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SOME RESULTS RELATED TO THE RAO–RUBIN CHARACTERIZATION OF THE POISSON DISTRIBUTION[1]

D. N. SHANBHAG
University of Sheffield

AND

J. PANARETOS
University of Bradford

1. Introduction

Let (X, Y) be a random vector of non-negative integer-valued components such that $P(X=n, Y=r)=g_nS(r\mid n)$, $r=0, 1, \ldots, n$; $n=0, 1, \ldots$, where $\{g_n: n=0, 1, \ldots\}$ and $\{S(r\mid n): r=0, 1, \ldots, n\}$ for each $n\geq 0$ are discrete probability distributions. Moran (1952) proved that if Y and $X-Y$ are non-degenerate and independent and there exists at least one integer i such that $P(Y=i)>0$, $P(X-Y=i)>0$, then for every n with $g_n>0$ the distribution $\{S(r\mid n)$: $r=0, 1, \ldots, n\}$ is binomial corresponding to n trials if and only if (iff) $\{g_n\}$ is a Poisson distribution (i.e. iff Y and $X-Y$ are Poisson random variables). (Note that what we have stated is a corrected version of Moran's theorem in a different notation; the theorem as stated by Moran without a specification concerning the index parameter of the binomial distribution is incorrect.) More recently, Rao and Rubin (1964) have shown that if $g_0<1$ and for every n with $g_n>0$

$$S(r\mid n)=\binom{n}{r}p^r(1-p)^{n-r}, \quad r=0, 1, \ldots, n, \tag{1.1}$$

with p as a fixed number lying in $(0, 1)$, then

$$P(Y=r)=P(Y=r\mid X=Y)=P(Y=r\mid X>Y), r=0, 1, \ldots \tag{1.2}$$

iff $\{g_n\}$ is a Poisson distribution. This latter result has been extended to a very general situation by Shanbhag (1977). Roughly speaking, he has shown that if $g_0<1$ and for every n with $g_n>0$

$$S(r\mid n)=\frac{a_rb_{n-r}}{c_n}, \quad r=0, 1, \ldots, n, \tag{1.3}$$

where $a_n>0$ for all $n\geq 0$, $b_0, b_1>0$ and $b_n\geq 0$, $n\geq 2$, then (1.2) is valid if and only if $g_n/c_n=(g_0/c_0)\alpha^n$, $n=1, 2, \ldots$ for some $\alpha>0$. As a by-product of Shanbhag's result, it follows that if (1.3) is satisfied, then the condition (1.2) is valid iff Y and $X-Y$ are independent. Noting that (1.1) is of the form (1.3), one can see that Moran's findings confirm those of Rao and Rubin. Srivastava and Srivastava (1970), Talwalker (1970), Shanbhag and Clark (1972), Shanbhag (1974), Shanbhag and Rajamannar (1974), Srivastava and Singh (1975), Patil and Ratnaparkhi (1975) and several others have given variants and extensions of the Rao-Rubin result; some of them have also given new techniques to arrive at the previous results. Haight (1967) has reviewed some of the earlier work related to Moran's result.

In the present note, we reveal that the result of Rao and Rubin (1964) does not remain valid, if the parameter p in (1.1) is allowed to depend on n and that if (X, Y) is such that X has a Poisson marginal distribution and (1.2) is satisfied, it is not necessary that (1.1) be satisfied. This latter result contrasts with the main result of Srivastava and Srivastava (1970). The note also disproves two conjectures in the literature and gives an improved version of a result of Shanbhag and Rajamannar (1974).

