CHARACTERIZATION OF THE COMPOUND POISSON DISTRIBUTION

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INTRODUCTION

Consider two non-negative integer-valued r.v.'s X,Y with $X \geq Y$. Suppose that the conditional distribution of $Y|X$ is binomial with parameters (n,p), $n=0,1,2,\ldots$; $0<p<1$ and p independent of n. It is known , and can be checked easily, that under the above assumption the distribution of Y is Poisson with parameter λp, $\lambda>0$ (Poisson(λp)) if and only if (iff) X is Poisson (λ). This model has been extensively used in the literature under different names in many practical situations.

One area in which the model has found considerable use is that of accident theory. One possible interpretation has been given by Leiter and Hamdan (1973). They have considered the bivariate distribution of (X,Y) as a model for highway accidents. X represents the number of highway accidents in a given locality and for a given period and Y is the number of fatal accidents among these X accidents. However, they found the fit to a certain set of accident data unsatisfactory. It is possible, therefore, to question the model for either $Y|X$ or X or both. One may consider, for example, that either p or λ or both are not constants. Instead, they may be assumed as r.v.'s with distribution functions $F(p)$ and $G(\lambda)$, respectively. The corresponding distributions then will have a compound form. The distribution of $Y|X$ will be binomial compounded by $F(p)$ (binomial (p)-$F(p)$) and that of X will be Poisson (λ)-$G(\lambda)$. The question arises then whether the distribution of Y can be characterized from the form of the distribution of X. This will simplify the use of the model. It will also provide many alternative models depending on the form of $F(p)$ and $G(\lambda)$. The following theorems look into the problem.

THE MAIN RESULTS

Theorem 1. Suppose that for the non-negative, integer-valued r.v.'s X,Y we have that

$$P(Y=r|X=n) = \int_0^1 \binom{n}{r} p^r q^{n-r} dF(p) \qquad r=0,1,... \tag{1}$$

(i.e., binomial (p)∼F(p)). Then Y is Poisson (λp)∼F(p) iff X is Poisson (λ).

Proof: The "if" part is straightforward. For the "only if" part suppose that Y is Poisson (λp)∼F(p), i.e., that

$$P(Y=r) = \int_0^1 \exp(-\lambda p) \frac{(\lambda p)^r}{r!} dF(p) \qquad r=0,1,... . \tag{2}$$

On the other hand we have

$$P(Y=r) = \sum_{n=r}^{\infty} P(X=n)P(Y=r|X=n) = \sum_{n=r}^{\infty} P(X=n) \int_0^1 \binom{n}{r} p^r q^{n-r} dF(p) \quad r=0,1,... \tag{3}$$

The x-th factorial moments (f.m.'s) of the distributions given by (2) and (3) are, respectively

$$\mu_{(x)}(r) = \int_0^1 \sum_{r=x}^{\infty} \exp(-\lambda p)\frac{(\lambda p)^r}{r!} r(r-1)...(r-x+1)dF(p) = \int_0^1 (\lambda p)^x dF(p) \tag{4}$$

and

$$\mu_{(x)}(r) = \sum_{r=x}^{\infty} r(r-1)...(r-x+1) \int_0^1 \sum_{n=r}^{\infty} P(X=n)\binom{n}{r} p^r q^{n-r} dF(p)$$

$$= \int_0^1 \sum_{n=r}^{\infty} \left[\sum_{r=x}^{n} r(r-1)...(r-x+1)\binom{n}{r} p^r q^{n-r}\right] P(X=n)dF(p)$$

$$= \int_0^1 \left[\sum_{n=x}^{\infty} n(n-1)...(n-x+1)p^x\right] P(X=n)dF(p)$$

$$= \mu_{(x)}(n) \int_0^1 p^x dF(p). \tag{5}$$

(where $\mu_{(x)}(n)$ denotes the x-th f.m. of P(X=n)).

