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A CHARACTERIZATION OF A GENERAL CLASS OF  
MULTIVARIATE DISCRETE DISTRIBUTIONS

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1. INTRODUCTION

Let  $X, Y$  be two non-negative integer-valued random variables (r.v.'s). RAO and RUBIN [1] have shown that if the conditional distribution of  $Y|X$  is binomial with parameters  $n$  and  $p$ , i.e.

$$(1.1) \quad P(Y = r | X = n) = \binom{n}{r} p^r q^{n-r} \\ (r=0, 1, \dots, n; n=0, 1, \dots)$$

with  $p$  a fixed number lying in  $(0, 1)$ ,  $q=1-p$ , then the Rao-Rubin condition (R-R condition), namely,

$$(1.2) \quad P(Y = r) = P(Y = r | X = Y) \quad (r=0, 1, \dots)$$

holds if and only if (iff) the distribution of  $X$  is Poisson. Later, other authors (e.g. TALWALKER [4], R.C. SRIVASTAVA and A.B.L. SRIVASTAVA [3]) used the R-R condition to obtain characterizations for other discrete distributions. They have also extended some of the results

to the multivariate case.

SHANBHAG [2] gave a generalization of Rao and Rubin's result using a technique existing in the renewal theory. Shanbhag's result provides most of the relevant results existing in the literature as special cases.

In Section 2 of this paper we state the result of Shanbhag. Our main result, i.e. the multivariate extension of Shanbhag's result, is presented in Section 3.

Finally, in Section 4 we illustrate our method by obtaining characterizations of some well-known multivariate discrete distributions. We also point out that an improved version of Talwalker's characterization [4] of the multiple Poisson distribution is a corollary of our main result.

## 2. SHANBHAG'S EXTENSION OF THE R-R CHARACTERIZATION

LEMMA 1 (Shanbhag [2]). Let  $\{(V_n, W_n), n=0,1,\dots\}$  be a sequence of vectors with non-negative real components such that  $V_n \neq 0$  for some  $n \geq 1$ ,  $W_1 \neq 0$ . Then

$$(2.1) \quad V_m = \sum_{n=0}^{\infty} V_{n+m} W_n \quad (m=0,1,\dots)$$

iff for some  $b > 0$

$$V_n = V_0 b^n \quad (n=1,2,\dots),$$

(2.2)

$$\sum_{n=0}^{\infty} W_n b^n = 1.$$

As a result of Lemma 1, Shanbhag obtained the following theorem.

THEOREM 1. Let  $\{(a_n, b_n) : n=0, 1, \dots\}$  be a sequence of real vectors with  $a_n > 0$  for every  $n \geq 0$  and  $b_0 > 0, b_1 > 0, b_n \geq 0$  for  $n \geq 2$ . Denote by  $\{C_n\}$  the convolution of  $\{a_n\}$  and  $\{b_n\}$ .

Let  $(X, Y)$  be a random vector of non-negative integer-valued components such that  $P(X=n) = P_n, n \geq 0$  with  $P_0 < 1$  and whenever  $P_n > 0$  we have

$$(2.3) \quad P(Y = r | X = n) = \frac{a_r b_{n-r}}{C_n} \quad (r=0, 1, \dots, n).$$

Then the R-R condition (1.2) holds iff for some  $\theta > 0$

$$(2.4) \quad \frac{P_n}{C_n} = \frac{P_0}{C_0} \theta^n \quad (n=1, 2, \dots).$$

PROOF. This follows from Lemma 1 if one defines the sequences  $V_n, W_n$  by

$$(2.5) \quad V_n = \frac{P_n}{C_n} \quad (n \geq 0), \quad W_n = b_n \sum_{n=0}^{\infty} P_n \frac{a_n}{C_n}.$$

These sequences satisfy all conditions set by Lemma 1. On the other hand it can be checked that (1.2) is equivalent to (2.1) and (2.4) to (2.2).

REMARK 1. Theorem 1 provides characterizations for many well-known discrete distributions such as the Poisson, binomial and negative binomial.

### 3. THE MULTIVARIATE EXTENSION

THEOREM 2. Let  $\{(a_{\underline{n}}, b_{\underline{n}}) : \underline{n} = (n_1, \dots, n_s), n_i = 0, 1, \dots; i=1, 2, \dots, s; s=1, 2, \dots\}$  be a sequence of real vectors such that  $a_{\underline{n}} > 0, b_{\underline{n}} \geq 0$  for every  $n_i \geq 0, i=1, 2, \dots, s$  with  $b_{\underline{0}} > 0, b_{0, \dots, 0, 1} > 0$  and

