CHAPTER 8.

SOME INTERESTING PROPERTIES AND CHARACTERIZATIONS BASED ON THE DAMAGE MODEL.

8.0 Introduction

In the previous chapter we examined the effect of mixing on the damage model. In this chapter we study the role of convolution in the model under investigation, and arrive at some interesting results. Then we look at the problem of characterizing the distribution of X using the distribution of Y, when $Y|X$ is known. Finally a more general characterization of the Poisson distribution is derived using a relation between $G_Y(t)$ and $G_{Y|X=Y}(t)$.

8.1. The Effect of the Convolution

If we assume that the distribution of $Y|X$ is Binomial then the following properties will hold.

Property 8.1.1

Let us suppose that the r.v's X_i i=1,...,s are independent and identically distributed each with p.g.f. G(t). Then, after binomial damage, the p.g.f.'s for Y_i and for $Y_i|X_i=Y_i$ are, respectively:

$$G_{Y_i}(t) = G(q+pt) \tag{8.1.1}$$

and

$$G_{Y_i|X_i=Y_i}(t) = \frac{G(pt)}{G(p)} . \tag{8.1.2}$$

Now consider $X = \sum_{i=1}^{s} X_i$ as the original r.v. and let $Y = \sum_{i=1}^{s} Y_i$,

then the p.g.f. of the distribution of X, the p.g.f. of the distribution of Y and the p.g.f. of the distribution of $Y|X=Y$ are, respectively,

$$G_X(t) = \{G(t)\}^s \tag{8.1.3}$$

$$G_Y(t) = G_X(q+pt) = \{G(q+pt)\}^s \tag{8.1.4}$$

and

$$G_{Y|X=Y}(t) = \frac{G_X(pt)}{G_X(p)} = \left\{\frac{G(pt)}{G(p)}\right\}^s \tag{8.1.5}$$

Note that if the X_i 's are not identically distributed, then

$G_X(t) = \prod_{i=1}^{s} G_{X_i}(t)$, and $G_Y(t) = G_X(q+pt) = \prod_{i=1}^{s} G_{X_i}(q+pt) = \prod_{i=1}^{s} G_{Y_i}(t)$;

also $G_{Y|X=Y}(t) = \prod_{i=1}^{s} G_{Y_i|X_i=Y_i}(t)$.

Property 8.1.2.

Let us now study the case where the r.v.'s X_i i=1,2 are distributed as Poisson $(\lambda_i) \wedge f(\lambda_i)$ i=1,2 where $0 < \lambda_i < \infty$, and $f(\lambda_i)$ is absolutely continuous. Then

$$G_{Y_i}(t) = \int_0^\infty e^{\lambda_i p(t-1)} f(\lambda_i)d\lambda_i \qquad i=1,2 \tag{8.1.6}$$

and

$$G_{Y_i|X_i=Y_i}(t) = \frac{\int_0^\infty e^{\lambda_i (pt-1)} f(\lambda_i)d\lambda_i}{\int_0^\infty e^{\lambda_i (p-1)} f(\lambda_i)d\lambda_i} \qquad i=1,2. \tag{8.1.7}$$

Consider now the Model where $X \sim \text{Poisson}(\lambda) \wedge F(\lambda)$ where $\lambda = \lambda_1 + \lambda_2$ and λ_1, λ_2 are identically distributed. Then

$$\lambda \sim \int_0^\infty f(\lambda-\lambda_2)\, f(\lambda_2)\, d\lambda_2 \tag{8.1.8}$$

$$G_Y(t) = \int_0^\infty\int_0^\infty e^{\lambda p(t-1)}\, f(\lambda-\lambda_2)\, f(\lambda_2)\, d\lambda_2\, d\lambda \tag{8.1.9}$$

and

$$G_{Y|X=Y} = \frac{\int_0^\infty\int_0^\infty e^{\lambda(pt-1)}\, f(\lambda-\lambda_2)\, f(\lambda_2)\, d\lambda_2\, d\lambda}{\int_0^\infty\int_0^\infty e^{\lambda(p-1)}\, f(\lambda-\lambda_2)\, f(\lambda_2)\, d\lambda_2\, d\lambda} \; . \tag{8.1.10}$$