2. Two Results Related to the Rao–Rubin Characterization

We shall now establish the following:

Theorem 1. *Let (X, Y) be a random vector of non-negative integer-valued components such that for each n with $P(X=n)>0$ the conditional distribution of Y given that $X=n$ is binomial corresponding to n trials (non-degenerate for all $n \geq 1$), and the Rao–Rubin condition (1.2) is satisfied. Then it is not necessary that X be Poisson distributed.*

Proof. Take (X, Y) such that

$$P(X=0)=c\left\{1+\frac{3\sqrt{e}-1}{8(\sqrt{e}-1)^2}\right\},$$

$$P(X=1)=c\left\{1+\frac{1}{2(\sqrt{e}-1)}\right\}, \quad P(X=2)=\frac{9c}{8}$$

and $P(X=n)=c/n!$, $n \geq 3$, where c is the normalizing constant, and

$$P(Y=r \mid X=n)=\binom{n}{r}\left(\frac{1}{2}\right)^n \quad 0 \leq r \leq n; \quad n \neq 2, \quad \text{and} \quad =\binom{n}{r}\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{n-r}, \quad r=0,1,2; \quad n=2.$$

It readily follows that the vector satisfies (1.2) and hence the theorem is established.

Theorem 2. *Let (X, Y) be a random vector of non-negative integer-valued components such that the marginal distribution of X is*

Poisson with mean $\lambda_0(>0)$, $P(X \geq Y)=1$ and $0<P(X=Y)<1$ and the Rao–Rubin condition (1.2) is satisfied. Then it is not necessary that the conditional distribution of Y given that $X=n$ be binomial corresponding to n trials for every n.

Proof. Take (X, Y) such that X is Poisson with mean λ_0 and $P(Y=r \mid X=n)=a_n$ if $r=0$ and $1-a_n$ if $r=1$ where $a_0=1$, $a_2=\frac{1}{3}$ and $a_r=\frac{1}{2}$, $r \neq 0, 2$ and λ_0 is a positive real number satisfying

$$\frac{1}{2}\left(1-\frac{\lambda^2}{6}+e^{\lambda}\right)=e^{\lambda}\left(1+\frac{\lambda}{2}\right)^{-1}. \tag{2.1}$$

Since both $a(\lambda)=\frac{1}{2}(1-\lambda^2/6+e^{\lambda})$ and $b(\lambda)=e^{\lambda}(1+\lambda/2)^{-1}$ are continuous functions of λ for $\lambda>0$, and $a(\lambda)<b(\lambda)$ for small $\lambda>0$ and $b(\lambda)<a(\lambda)$ for large $\lambda>0$ it follows that (2.1) has the required solution. The vector (X, Y) considered satisfies the Rao–Rubin condition and hence the theorem follows.

Remark 1. Theorem 1 implies that Moran's theorem does not remain valid if the assumption that X and Y are independent is replaced by $P(Y=0)>0$, $P(X=r)=P(X=r \mid Y=0)$, $r=0, 1, \ldots,$ and Theorem 2 implies that Theorem 2.1 of Srivastava and Srivastava does not remain valid if their parameter λ is considered fixed.

3. Two Counter-Examples

Consider the random vector (X, Y) of non-negative integer-valued components such that $P(X \leq n_1+n_2)=1$ and for every n with $P(X=n)>0$

$$P(Y=r \mid X=n)=\frac{\binom{n_1}{r}\binom{n_2}{n-r}}{\binom{n_1+n_2}{n}}, \quad r=0, 1, \ldots, n, \tag{3.1}$$

where n_1 and n_2 are positive integers and $\binom{k}{m}$ is defined to equal zero if $m>k$. It is clear that if the marginal distribution of X is binomial (n_1+n_2, p) for some $0<p<1$, then Y and $X-Y$ are independent and hence (1.2) is satisfied. The following example shows that if a vector (X, Y) satisfies (1.2) and (3.1) then it is not necessary that X be binomial. This answers a question raised by Patil and Ratnaparkhi (1975).

Example 1. Let (X, Y) be such that for some positive integer $m(\geq 3)$ we have $P(X=0)=P(X=1)=m^2/(2m^2+m-1)$ and $P(X=m)=(m-1)/(2m^2+m-1)$ and (3.1) is satisfied with $n_1=1$ and $n_2=m-1$. Observe that the vector satisfies the Rao–Rubin condition (1.2).

The reader may note that even though the conditional distribution in (3.1) is of the form $\{a_r b_{n-r}/c_n;\ r=0,1,\ldots,n\}$ the present example is not covered by Shanbhag's extension of the Rao–Rubin result. This is, particularly, so because in the present case Shanbhag's requirement that $a_r>0$ for every non-negative integer r is not met.