some  $b_{0,0,\dots,0,1,n_s} > 0$ , some  $b_{0,0,\dots,1,n_{s-1},n_s} > 0$ ,  
 ..., some  $b_{1,n_2,n_3,\dots,n_s} > 0$ . Define  $\{C_{\underline{n}}\}$  to be the  
 convolution of  $\{a_{\underline{n}}\}$  and  $\{b_{\underline{n}}\}$  given by  $C_{\underline{n}} = \sum_{\underline{r}=\underline{0}}^{\underline{n}} a_{\underline{r}} b_{\underline{n}-\underline{r}}$   
 where  $a_{\underline{r}} = a_{r_1,\dots,r_s}$  and  $\sum_{\underline{r}=\underline{0}}^{\underline{n}}$  denoting  $\sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \dots$   
 $\dots \sum_{r_s=0}^{n_s}$ .

Consider a random vector  $(\underline{X}, \underline{Y})$  where  $\underline{X} = (X_1, \dots, X_s)$ ,  $\underline{Y} = (Y_1, \dots, Y_s)$  with  $X_i, Y_i$  ( $i=1,2,\dots,s$ )  
 non-negative integer-valued r.v.'s such that  $P_{\underline{n}} = P(X_1 = n_1, \dots, X_s = n_s) = P_{n_1, \dots, n_s} > 0$  for some  $n_i$  and  
 for every  $i=1,2,\dots,s$  and whenever  $P_{\underline{n}} > 0$

$$(3.1) \quad P(\underline{Y} = \underline{r} | \underline{X} = \underline{n}) = \frac{a_{\underline{r}} b_{\underline{n}-\underline{r}}}{C_{\underline{n}}} \quad (r_i=0,1,\dots,n_i; i=1,2,\dots,s)$$

Also define  $X^{(j)} = (X_1, \dots, X_j)$ ,  $Y^{(j)} = (Y_1, \dots, Y_j)$   
 ( $j=2,3,\dots,s$ ) and let  $X^{(j)} > Y^{(j)}$  denote that  $(X_k = Y_k, k=1,2,\dots,j-1$  and  $X_j > Y_j)$ . Then

$$(3.2) \quad P(\underline{Y} = \underline{r}) = P(\underline{Y} = \underline{r} | \underline{X} = \underline{Y}) = P(\underline{Y} = \underline{r} | X^{(j)} > Y^{(j)}) \quad (j=2,3,\dots,s)$$

iff for some  $\theta_1, \dots, \theta_s > 0$

$$(3.3) \quad \frac{P_{\underline{n}}}{C_{\underline{n}}} = \frac{P_{\underline{0}}}{C_{\underline{0}}} \prod_{i=1}^s \theta_i^{n_i}$$

Also if (3.3) is true then  $\underline{Y}$  and  $\underline{X}-\underline{Y}$  are independent.

PROOF. If we use the notation  $x^{(0)} = y^{(0)}$  to denote  $P(\underline{Y} = \underline{r} | x^{(0)} = y^{(0)}) = P(\underline{Y} = \underline{r})$  we can see that (3.2) is equivalent to

$$(3.4) \quad P(\underline{Y} = \underline{r} | \underline{X} = \underline{y}) = \\ = P(\underline{Y} = \underline{r} | x^{(\ell-1)} = y^{(\ell-1)}) \quad (\ell=1, 2, \dots, s).$$

Now define the sequences

$$(3.5) \quad v_{n_s} = \frac{P_{r_1, r_2, \dots, r_{s-1}, n_s}}{C_{r_1, r_2, \dots, r_{s-1}, n_s}}, \\ w_{n_s} = \frac{b_{0, 0, \dots, 0, n_s} P(\underline{X} = \underline{y})}{b_0 P(x^{(s-1)} = y^{(s-1)})}$$

for fixed  $r_i > 0, i=1, 2, \dots, s-1$  and  $n_s \geq 0$ . In this case we have that for  $\ell=s$  (3.4) is equivalent to

$$\sum_{n_s=0}^{\infty} v_{n_s} + r_s w_{n_s} = v_{r_s}$$

and hence using Lemma 1 we come to

the conclusion that (3.4) holds for  $\ell=s$  iff

$$\frac{P_{r_1, \dots, r_{s-1}, n_s}}{C_{r_1, \dots, r_{s-1}, n_s}} = \frac{P_{r_1, \dots, r_{s-1}, 0}}{C_{r_1, \dots, r_{s-1}, 0}} \theta_s^{n_s}$$

for some  $\theta_s > 0$ ,

every  $n_s > 0$  and every  $r_i > 0 (i=1, 2, \dots, s-1)$  (since  $r_i$  were fixed but arbitrary). Consequently (3.4) for  $\ell=s$  holds iff

$$(3.6) \quad \frac{P_{\underline{n}}}{C_{\underline{n}}} = \frac{P_{n_1, \dots, n_{s-1}, 0}}{C_{n_1, \dots, n_{s-1}, 0}} \theta_s^{n_s} \quad \text{for some } \theta > 0 \text{ and}$$

every  $n_i > 0$  ( $i=1,2,\dots,s$ ). It can also be verified that whenever (3.6) is valid we have that, conditional on  $X^{(s-1)} = Y^{(s-1)}$ ,  $\underline{Y}$  and  $X_s - Y_s$  are independent.