If we denote by

$$\mathcal{L}\{F(\lambda),t\} = \int_0^\infty e^{-\lambda t} F(\lambda)\, d\lambda \tag{8.1.11}$$

the Laplace transform of $F(\lambda)$, then (8.1.6), (8.1.7), (8.1.9) and (8.1.10) become, respectively

$$G_{Y_i}(t) = \mathcal{L}\{f(\lambda_i),\, p(1-t)\}$$

$$G_{Y_i|X_i=Y_i}(t) = \frac{\mathcal{L}\{f(\lambda_i),\, 1-pt\}}{\mathcal{L}\{f(\lambda_i),\, 1-p\}} \tag{8.1.12}$$

$$G_Y(t) = \mathcal{L}\left\{\int_0^\infty f(\lambda-\lambda_2)\, f(\lambda_2)\, d\lambda_2,\; p(1-t)\right\} \tag{8.1.13}$$

$$G_{Y|X=Y}(t) = \frac{\mathcal{L}\left\{\int_0^\infty f(\lambda-\lambda_2)\ f(\lambda_2)\ d\lambda_2,\ 1-pt\right\}}{\mathcal{L}\left\{\int_0^\infty f(\lambda-\lambda_2)\ f(\lambda_2)\ d\lambda_2,\ 1-p\right\}} \quad . \tag{8.1.14}$$

But it is known that the Laplace transform of the convolution of two r.v's is equal to the product of the Laplace transforms of the p.d.f.'s of the two r.v's.

Consequently, we can rewrite (8.1.13) and (8.1.14) as follows:

$$G_Y(t) = \mathcal{L}\{f(\lambda_1),\ p(1-t)\} \times \mathcal{L}\{f(\lambda_2),\ p(1-t)\} \tag{8.1.15}$$

$$G_{Y|X=Y}(t) = \frac{\mathcal{L}\{f(\lambda_1),\ 1-pt\} \times \mathcal{L}\{f(\lambda_2),\ 1-pt\}}{\mathcal{L}\{f(\lambda_1),\ 1-p\} \times \mathcal{L}\{f(\lambda_2),\ 1-p\}} \tag{8.1.16}$$

or, by using (8.1.11), (8.1.12)

$$G_Y(t) = G_{Y_1}(t)\ G_{Y_2}(t) \tag{8.1.17}$$

$$G_{Y|X=Y}(t) = G_{Y_1|X_1=Y_1}(t) \quad G_{Y_2|X_2=Y_2}(t). \tag{8.1.18}$$

Hence, (8.1.17) and (8.1.18) show that if X is Poisson $(\lambda) \wedge f(\lambda)$ with $\lambda=\lambda_1+\lambda_2$, then the p.g.f.'s of Y and $Y|X=Y$ can be obtained directly as the product of the corresponding p.g.f.'s of Y_1, Y_2 and $Y_1|X_1=Y_1$, $Y_2|X_2=Y_2$.

<u>Note</u> Property (8.1.2) can be extended to the situation where X ~ Poisson $\wedge$ $f(\lambda)$ with

$$\lambda = \lambda_1+\lambda_2+\ldots+\lambda_s, \quad 0 < \lambda_i < \infty, \quad i=1,2,\ldots,s$$

with λ_i identically distributed. This follows since

$$f(\lambda) = \int_0^\infty f(\lambda_s) \int_0^\infty f(\lambda_{s-1}) \ldots \int_0^\infty f(\lambda_2)\, f\left(\lambda - \sum_{k=2}^{s} \lambda_k\right) d\lambda_2 \ldots d\lambda_s \tag{8.1.19}$$