Combining (2),(3),(4),(5) we find that

$$\mu_{(x)}(n) = \lambda^x \qquad x=1,2,... .$$

Consequently, the f.m.'s of the distribution of the r.v. X are the same as those corresponding to the Poisson distribution. This implies that the r.v. X has the same moments as a Poisson variable.

Since the Poisson distribution is uniquely determined by its moments the result follows.

Theorem 2. Let X,Y be as in theorem 1. In addition, assume that the distribution of X is determined uniquely by its f.m.'s and that

$$\int_0^\infty \lambda^x dG(\lambda) < \infty \text{ for } \lambda > 0,\ x=0,1,\dots .$$

Then Y is Poisson (λp)∧G(λ)∧F(p) iff X is Poisson (λ)∧G(λ).

Proof: The "if" part of the proof is straight forward. As far as the "only if" part is concerned, by following the argument of the proof of theorem 1 we have that the x-th f.m. of the distribution of X is given by

$$\mu_{(x)}(n) = \int_0^\infty \lambda^x dG(\lambda) \qquad (6)$$

It is now known that the x-th f.m. of the CPD is of the form (6). Since we have assumed that the distribution of X is uniquely determined by its f.m.'s the result follows.

REMARKS

1. It can be observed that if G(λ) is degenerate, theorem 2 reduces to theorem 1.

2. An interesting problem concerning theorem 2 is that of relaxing the condition that the distribution of X is uniquely determined by its f.m.'s.

3. It is clear that for different forms of F(p) and G(λ) theorems 1 and 2 provide characterizations for different forms of CPD's. If, for example, F(p)~beta(a,b) and G(λ)~gamma(a+b,m) then, by theorem 2,Y~negative binomial(a,m/m+1) iff X~negative binomial(a+b,m/m+1

4. The model with Y|X~compound binomial can also be viewed as an extension of the damage model considered by Rao (1963). In this model X represents an original observation produced by a natural process (e.g., number of eggs). Y|X is the damage process and Y is the resulting observation. Some of the aspects regarding an extension of the damage model have been examined by one of the authors (Panaretos (1977)).

5. In the damage model set-up Rao (1963) observed that, with binomial damage, Y is negative binomial iff X is negative binomial.- Remark 3 indicates that this property of the damage model is preserved if one allows the parameter p of the binomial damage process to vary according to a beta law.

6. The CPD which results from compounding the Poisson by a gamma distribution was first adopted by Greenwood and Woods (1919) under the assumption that the accident experience of each individual was Poisson with mean value λ varying from individual to individual.

This led to the introduction of accident proneness. The results of theorems 1 and 2 may be of some interest in this direction of accident theory, especially in connection with actuarial studies. In this context X will denote the number of accidents incurred and Y will be the number of reported accidents. Here, one is justified to assume that each accident is reported with probability p which varies from accident to accident. (Social, legal and financial pressures may encourage a tendency towards the underreporting of individual accident involvement). Consequently, the model with $Y|X$ following a CBD might be appropriate. In this case, let the distribution of p be known and assume that there is evidence to suggest that Y is compound Poisson distributed. Then, by theorems 1 and 2, testing for a proneness factor in the distribution of X will be equivalent to testing for the form of the compounding process in the distribution of Y.

7. Going back to the accident situation examined by Leiter and Hamdan (1973) one may ask whether any of the compound models provided by theorems 1 and 2 can be employed to give a satisfactory answer. The present authors, currently investigating these possibilities, are in a position to claim that one of the models (namely the one with $F(p)$ degenerate and $G(\lambda)$~Gamma) provides a satisfactory fit.

REFERENCES

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RÉSUMÉ

Les distributions Poissoniennes composites (DPC) ont été adoptées largement pour la résolution des problèmes où la distribution Poissonienne simple n'est pas été trouvée adéquate à utiliser. Dans cet article on démontre une propriété intéressante que la DPC possède à la forme des deux charactérisations. La supposition basilaire est que la distribution conditionelle des deux variables aléatoires a la forme d'une distribution binomiale composite. Les résultats obtenus sont examinés au cas d'un problème dans la théorie des accidents. On considère enfin des autres applications possibles.