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A Simulation Study on the Performance of Extreme-Value Index Estimators and Proposed Robustifying Modifications

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***Abstract*- The key issue of extreme-value theory is the estimation of a parameter γ, known as extreme-value index. In this paper we review several extreme-value index estimators, ranging from the oldest ones to the most recent developments. Moreover, a smoothing procedure of these estimators are presented. A simulation study is conducted in order to compare the behaviour of the estimators and their smoothed alternatives. Maybe the most prominent result of this study is that no uniformly best estimator exist and that the behaviour of estimators depends on the value of the parameter γ itself.**

***Index Terms*- extreme value index; semi-parametric estimation; smoothing modification.**

I. Introduction

Extreme value theory is an issue of major importance in many fields of application where extreme values may appear and have detrimental effects. Such fields range from hydrology (Smith [31], Davison and Smith [8], Coles and Tawn [5], Barão and Tawn [2]) to insurance (Beirlant et al. [3], Mikosch [25], McNeil [22], Rootzen and Tajvidi [29]) and finance (Danielsson and de Vries [6], McNeil [23] and [24], Embrechts et al. [13],[14], Embrechts [12]). Actually, extreme value theory is a blend of a variety of applications and sophisticated mathematical results on point processes and regular varying functions.

Actually, extreme value theory is a blend of a variety of applications and sophisticated mathematical results on point processes and regular varying functions.

The cornerstone of extreme value theory is Fisher-Tippet's theorem for limit laws for maxima (Fisher and Tippet, [16]). According to this. if the maximum value of a distribution function (d.f.) tends (in distribution) to a non-degenerate d.f. then this limiting d.f. can ***only*** be the Generalized Extreme Value (GEV) distribution:

$$H_\theta(x) = H_{\gamma;\mu,\sigma}(x) = \exp\left\{-\left(1+\gamma\frac{x-\mu}{\sigma}\right)^{-1/\gamma}\right\},$$

$$1+\gamma\frac{x-\mu}{\sigma} > 0,\ \theta=(\gamma,\mu,\sigma)\in\Re\times\Re\times\Re_-.$$

In this paper we deal with the estimation of the parameter (extreme-value index) γ. Particularly, in section 2 several existing estimators for γ are presented, while in section 3 a smoothing method on specific estimators is given and extended to other estimators, too. A simulation comparison of the presented extreme-value index estimators along with their smoothing alternatives is analytically described in section 4. Finally, concluding remarks can be found in section 5.

II. Semi-Parametric Extreme-Value Index Estimation

The most popular estimation approach in the context of extreme value analysis is the so-called 'Maximum Domain of Attraction Approach' (Embrechts et al., [13]), or Non-Parametric. In the present context we prefer the term '*semi-parametric*' since this term reflects the fact that we make only partly assumptions about the unknown d.f. F.

Here we are interested in the distribution of the maximum (or minimum) value. According to the Fisher-Tippet theorem, the limiting d.f. of the (normalized) maximum value (if it exists) is the GEV d.f. $H_\theta = H_{\gamma;\mu,\sigma}$. So, without making any assumptions about the unknown d.f. F (apart from some continuity conditions), extreme-value theory provides us with a fairly sufficient tool for describing the behaviour of extremes of the distribution that the data in hand stem from. The only issue that remains to be resolved is the estimation of the parameters of the GEV d.f. $\theta=(\gamma,\mu,\sigma)$.

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The procedure followed in practice is that we assume that the asymptotic approximation is achieved for the largest k observations (where k is large but not as large as the sample size n), which we subsequently use for the estimation of the parameters. However, the choice of k is not an easy task. On the contrary, it is a very controversial issue. In this section, we give the most prominent answers to the issue of parameter estimation. We mainly concentrate to the estimation of the *shape parameter* γ (also called *tail index* or *extreme-value index*), since this is the parameter that determines the behaviour of extremes. We describe the most well-known proposals, ranging from the first contributions, of 1975, in the area to very recent modifications and new developments.

Pickands estimator (Pickands, [26]) is the first suggested estimator for the parameter $\gamma \in \Re$ of GEV d.f and is given by the formula $\hat{\gamma}_P = \frac{1}{\ln 2} \ln\left(\frac{X_{(k/4):n} - X_{(k/2):n}}{X_{(k/2):n} - X_{k:n}} \right)$, where $X_{1n} \geq X_{2n} \geq \ldots \geq X_{nn}$ are the descending order statistics of the corresponding sample of observations. A particular characteristic of Pickands estimator is the fact that the largest observation is not explicitly used in the estimation. One can argue that this makes sense since the largest observation may add too much uncertainty. The properties of Pickands estimator were mainly explored by Dekkers and de Haan [10], who proved, under certain conditions, weak and strong consistency, as well as asymptotic normality.