Thus

$$G_Y(t) = \int_0^\infty e^{\lambda p(t-1)} \int_0^\infty f(\lambda_s) \int_0^\infty f(\lambda_{s-1}) \ldots \int_0^\infty f(\lambda_2)\, f\left(\lambda - \sum_{k=2}^{s} \lambda_k\right) d\lambda_2 \ldots d\lambda_s\, d\lambda$$

$$= \mathcal{L}\left\{\int_0^\infty f(\lambda_s) \int_0^\infty f(\lambda_{s-1}) \ldots \int_0^\infty f(\lambda_2)\, f\left(\lambda - \sum_{k=2}^{\infty} \lambda_k\right) d\lambda_2 \ldots d\lambda_s,\ p(1-t)\right\}$$

$$= \prod_{i=1}^{s} \mathcal{L}\{f(\lambda_i),\ p(1-t)\},$$

and hence

$$G_Y(t) = \prod_{i=1}^{s} G_{Y_i}(t). \tag{8.1.20}$$

Similarly

$$G_{Y|X=Y}(t) = \prod_{i=1}^{s} G_{Y_i|X_i=Y_i}(t). \tag{8.1.21}$$

<u>Property 8.1.3</u>

We now examine the more general situation, in which the distribution of X is Poisson $(\lambda) \wedge f(\lambda)$ with $\lambda=\lambda_1+\lambda_2$ and λ_1, λ_2 are not necessarily identically distributed. Then, if $f_i(\lambda_i)$ denotes the p.d.f. of λ_i, $i=1,2$.

$$f(\lambda) = \int_0^\infty f_2(\lambda-\lambda_1)\, f_1(\lambda_1)\, d\lambda_1 \tag{8.1.22}$$

and hence

$$G_Y(t) = \int_0^\infty\int_0^\infty e^{\lambda p(t-1)} f_2(\lambda-\lambda_1)\ f_1(\lambda_1)\ d\lambda_1 d\lambda$$

$$= \mathcal{L}\left\{\int_0^\infty f_2(\lambda-\lambda_1)\ f_1(\lambda_1)\ d\lambda_1,\ p(1-t)\right\}$$

$$= \mathcal{L}\{f_2(\lambda_2), p(1-t)\} \times \mathcal{L}\{f_1(\lambda_1), p(1-t)\}\ .$$

So,

$$G_Y(t) = G_{Y_1}(t) \times G_{Y_2}(t). \qquad (8.1.23)$$

In the same way

$$G_{Y|X=Y}(t) = G_{Y_1|X_1=Y_1}(t) \quad G_{Y_2|X_2=Y_2}(t) \qquad (8.1.24)$$

In general, if $X \sim$ Poisson $\wedge$ $f(\lambda)$ with

$$\lambda = \lambda_1+\ldots+\lambda_s \qquad 0 < \lambda_i < \infty$$

and λ_i is r.v. with p.d.f. $f_i(\lambda_i)$ $i=1,2,\ldots,s$ we have

$$f(\lambda) = \int_0^\infty f_s(\lambda_s) \int_0^\infty f_{s-1}(\lambda_{s-1})\ldots\int_0^\infty f_2(\lambda_2)\ f_1\left(\lambda - \sum_{i=2}^{s} \lambda_i\right) d\lambda_2\ldots d\lambda_s$$

which implies that

$$G_Y(t) = \int_0^\infty e^{\lambda p(t-1)} \int_0^\infty f_s(\lambda_s)\ldots\int_0^\infty f_2(\lambda_2)\ f_1\left(\lambda - \sum_{k=2}^{n} \lambda_k\right) d\lambda_2\ldots d\lambda_s d\lambda$$

$$= \prod_{i=1}^{s} \mathcal{L}\{f(\lambda_i), p(1-t)\}.$$

This gives

$$G_Y(t) = \prod_{i=1}^{s} G_{Y_i}(t).$$

A similar expression holds for $G_{Y|X=Y}(t)$.