Consider now X as a random variable distributed with a power-series Poisson distribution truncated at $c-1$ with parameter λ, and Y a random variable such that $P(Y)=r \mid X=n)=S(r \mid n)$, $r=0,\ 1,\ldots,n$; $n=c,\ c+1,\ldots,$ where $\{S(r|n): r=0,\ldots,n\}$ is a discrete probability distribution with the terms independent of λ at least for $r\geqslant c$ for each n. Srivastava and Singh (1975) have conjectured that if (for all values of λ)

$$(3.2)\quad P(Y=r \mid Y\geqslant c)=P(Y=r \mid X=Y)=P(Y=r \mid X>Y,\ Y\geqslant c),\qquad r=c,c+1,\ldots$$

then every $\{S(r \mid n): r=0,1,\ldots,n\}$ has terms for $r\geqslant c$ to be the same as those of the binomial distribution (n,p) for some $0<p<1$. The following example shows that this conjecture is false.

Example 2. Let (X,Y) be a random vector of non-negative integer-valued components such that $P(X=n)=(\lambda^n/n!)/(e^\lambda-1)$, $n=1,2,\ldots$ and for each $n\geqslant 1$, $P(Y=r \mid X=n)=\binom{n}{r}(\frac{1}{2})^n$ if $r=1,2,\ldots,n-1$ and $=(\frac{1}{2})^{n-1}$ if $r=n$. Observe that $P(Y=r \mid Y\geqslant 1)=P(Y=r \mid X=Y)=P(Y=r \mid X>Y,\ Y\geqslant 1)$ $r=1,2,\ldots$ but in this case the conditional distribution of Y given $X=n$ is not of the form conjectured by Srivastava and Singh (1975).

4. An Improved Version of a Result of Shanbhag and Rajamannar

The following theorem is an improved version of a result of Shanbhag and Rajamannar.

Theorem 3. *Let* $(X_1,X_2;Y_1,Y_2)(=(X_1^{(\theta_1,\theta_2)},X_2^{(\theta_1,\theta_2)};\ Y_1^{(\theta_1,\theta_2)},Y_2^{(\theta_1,\theta_2)})$ *where* $0<\theta_1<a_1, 0<\theta_2<a_2)$ *be a random vector such that* X_1 *and* X_2 *are non-negative integer-valued and non-degenerate random variables with the power series joint distribution* $\{P_{uv}\}$, *where*

$$(4.1)\qquad P_{uv}=\frac{a_{uv}\theta_1^u\theta_2^v}{A(\theta_1,\theta_2)}\quad (u,v=0,1,\ldots),$$

and Y_1, Y_2 *are two non-negative random variables such that for every* r,s *with* $a_{rs}>0$ *we have* $P(Y_1=r,\ Y_2=s \mid X_1=r, X_2=s)=g(r,s)$.

Further let $E(Y_i)=\pi_i E(X_i)$, $i=1,2$ *and* $E(Y_i(Y_i-1))=\pi_i^2 E\{X_i(X_i-1)\}$, $i=1,2$ *where* $0<\pi_i<1$, $i=1,2$. *Take* π_1, π_2 *and* $g(r,s)$ *to be independent of* (θ_1,θ_2) *and* $0<P(X_1=Y_1, X_2=Y_2)<1$. *Then if* π_1 *and* π_2 *are distinct or* $g(r,s)$ *is of the form* $g_1(r)g_2(s)$, *then* *(for every* (θ_1, θ_2) *and) for* $i = 1, 2$

$$E(Y_i) = E(Y_i \mid X_1 = Y_1, X_2 = Y_2), \qquad \text{var}(Y_i) = \text{var}(Y_i \mid X_1 = Y_1, X_2 = Y_2) \tag{4.2}$$

iff $\{P_{uv}\}$ *is the joint distribution of independent Poisson random variables and* $g(r, s) = g(0, 0)\pi_1^r \pi_2^s$, $r, s > 0$.

Proof. The 'if' part of the Theorem is obvious. In what follows we shall establish the 'only if' part.