However, the most popular tail index estimator is the **Hill estimator**, (Hill, [18]), which though is restricted to the case $\gamma > 0$. Hill estimator is provided by the formula

$$\hat{\gamma}_H = \frac{1}{k} \sum_{i=1}^{k} \ln X_{i:n} - \ln X_{k+1:n}.$$

Weak and strong consistency as well as asymptotic normality of Hill estimator hold under the assumption of i.i.d. data (Embrechts et al., [13]). Though the Hill estimator has the apparent disadvantage that is restricted to the case $\gamma>0$, it has been widely used in practice and extensively studied by statisticians. Its popularity is partly due to its simplicity and partly due to the fact that in most of the cases where extreme-value analysis is called for, we have long-tailed d.f.'s (i.e. $\gamma>0$).

The popularity of Hill estimator made a tempting problem to try to extend the Hill estimator to the general case $\gamma \in \Re$. Such an attempt, led Beirlant et al. [4] to the so-called **adapted Hill estimator**, which is applicable for any γ in the range of real numbers:

$$\hat{\gamma}_{adH} = \frac{1}{k} \sum_{i=1}^{k} \ln(U_i) - \ln(U_{k+1}), \text{ where}$$

$$U_i = X_{(i+1):n} \left(\frac{1}{i} \sum_{j=1}^{i} \ln X_{j:n} - \ln X_{(i+1):n} \right).$$

Another estimator that can be considered as an adaptation of Hill estimator, in order to obtain consistency for all $\gamma \in \Re$, has been proposed by Dekkers et al. [10],[11]. This is the **Moment estimator**, given by

$$\hat{\gamma}_M = M_1 + 1 - \frac{1}{2}\left(1 - \frac{(M_1)^2}{M_2} \right)^{-1}, \text{ where}$$

$$M_j = \frac{1}{k} \sum_{i=1}^{k} \left(\ln X_{i:n} - \ln X_{(k+1):n} \right)^j, \; j=1,2.$$

Weak and strong consistency, as well as asymptotic normality of the Moment estimator have been proven by Dekkers et al. [10],[11].

One of the approaches concerning Hill's derivation is the 'QQ-plot' approach (Beirlant et al., [4]). A more precise estimator, under this approach, has been suggested by Kratz and Resnick [20], who derived the following estimator of γ:

$$\hat{\gamma}_{qq} = \frac{\sum_{i=1}^{k} \ln \frac{i}{k+1} \left\{ \sum_{j=1}^{k} \ln X_{j:n} - k \ln X_{i:n} \right\}}{k \sum_{i=1}^{k} \left(\ln \frac{i}{k+1} \right)^2 - \left(\sum_{i=1}^{k} \ln \frac{i}{k+1} \right)^2}.$$

The authors proved weak consistency and asymptotic normality **of QQ-estimator** (under conditions similar to the ones imposed for the Hill estimator). However, the asymptotic variance of qq-estimator is twice the asymptotic variance of Hill estimator, while similar conclusions are drawn from simulations of small samples.

Concentrating on cases where $\gamma > 0$, the main disadvantage of Hill estimator is that it can be severely biased, depending on the 2^{nd} order behaviour of the underlying d.f. F. Based on an asymptotic 2^{nd} order expansion of the d.f. F, Danielsson et al. [7] proposed the **Moments**

Ratio estimator : $\hat{\gamma}_{MR} = \frac{1}{2} \cdot \frac{M_2}{M_1}$. They proved that $\hat{\gamma}_{MR}$ has lower asymptotic square bias than the Hill estimator (when evaluated at the same threshold, i.e. for the same k), though the convergence rates are the same.

An estimator related to the Moment estimator $\hat{\gamma}_M$ is **Peng's estimator**, suggested by Deheuvels et al. [9]:

$$\hat{\gamma}_L = \frac{M_2}{2M_1} + 1 - \frac{1}{2}\left(1 - \frac{(M_1)^2}{M_2}\right)^{-1}.$$

This estimator has been designed to somewhat reduce the bias of the moment estimator. Another related estimator suggested by the same authors is the **W estimator**:

$$\hat{\gamma}_W = 1 - \frac{1}{2}\left(1 - \frac{(L_1)^2}{L_2}\right)^{-1}, \text{ where}$$

$$L_j \equiv \frac{1}{k}\sum_{i=1}^{k}\left(X_{i:n} - X_{(k+1):n}\right)^j, \text{ j=1, 2.}$$

As Deheuvels et al. [9] mentioned, $\hat{\gamma}_L$ is consistent for any $\gamma \in \Re$ (under the usual conditions), while $\hat{\gamma}_W$ is consistent only for $\gamma < 1/2$. Moreover, under appropriate conditions on F and $k(n)$, $\hat{\gamma}_L$ is asymptotically normal. Normality holds for $\hat{\gamma}_W$ only for $\gamma < 1/4$.