Property 8.1.4

Let us now consider the general case where for $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ we define $\underset{\sim}{X} = (X_1, X_2, \ldots, X_s)$ to be a vector of s-independent Poisson variables such that $E(X_j) = \lambda_j$, $j=1,\ldots,s$. Let $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_s)$ be a vector of independent and non-negative integer-valued r.v.'s such that

$$P(Y_j = r_j \mid X_j = n_j) = \binom{n_j}{r_j} p_j^{r_j} q_j^{n_j - r_j}, \quad j=1,\ldots,s. \tag{8.1.25}$$

Further assume that $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a random vector of positive components with distribution function $F(\lambda_1, \lambda_2, \ldots, \lambda_s)$. Then the p.g.f. of the distribution of $(\underset{\sim}{X}, \underset{\sim}{Y})$ will be given by

$$G_{\underset{\sim}{X},\underset{\sim}{Y}}(\underset{\sim}{t},\underset{\sim}{z}) = \sum_{n_1,\ldots,n_s} G_{\underset{\sim}{Y}|\underset{\sim}{X}}(\underset{\sim}{z})\, P_{n_1,\ldots,n_s}\, t_1^{n_1}\cdots t_s^{n_s}$$

$$= \sum_{n_1,\ldots,n_s} \prod_{j=1}^{s} (q_j + p_j z_j)^{n_j}\, P_{\underset{\sim}{n}}\, t_1^{n_1}\cdots t_s^{n_s}$$

$$= G_{\underset{\sim}{X}}(\underset{\sim}{t}\,(\underset{\sim}{q}+\underset{\sim}{p}\underset{\sim}{z})) = \int G_{\underset{\sim}{X}|\underset{\sim}{\lambda}}\, dF(\underset{\sim}{\lambda}) \quad (\text{where } \underset{\sim}{u}\underset{\sim}{v} = (u_1 v_1, \ldots, u_s v_s))$$

$$= E(G_{\underset{\sim}{X}|\underset{\sim}{\lambda}}) = E\left(\prod_{i=1}^{s} e^{-\lambda_i (1 - t_i (q_i + p_i z_i))}\right)$$

i.e. by

$$G_{\underset{\sim}{X},\underset{\sim}{Y}}(\underset{\sim}{t},\underset{\sim}{z}) = E\left[\prod_{i=1}^{s} e^{-\lambda_i \{1 - t_i (q_i + p_i z_i)\}}\right]. \tag{8.1.26}$$

For the p.g.f. of $\underset{\sim}{Y}$ we have

$$G_{\underset{\sim}{Y}}(\underset{\sim}{z}) = E\left[\prod_{i=1}^{s} e^{-\lambda_i p_i (1-z_i)}\right]. \tag{8.1.27}$$

We also have $\left(\text{for } \sum_{i=1}^{s} Y_i = Y,\ \lambda = \sum_{i=1}^{s} \lambda_i\right)$

$$G_Y(z) = E\left[\prod_{i=1}^{s} e^{-\lambda_i p_i (1-z)}\right] \tag{8.1.28}$$

which if all $p_i = p$, simplifies to,

$$G_Y(z) = E\left\{e^{-\lambda p(1-z)}\right\}^{s}. \tag{8.1.29}$$

8.2 Some Characterizations of the Distribution of X when the Distribution of Y|X is Given.

As we mentioned in the introduction, many authors have derived characterizations of the distribution of X using the distribution of Y or else the one of $Y|X$, provided that the latter has a given form. In what follows we derive general characterizations of this kind using mixing forms for the distribution of $Y|X$.

Theorem 8.2.1 (Characterization of the Poisson Distribution)

Suppose that for the non-negative, integer-valued r.v.'s X and Y we have that

$$P(Y=r|X=n) = \int_0^1 \binom{n}{r} p^r q^{n-r}\, dF(p) \tag{8.2.1}$$

i.e., $Y|X$ is Binomial $\wedge$ F(p).

Then X is Poisson (λ) iff Y is Poisson (λp) $\wedge$ F(p).