Let us now define the sequences

$$(3.7) \quad v_{n_\ell} = \frac{P_{r_1, \dots, r_{\ell-1}, 0, \dots, 0}}{C_{r_1, \dots, r_{\ell-1}, 0, \dots, 0}} \quad (r_i > 0 \text{ fixed, } i = 1, 2, \dots, \ell-1)$$

and every  $n_\ell > 0$  and

$$(3.8) \quad w_{n_\ell} = \sum_{n_{\ell+1}=0}^{\infty} \dots \sum_{n_s=0}^{\infty} b_{0, \dots, 0, n_\ell, n_{\ell+1}, \dots, n_s} \times \theta_{\ell+1}^{n_{\ell+1}} \dots \theta_s^{n_s} \frac{P(\underline{X} = \underline{Y})}{b_0 P(X^{(\ell-1)} = Y^{(\ell-1)})}$$

for  $\theta_{\ell+1}, \dots, \theta_s > 0$ ,  $\ell=1, \dots, s-1$ . Assume that (3.4) holds for  $\ell=k, k+1, \dots, s$ ;  $2 \leq k \leq s$  and is equivalent to

$$(3.9) \quad \frac{P_{n_1, \dots, n_{k-1}, n_k, \dots, n_s}}{C_{n_1, \dots, n_{k-1}, n_k, \dots, n_s}} = \frac{P_{n_1, \dots, n_{k-1}, 0, \dots, 0}}{C_{n_1, \dots, n_{k-1}, 0, \dots, 0}} \times \theta_k^{n_k} \dots \theta_s^{n_s}$$

for some  $\theta_k, \dots, \theta_s > 0$  and every  $n_i > 0$  ( $i=1,2,\dots,s$ ).

(Note that if (3.9) is valid then, conditional on  $X^{(k-1)} = Y^{(k-1)}$ ,  $\underline{Y}$  and  $(X_k - Y_k, X_{k+1} - Y_{k+1}, \dots, X_s - Y_s)$  are independent.) Under these circumstances it can be shown that, for  $\ell=k-1$ , (3.4) is equivalent to

$$(3.10) \quad \frac{P_{n_1, \dots, n_{k-1}, \dots, n_s}}{C_{n_1, \dots, n_{k-1}, \dots, n_s}} = \frac{P_{n_1, \dots, n_{k-2}, 0, \dots, 0}}{C_{n_1, \dots, n_{k-2}, 0, \dots, 0}} \theta_{k-1}^{n_{k-1}} \times$$

$$\times \theta_k^{n_k} \dots \theta_s^{n_s}$$

for some  $\theta_{k-1}, \dots, \theta_s$  and for every  $n_i > 0$  ( $i=1, 2, \dots, \dots, s$ ;  $2 \leq k \leq s$ ). This is so because with the help of Lemma 1 we can see that, for  $\ell=k-1$ , (3.4) holds iff

$$(3.11) \quad \frac{P_{n_1, \dots, n_{k-1}, 0, \dots, 0}}{C_{n_1, \dots, n_{k-1}, 0, \dots, 0}} = \frac{P_{n_1, \dots, n_{k-2}, 0, \dots, 0}}{C_{n_1, \dots, n_{k-2}, 0, \dots, 0}} \theta_{k-1}^{n_{k-1}}$$

$$(2 \leq k \leq s)$$

i.e. (by combining (3.9) and (3.11)), iff (3.10) holds. We may also observe that if (3.10) is valid then conditional on  $X^{(k-2)} = Y^{(k-2)}$ ,  $\underline{Y}$  and  $(X_{k-1} - Y_{k-1}, X_k - Y_k, \dots, X_s - Y_s)$  will be independent ( $2 \leq k \leq s$ ).

Consequently, we can say that (3.4) (and hence (3.2)) is equivalent to (3.3). Also we have that if (3.3) (i.e. (3.10) for  $k=2$ ) holds,  $\underline{Y}$  and  $\underline{X-Y}$  are independent. Hence Theorem 3 is established.

#### 4. CHARACTERIZATION OF THE MULTIPLE POISSON, BINOMIAL AND NEGATIVE BINOMIAL DISTRIBUTIONS

As a result of Theorem 2 the following corollaries can be established.

