim_{k\to\infty} {}_lc_k(t) = \sum_{i=0}^{\infty} {}_lc(t+iq)\, e^{-\varepsilon iq} \equiv A_l(t), \tag{2.31}$$

say. Then, combining (2.30), (2.25) and (2.31) yields (2.24). It can be noted here that

$$|\, A_l(t)\,| \leqslant \theta\beta^l. \tag{2.32}$$

If we take the l-th order derivative of both sides of (2.22), we have

$$G^{(l)}(t) = (e^{\alpha t}\Lambda(t))^{(l)} + (e^{\beta t}A(t))^{(l)}. \tag{2.33}$$

Using (2.20) and (2.12) we find that

$$(e^{\beta t}A(t))^{(l)} = \sum_{i=0}^{\infty} \psi^{(l)}(t+iq)\, e^{-\alpha iq}. \tag{2.34}$$

On the other hand, taking into account (2.31) and (2.26) we see that

$$e^{\beta t}A_l(t) = \sum_{i=0}^{\infty} \psi^{(l)}(t+iq)\, e^{-\alpha iq}. \tag{2.35}$$

Comparison of (2.34) and (2.35) results in

$$(e^{\beta t}A(t))^{(l)} = e^{\beta t}A_l(t). \tag{2.36}$$

Furthermore, (2.24), (2.33) and (2.36) give

$$\Lambda_l(t)\, e^{\alpha t} = (e^{\alpha t}\Lambda(t))^{(l)}, \tag{2.37}$$

or, by setting

$$G_0(t) = \Lambda(t)\, e^{\alpha t}, \tag{2.38}$$

$$G_0^{(l)}(t) = \Lambda_l(t)\, e^{\alpha t}. \tag{2.39}$$

We will now show that $\Lambda(t)$ is constant, say Λ. For $l = 1, 2, \ldots$ we have (using (2.22) and (2.24) and the fact that $A_l(t)$ is bounded and $\Lambda_l(t)$ is periodic)

$$\lim_{k\to\infty} \frac{G^{(l)}(t+kq)}{G(t+kq)} = \lim_{k\to\infty} \frac{\Lambda_l(t+kq)\, e^{\alpha(t+kq)} + A_l(t+kq)\, e^{\beta(t+kq)}}{\Lambda(t+kq)\, e^{\alpha(t+kq)} + A(t+kq)\, e^{\beta(t+kq)}} =$$

$$= \lim_{k\to\infty} \frac{\Lambda_l(t)\, e^{\alpha t} + A_l(t+kq)\, e^{\beta t} e^{-\varepsilon kq}}{\Lambda(t)\, e^{\alpha t} + A(t+kq)\, e^{\beta t} e^{-\varepsilon kq}} = \frac{\Lambda_l(t)}{\Lambda(t)} = \frac{G_0^{(l)}(t)}{G_0(t)} \tag{2.40}$$

(because of (2.39)). Specializing (2.40) for $l = 1$ and $l = 2$ and using the technique of Shanbhag [4] we obtain

$$\frac{d}{dt}\left\{\frac{G_0'(t)}{G_0(t)}\right\} = \frac{G_0''(t)}{G_0(t)} - \left\{\frac{G_0'(t)}{G_0(t)}\right\}^2 = \lim_{k\to\infty}\frac{G''(t+kq)}{G(t+kq)} - \left\{\lim_{k\to\infty}\frac{G'(t+kq)}{G(t+kq)}\right\}^2 =$$

$$= \lim_{k\to\infty}\left[\frac{\sum\limits_{j=1}^{\infty} j^2 p_j (t+kq)^{j-2}}{\sum\limits_{j=0}^{\infty} p_j (t+kq)^j} - \left\{\frac{\sum\limits_{j=1}^{\infty} j p_j (t+kq)^{j-1}}{\sum\limits_{j=1}^{\infty} p_j (t+kq)^j}\right\}^2\right] \geqslant 0. \tag{2.41}$$

This means that $\log G_0(t)$ is a convex function, i. e. (from (2.39)), $\log \Lambda(t) + \alpha(t)$ is convex. This can happen if and only if $\Lambda(t)$ is a constant function (as $\Lambda(t)$ is periodic). So

$$\Lambda(t) = \Lambda, \text{ a constant.} \tag{2.42}$$

Then (2.37) and (2.42) give

$$\Lambda_l(t) = \alpha^l \Lambda. \tag{2.43}$$

Thus, using (2.23) and (2.43) successively, we have for $n = 0, 1, 2, \ldots$,

$$\mathrm{P}(X = n) = \frac{1}{n!} G^{(n)}(0) = \frac{1}{n!}(\Lambda_n(0) + A_n(0)) = \frac{1}{n!}(\alpha^n \Lambda + A_n(0)). \tag{2.44}$$

Putting $\Lambda(t) = \Lambda$ and $t = 1$ we obtain from (2.22)

$$\Lambda = e^{-\alpha} - A(1)e^{-\varepsilon}. \tag{2.45}$$

Consequently, by taking into account (2.45), (2.21), (2.32),

$$\left|\mathrm{P}(X = n) - e^{-\alpha}\frac{\alpha^n}{n!}\right| = \left|-\frac{1}{n!}A(1)e^{-\varepsilon} + \frac{1}{n!}A_n(0)\right| \leqslant$$

$$\leqslant \frac{1}{n!}\{e^{-\varepsilon}|A(1)| + |A_n(0)|\} \leqslant \frac{\theta}{n!}(e^{-\varepsilon} + \beta^n).$$

This completes the proof of the theorem.

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