Proof

The "only if" part is straightforward. For the "if" part, suppose that Y is Poisson (λp) $\wedge$ F(p), i.e.

$$P(Y=r) = \int_0^1 e^{-\lambda p} \frac{(\lambda p)^r}{r!} dF(p) \qquad r=0,1,\ldots \tag{8.2.2}$$

On the other hand, we have in general

$$P(Y=r) = \sum_{n=r}^{\infty} P(X=n)\, P(Y=r|X=n)$$

which in our case, because of (8.2.1) becomes

$$P(Y=r) = \sum_{n=r}^{\infty} P_n \int_0^1 \binom{n}{r} p^r q^{n-r}\, dF(p) \qquad r=0,1,\ldots \tag{8.2.3}$$

(8.2.2) and (8.2.3) give

$$\int_0^1 \sum_{n=r}^{\infty} P_n \binom{n}{r} p^r q^{n-r}\, dF(p) = \int_0^1 e^{-\lambda p} \frac{(\lambda p)^r}{r!} dF(p) \qquad r=0,1,\ldots \tag{8.2.4}$$

The x-th factorial moment of the distribution (8.2.2) is given by

$$\mu_{[x]}(r) = \int_0^1 \sum_{r=x}^{\infty} e^{-\lambda p} \frac{(\lambda p)^r}{r!} r(r-1)\ldots(r-x+1)\, dF(p)$$

i.e.

$$\mu_{[x]}(r) = \int_0^1 (\lambda p)^x\, dF(p). \tag{8.2.5}$$

Similarly the x-th factorial moment of the distribution (8.2.3) is given by

$$\mu_{[x]}(r) = \sum_{r=x}^{\infty} r(r-1)\ldots(r-x+1) \int_0^1 \sum_{n=r}^{\infty} P_n \binom{n}{r} p^r q^{n-r} \, dF(p)$$

$$= \int_0^1 \sum_{r=x}^{\infty} r(r-1)\ldots(r-x+1) \sum_{n=r}^{\infty} P_n \binom{n}{r} p^r q^{n-r} \, dF(p)$$

$$= \int_0^1 \sum_{n=x}^{\infty} \left\{ \sum_{r=x}^{n} r(r-1)\ldots(r-x+1) \binom{n}{r} p^r q^{n-r} \right\} P_n \, dF(p)$$

$$= \int_0^1 \left\{ \sum_{n=x}^{\infty} n(n-1)\ldots(n-x+1) \, p^x \right\} P_n \, dF(p) \tag{8.2.6}$$

(because the x-th factorial moment of the binomial distribution is $n(n-1)\ldots(n-x+1)p^x$). (8.2.6) gives

$$\mu_{[x]}(r) = \mu_{[x]}(n) \int_0^1 p^x \, dF(p), \tag{8.2.7}$$

where $\mu_{[x]}(n)$ denotes the r-th factorial moment of $\{P_n\}$.

Combining (8.2.4), (8.2.5) and (8.2.7) we find that

$$\mu_{[x]}(n) = \lambda^x \qquad x=1,2,\ldots \tag{8.2.8}$$

Consequently the factorial moments of the distribution of the random variable X are the same to those corresponding to Poisson distribution. This implies that the random variable X has the same moments as a Poisson variable. Since the Poisson distribution is uniquely determined by its moments the result follows.

Applying Theorem 8.2.1 to the various mixtures that we have studied previously, the following results can be obtained as special cases.

Corollary 8.2.1

Suppose that $P(Y=r|X=n)$ is Binomial ^ Beta as in 7.1.1. Then

$$G_Y(t) = {}_1F_1\{\alpha;\alpha+\beta;\lambda(t-1)\}$$

iff X is Poisson.

Corollary 8.2.2

Suppose that $Y|X$ is Binomial $\wedge$ Beta Truncated to the right, as in 7.1.2. Then,

$$G_Y(t) = \frac{\Phi_1[\alpha,1-\beta;\alpha+1;x,\lambda x(t-1)]}{{}_2F_1(\alpha,1-\beta;\alpha+1;x)}$$

iff $X \sim$ Poisson (λ).

Corollary 8.2.3

Suppose that $Y|X$ is Binomial $\wedge$ Right Truncated exponential (as examined in Section 7.1.3). Then,

$$G_Y(t) = \frac{{}^1/_\theta\left[e^{\lambda t-\frac{1}{\theta}-\lambda}-1\right]}{\left(\lambda t-\frac{1}{\theta}-\lambda\right)\left(1-e^{-\frac{1}{\theta}}\right)}$$

iff $X \sim$ Poisson (λ).