The aforementioned estimators share some common desirable properties, such as weak consistency and asymptotic normality (though these properties may hold under slightly different, unverifiable in any case, conditions). On the other hand, simulation studies or applications on real data can end up in large differences among these estimators. In any case, there is no 'uniformly best' estimator. Of course, Hill, Pickands and Moment estimators are the most popular ones. This could be partly due to the fact that they are the oldest ones. Actually, most of the rest have been introduced as alternatives to Hill, Pickands or Moment estimator and some of them have been proven to be superior in some cases only. In the literature, there are several comparison studies of extreme-value index estimators (either theoretically or via Monte-Carlo methods), such as Deheuvels et al. [9] and Rosen and Weissman [30]. Still, these studies are confined to a small number of estimators.

III. Smoothing Procedures for Semi-Parametric Extreme-Value Index Estimators

One of the most serious objections one could raise against the aforementioned semi-parametric estimators is their sensitivity towards the choice of k (number of upper order statistics used in the estimation). The well-known phenomenon of bias-variance trade off turns out to be unresolved, and choosing k seems to be more of an art than a science. In this paper we present an approach aiming to confront this issue.

An exploratory way to subjectively choose the number k is based on the plot of the estimator $\hat{\gamma}(k)$ versus k. A stable region of the plot indicates a valid value for the estimator. The search for a stable region in the plot is a standard but problematic and ill-defined practice. The need for a stable region results from adapting theoretical limit theorems which are proved subject to the conditions that $k(n) \to \infty$ but also $k(n)/n \to 0$. But, since extreme events by definition are rare, there is only little data (few observations) that can be utilised and this inevitably involves an added large statistical uncertainty. A possible solution would be to smooth 'somehow' the estimates with respect to the choice of k (i.e. make it more insensitive to the choice of k), leading to a more stable plot and a more reliable estimate of γ. Such a method was proposed by Resnick and Stărică [27], [28] for smoothing Hill and Moment estimators, respectively.

Resnick and Stărică [27] proposed a simple averaging technique that reduces the volatility of the Hill-plot. The smoothing procedure consists of averaging the Hill estimator values corresponding to different values of order statistics p. The formula of the proposed averaged-Hill estimator is :

$$av\hat{\gamma}_H(k) = \frac{1}{k - [ku]}\sum_{p=[ku]+1}^{k}\hat{\gamma}_H(p),$$

where u<1, and [x] the smallest integer greater than or equal to x.

The authors proved that through averaging, the variance of the Hill estimator can be considerably reduced and the volatility of the plot tamed. The smoothing techniques make no (additional) unrealistic or uncheckable assumptions and are always available to complement the Hill plot. Obviously, when considerable bias is present, the averaging technique offers no improvement. They derived the adequacy (consistency and asymptotic normality) of the averaged-Hill estimator, as well as its improvement over Hill estimator (smaller asymptotic variance). Since the asymptotic variance is a decreasing function of u, one would like to choose u as big as possible to ensure the maximum decrease of the variance. However, the choice of u is limited by the sample size. Due to the averaging, the larger the u, the fewer the points one gets on the plot of averaged Hill. Therefore, an equilibrium should be reached between variance reduction and a comfortable number of points on the plot.

Resnick and Stărică [28] also applied their idea of smoothing to the more general Moment estimator $\hat{\gamma}_M$. The formula of the proposed averaged-moment estimator is :

$$av\hat{\gamma}_M(k) = \frac{1}{k-[ku]}\sum_{p=[ku]+1}^{k}\hat{\gamma}_M(p) \cdot$$

In practice, the authors suggest to take u=0.3 or u=0.5 depending on the sample size (the smaller the sample size the larger u should be).

In this case the consequent reduction in asymptotic variance is not so profound. The authors actually showed that through averaging (using the above formula), the variance of the moment estimator can be considerably reduced only in the case $\gamma < 0$. In the case $\gamma > 0$ the simple moment estimator turns out to be superior. For $\gamma \approx 0$ the two moment estimators (simple and averaged) are almost equivalent. These conclusions hold asymptotically, and have been shown via a graphical comparison.

IV. Simulation Comparison of Extreme-Value Index Estimators

A. Details of Simulation Study

In this section, we try to investigate and compare, via Monte Carlo methods, the performance of the extreme-value index estimators introduced, as well as the performance of the modifications suggested previously. Apart from the standard form of estimators, we apply to all these the averaging procedure presented in section III. Resnick and Stărică [27], [28] suggested (and proved the adequacy and good properties of) this procedure only in the context of Hill and moment estimator. We apply the procedure to other extreme-value index estimators, so as to empirically evaluate its effect on these estimators. In addition, apart from these mean-averaged estimators, we apply analogously a median-averaging procedure to our estimators.