**COROLLARY 1** (Characterization of the multiple Poisson). Suppose that for the random vector  $(\underline{X}, \underline{Y})$  we know that

$$(4.1) \quad P(\underline{Y} = \underline{r} | \underline{X} = \underline{n}) = \prod_{i=1}^s \binom{n_i}{r_i} p_i^{r_i} q_i^{n_i - r_i}$$

$$(0 < p_i < 1, q_i = 1 - p_i, 0 \leq r_i \leq n_i, n_i \geq 0,$$

$$i=1, 2, \dots, s)$$

(i.e. multiple binomial) then condition (3.2) holds iff

$$(4.2) \quad P_{\underline{n}} = e^{-\lambda} \prod_{i=1}^s \frac{\lambda_i^{n_i}}{n_i!} \quad (i=1, \dots, s; \lambda = \sum_{i=1}^s \lambda_i, \lambda_i > 0)$$

(i.e. multiple Poisson).

PROOF. Observe that (4.1) is of the form (3.1) with

$$a_{\underline{n}} = \prod_{i=1}^s \frac{p_i^{n_i}}{n_i!} \quad \text{and} \quad b_{\underline{n}} = \prod_{i=1}^s \frac{q_i^{n_i}}{n_i!} \quad (n_i=0, 1, \dots).$$

Since the corresponding  $c_{\underline{n}} = \prod_{i=1}^s \frac{1}{n_i!}$  for  $n_i \geq 0$  the Corollary follows.

REMARK 2. TALWALKER [4] derived a similar characterization of the multiple Poisson distribution using a condition similar but more complicated than our condition (3.2).

COROLLARY 2 (Characterization of the multiple binomial). Suppose that  $P_{\underline{n}}$  is multiple Poisson of the form (4.2) and that the conditional distribution of  $\underline{Y} | \underline{X}$  can be written in the form (3.1). Then condition (3.2) is true iff  $P(\underline{Y} = \underline{r} | \underline{X} = \underline{n})$  is multiple binomial of the form (4.1).

PROOF. The necessary part of the proof is straightforward and is contained in Corollary 1. For "sufficiency" we observe that Theorem 2 implies that condition (3.2) holds iff  $c_{\underline{n}} = c_0 \prod_{i=1}^s (\lambda_i \theta_i)^{n_i} / n_i!$ . Using TEICHER's [5] extension of Raikov's theorem we see that this is so iff  $a_{\underline{n}} = a_0 \prod_{i=1}^s (\alpha_i)^{n_i} / n_i!$  and  $b_{\underline{n}} = b_0 \prod_{i=1}^s (\beta_i)^{n_i} / n_i!$  ( $\alpha_i, \beta_i > 0; \alpha_i + \beta_i = \lambda_i \theta_i$ ). Since  $\{a_{\underline{n}}\}, \{b_{\underline{n}}\}$  should

satisfy the latter conditions it is immediate that we should have the distribution of  $\underline{Y}|\underline{X}$  to be multiple binomial of the form (4.1), for some  $(p_1, \dots, p_s) \in (0, 1)$ .

**COROLLARY 3** (Characterization of the multiple negative binomial). Suppose that the vector  $(\underline{X}, \underline{Y})$  is such that

$$(4.3) \quad P(\underline{Y} = \underline{r} | \underline{X} = \underline{n}) = \prod_{i=1}^s \frac{\binom{-m_i}{r_i} \binom{-\rho_i}{n_i - r_i}}{\binom{-m_i - \rho_i}{n_i}}$$

$$(r_i \leq n_i; m_i, \rho_i > 0, i=1, 2, \dots, s)$$

(i.e. multiple negative hypergeometric). Then, condition (3.2) holds iff  $P_{\underline{n}}$  is multiple negative binomial of the form

$$(4.4) \quad P_{\underline{n}} = \prod_{i=1}^s \binom{-N_i}{n_i} p_i^{N_i} (-q_i)^{n_i} \quad (N_i = m_i + \rho_i).$$

**PROOF.** The proof follows easily if one observes that (4.3) is of the form (3.1) with

$$(4.5) \quad a_{\underline{n}} = \prod_{i=1}^s \binom{m_i + n_i - 1}{n_i} q_i^{n_i}$$

$$b_{\underline{n}} = \prod_{i=1}^s \binom{\rho_i + n_i - 1}{n_i} q_i^{n_i} \quad (\rho_i, m_i > 0)$$

in which case  $c_{\underline{n}} = \prod_{i=1}^s \binom{m_i + \rho_i + n_i - 1}{n_i} q_i^{n_i}$ .

REMARK 3. It is clear that for different forms of the sequence  $\{a_{\underline{n}}, b_{\underline{n}}\}$  characterizations for other forms of multivariate distributions can be obtained.

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