Corollary 8.2.4

Suppose that $Y|X$ is Binomial $\wedge$ Right Truncated Gamma (see Section 7.1.4). Then,

$$G_Y(t) = \frac{{}_1F_1(\alpha;\alpha+1;\lambda(t-1)-\frac{1}{\beta})}{{}_1F_1(\alpha;\alpha+1;-\frac{1}{\beta})}$$

iff X is Poisson (λ).

Remark Since Theorem 8.2.1, holds for any form of $F(p)$ continuous or discrete it is apparent that it can be used to provide characterizations for many other distributions.

We now move to a slightly more general case and establish the following theorem.

<u>Theorem 8.2.2</u> (Characterization of the Mixed Poisson distribution)

Let us assume that

$$P(Y=r|X=n) = \int_0^1 \binom{n}{r} p^r q^{n-r} dF_2(p) \qquad (8.2.9)$$

i.e. Binomial $\wedge$ $F_2(p)$.

Suppose that the distribution of X is uniquely determined by its factorial moments and that

$$\int_0^\infty \lambda^x dF_1(\lambda) < \infty \quad \text{for } \lambda > 0, \quad x=0,1,.... \qquad (8.2.10)$$

Then, the distribution of Y is Poisson (λp) $\wedge$ $F_1(\lambda)$ $\wedge$ $F_2(p)$ iff X ~ Poisson (λ) $\wedge$ $F_1(\lambda)$.

<u>Proof</u>

The "if" part of the proof is straightforward. As far as the "only if" part is concerned, by following the argument of the proof of Theorem 8.2.1 we have that the f.m.g.f. of the distribution of X is given by

$$\mu_{[x]}(n) = \int_0^\infty \lambda^x dF_1(\lambda). \qquad (8.2.11)$$

It is now known that the f.m.g.f. of the mixed Poisson distribution is of the form (8.2.11). Since we have assumed that the distribution of X is uniquely determined by its f.m., the result follows.

<u>Note</u> It can be observed that if F_1 is degenerate, then Theorem 8.2.2 reduces to Theorem 8.2.1.

8.3 Characterization of the Distribution of $Y|X$

In this section we examine characterizations of the distribution of $Y|X$, when it is given that X is of Poisson type. First we present the following general result.

Theorem 8.3.1 (Characterization of the Mixed Binomial)

Suppose that the distribution of X is Poisson with parameter λ and that the distribution of $Y|X$ is independent of λ. Then $Y \sim$ Poisson $(\lambda p) \wedge F(p)$ iff $Y|X$ is Binomial $\wedge F(p)$.

Proof "If" part. ("Only if" part already proved in Theorem 8.2.1.)

We have been given that

$$P(Y=r) = \int_0^1 e^{-\lambda p} \frac{(\lambda p)^r}{r!} dF(p). \qquad (8.3.1)$$

and

$$P(Y=r) = \sum_{n=r}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} P(Y=r|X=n). \qquad (8.3.2)$$

Hence

$$\sum_{n=r}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} P(Y=r|X=n) = \int_0^1 e^{-\lambda p} \frac{(\lambda p)^r}{r!} dF(p) . \qquad (8.3.3)$$

This is a functional equation in $P(Y=r|X=n)$. The Binomial $\wedge F(p)$ is a solution of (8.3.3). This is so because for $P(Y=r|X=n) \sim$ Binomial $\wedge F(p)$ the L.H.S. of (8.3.3) can be written as

$$\sum_{n=r}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \int_0^1 \binom{n}{r} p^r q^{n-r} dF(p)$$

$$= \int_0^1 e^{-\lambda} \frac{(\lambda p)^r}{r!} \sum_{n=r}^{\infty} \frac{(\lambda q)^{n-r}}{(n-r)!} dF(p)$$

$$= \int_0^1 e^{-\lambda p} \frac{(\lambda p)^r}{r!} dF(p).$$

To show that this is the only solution, suppose that there is another one $P^*(Y=r|X=n)$. Then we would have

$$P(Y=r) = \sum_{n=r}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} P^*(Y=r|X=n). \tag{8.3.4}$$