The distributions used in the simulation study, range from distributions with finite upper endpoint ($\gamma<0$, such as in Beta, Uniform) to long-tailed distributions ($\gamma=0$, such as Gamma, Normal, Log-Normal, Exponential and Weibull or $\gamma>0$, such as Burr, Fréchet, Log-Gamma, Log-Logistic, Pareto). From each of these distributions, 1000 samples were generated of moderate size (n=100) and 500 samples of large size (n=1000), based on which the performance of the estimators is examined. In our study, the performance of any estimator of γ, is evaluated in terms of the bias, standard error and root mean square error of the estimator based on k upper order statistics (where k ranges from 1 up to sample size). The root mean square error (rmse), being a combination of standard deviation and bias, is essentially the basis for comparisons of estimators.

B. Discussion of Simulation Results

Before proceeding to the discussion of the results, it should be noted that the performance of the estimators did not seem to remain stable for data stemming from different distributions. For this reason, in the sequel we provide the main findings of the simulation study distinguishing for each different class of distributions.

- For $\gamma>0$

For large sample sizes Moment-Ratio seems to be the most preferable estimator. It is usually the best estimator, in terms of minimum rmse. Even in cases that other estimators outperform it, it is one of the bests, while in no case does it display very unsatisfactory performance. It is interesting to note that the W estimator tends to be appropriate for distributions with extreme-value index γ larger than 1, though for smaller values of γ its performance can be very unsatisfactory. So, it may be risky to use this estimator, since in real-life applications the value of γ is unknown. For small samples (in our case n=100), Hill estimator turns out to be the best choice, while Moment-Ratio and Moment estimators can also be regarded as safe options. Among averaging procedures, only the mean averaging of Pickands estimator is effective. However, the improvement is not large enough to out-beat the other standard estimators.

- For $\gamma=0$

This class contains a wide range of distributions. Consequently there is not a uniformly superior estimator. However, by examining more carefully the simulation results one could deduce that Peng's is the most preferable estimator of the extreme-value index. Moment and (surprisingly) Moment-Ratio also display an adequate behaviour. The usefulness of averaging procedures in these cases should also be stressed out. These procedures have an obvious profitable impact on Pickands estimator, so that mean-averaged Pickands estimator can also be regarded as an adequate estimator of γ.

- For $\gamma<0$

This class contains upper-bounded distributions. Though the shape of distributions differs a lot from the distributions with $\gamma=0$, the behaviour of extreme-value index estimators in these two classes of d.f.'s displays great analogies. Here, Moment and Peng's estimators are undeniably the most preferable estimators and the beneficial effect of both mean and median averaging procedures is even more evident. Moreover, as we deviate from zero (and positive) values of γ, the inadequacy of estimators such as Hill, Moment-Ratio and so on, is more clear.

V. Discussion

The comparison (via simulation) of semi-parametric estimation methods for the extreme-value index and some smoothing alternatives has been the central issue of this paper. The simulation study conducted led to some very interesting results. The first is that, as one could naturally expect, the performance of estimators on a specific data-set depends on the distribution of the data. So, there is not a uniformly best estimator. Nevertheless, by looking more carefully at the results, some general conclusions may be reached. More specifically, in cases of long-tailed data (with an infinite upper end-point) Moment and Moment-Ratio estimators seem to estimate more satisfactorily the non-negative extreme-value index γ. However, when it comes to upper-bounded distributions (characterized by a negative value of γ) Peng's and Moment estimators are more preferable. As far as the impact of smoothing (averaging) procedures is concerned, we deduce that it is effective (improving the performance of standard estimators) in cases where the true value of extreme-value index is non-positive. Particularly, mean-averaging procedures improve greatly the performance of Pickands estimator (in case of zero γ), while median-averaging of Moment and Peng's also leads to improved estimators (for γ negative).

The dependence of estimators' performance on the distribution of data in hand can be alternatively seen as dependence on the true value of the index itself. So, before proceeding to the use of any estimation formula it would be useful if we could get an idea about the range of values where the true γ lies in. This can be achieved graphically via QQ and mean excess plots. Alternatively, there exist statistical tests which tests such hypothesis. See, for example, Hosking [19], Hasofer and Wang [17], Alves and Gomes [1] and Marohn [21].

Moreover, it should be pointed out that among the averaged estimators used in the simulation study only the mean-averaged Hill and Moment estimators have been theoretically explored by Resnick and Stărică [27], [28]. As we have seen, the median averaging procedure has also displayed some interesting effectiveness, implying that it may be worthy to be also studied theoretically (with special emphasis on Moment and Peng's estimators). The same holds for the mean-averaged Pickands estimator.

VI. References

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