Repeating the steps in Shanbhag and Clark (1972) with obvious changes, we see that the first set of equations in (4.2) implies

$$A^*(\theta_1, \theta_2) = K_1(\theta_2)(A(\theta_1, \theta_2))^{\pi_1} = K_2(\theta_1)(A(\theta_1, \theta_2))^{\pi_2}, \tag{4.3}$$

where $A^*(\theta_1, \theta_2) = \sum_{\{u, v : a_{uv} > 0\}} g(u, v) a_{uv} \theta_1^u \theta_2^v$ and $K_1(\theta_2)$ and $K_2(\theta_1)$ are certain functions of θ_2 and θ_1 only. Further it follows that the validity of the whole of (4.2) implies

$$A(\theta_1, \theta_2) = \exp\{c_1^{(1)}(\theta_2) + c_2^{(1)}(\theta_2)\theta_1\} = \exp\{c_1^{(2)}(\theta_1) + c_2^{(2)}(\theta_1)\theta_2\}, \tag{4.4}$$

where $c_1^{(1)}(\theta_2)$ and $c_2^{(1)}(\theta_2)$ are certain functions of θ_2 only and $c_1^{(2)}(\theta_1)$ and $c_2^{(2)}(\theta_1)$ are certain functions of θ_1 only. (4.4) cannot be valid unless

$$A(\theta_1, \theta_2) = \exp\{c_0 + c_1\theta_1 + c_2\theta_2 + c_3\theta_1\theta_2\} \tag{4.5}$$

for some c_0, c_1, c_2 and c_3. It also follows that if (4.5) is valid, then (4.3) cannot be valid unless $\pi_1 = \pi_2$ or $c_3 = 0$. If $c_3 = 0$ for this to happen we should have

$$A(\theta_1, \theta_2) = \exp\{c_0 + c_1\theta_1 + c_2\theta_2\}, \quad A^*(\theta_1, \theta_2) = \exp\{c_0' + c_1\pi_1\theta_1 + c_2\pi_2\theta_2\} \tag{4.6}$$

with $c_1, c_2 > 0$ and c_0' as some constant and hence we should have X_1, X_2 to be independent Poisson and $g(r, s) = g(0, 0)\pi_1^r\pi_2^s$, $r, s \geq 0$. Further, if $\pi_1 = \pi_2 = \pi$, the functions $K_1(\theta_2)$ and $K_2(\theta_1)$ should be independent of θ_2 and θ_1 respectively. Since under this situation, we take $g(r, s) = g_1(r)g_2(s)$, equating the coefficients of $\theta_1^r\theta_2^0$ and also $\theta_1^0\theta_2^s$ in (4.3) under the validity of (4.5), we observe that $g_1(r) = g_1(0)\pi^r$, $r \geq 0$, and $g_2(s) = g_2(0)\pi^s$, $s \geq 0$, and hence $g(r, s) = g(0, 0)\pi^{r+s}$. This cannot happen unless $c_3 = 0$. Because of what we observed earlier the theorem follows.

Remark 2. If for $i = 1, 2$ we have $E(Y_i \mid X_i) = \pi_i X_i$ and $\text{var}(Y_i \mid X_i) = X_i\pi_i(1 - \pi_i)$, then follows immediately that $E(Y_i) = \pi_i E(X_i)$ and $E\{Y_i(Y_i - 1)\} = \pi_i^2 E\{X_i(X_i - 1)\}$. However if we take the first set of conditions, then it is clear that the theorem is valid with $g(0, 0) = 1$. (Note that we do not assume here $P(Y_i \leq X_i, i = 1, 2) = 1$.) In the form we have presented it, the theorem does not require $E(Y_i \mid X_i)$ and $\text{var}(Y_i \mid X_i)$ to be independent of (θ_1, θ_2).

Remark 3. From the proof of Theorem 1 of Shanbhag and Rajamannar it is clear that they take $g_i(n_i) = g_i(n_i, n_i) = P(Y_i = n_i \mid X_i = n_i)$, $i = 1, 2$. The printed version of the paper has inadvertently missed specifying the notation.

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