From (8.3.2) and (8.3.4) we would then have

$$\sum_{n=r}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} [P^*(Y=r|X=n) - P(Y=r|X=n)] = 0$$

i.e.,

$$\sum_{n=r}^{\infty} \frac{\lambda^n}{n!} [P^*(Y=r|X=n) - P(Y=r|X=n)] = 0 . \tag{8.3.5}$$

But we know that $P(Y=r|X=n)$ is independent of λ. Thus equating the coefficients of $\frac{\lambda^n}{n!}$ of both sides of (8.3.5) we see that

$$P^*(Y=r|X=n) = P(Y=r|X=n), \quad n \geqslant r,$$

and the theorem follows.

As a result of Theorem 8.3.1 the following characterizations can be obtained.

Corollary 8.3.1 (Characterization of the Binomial ^ Beta)

Suppose that X ~ Poisson (λ); then $Y|X$ is Binomial ^ Beta iff Y is Gurland with p.g.f. given by (7.1.8).

Corollary 8.3.2 (Characterization of the Binomial ^ Right Truncated Beta)

Suppose that X ~ Poisson (λ) and that $Y|X$ is independent of λ. Then $Y|X$ is Beta truncated to the right iff the p.g.f. of Y has the form (7.1.12).

<u>Corollary 8.3.3</u> (Characterization of the Binomial ^ Right Truncated Exponential)

If X ~ Poisson (λ) and Y|X independent of λ, we have Y|X ~ Binomial ^ Right truncated Exponential iff the p.g.f. of Y is given by (7.1.16).

<u>Corollary 8.3.4</u> (Characterization of the Binomial ^ Right Truncated Gamma)

Assume that X ~ Poisson (λ) with λ independent of Y|X. Then Y|X is Binomial ^ Right Truncated Gamma iff Y follows a distribution with p.g.f. given by (7.1.19).

<u>Note</u> It is interesting to point out here that the result of Theorem 8.3.1 as far as its "only if" part is concerned is not valid any longer if one assumes that the distribution of X is Poisson ^ $F_1(\lambda)$ and the distribution of Y is Poisson ^ $F_1(\lambda)$ ^ $F_2(p)$.

In these circumstances an argument similar to the one used in Theorem 8.3.1 leads to the conclusion that the distribution of Y|X must satisfy the following equation

$$P(Y=r) \;=\; \sum_{n=r}^{\infty} \left\{ \int_0^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \, dF_1(\lambda) \right\} s(r,n) \; . \qquad (8.3.6)$$

Clearly the Binomial ^ $F_2(p)$ is a solution. To see whether it is unique, suppose that $s^*(r,n)$ gives a second solution. Then, we would have

$$\int_0^{\infty} \sum_{n=r}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \left\{ s(r,n) - s^*(r,n) \right\} dF_1(\lambda) \;=\; 0 \; . \qquad (8.3.7)$$

However (8.3.7) does not imply that

$$\sum_{n=r}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \left\{s(r,n)-s^*(r,n)\right\} = 0 \tag{8.3.8}$$

i.e. that $s(r,n) = s^*(r,n)$,

because if we consider $s^*(r,n)$ such that

$$s^*(r,n) = \begin{cases} s(r,n), & n=k+1,k+2,\dots. \\ s(r,n)-c_{r,n}, & n=0,1,2,\dots,k \end{cases}$$

then (8.3.7) is equivalent to

$$\int_0^{\infty} \sum_{n=r}^{k} e^{-\lambda} \frac{\lambda^n}{n!} c_{r,n} \, dF_1(\lambda) = 0$$

i.e. to

$$\sum_{n=r}^{k} c_{r,n} \xi_n = 0 \tag{8.3.9}$$

where

$$\xi_n = \int_0^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \, dF_1(\lambda) \tag{8.3.10}$$

Since we can find $c_{r,n} \neq 0$ for which (8.3.9) holds, we come to the conclusion that there exist solutions of (8.3.6) other than the Binomial $\hat{}$ F(p).

However a characterization of the mixed Binomial can be obtained in this case by making certain restrictions on the form of $F_1(\lambda)$.

This is done in the theorem that follows.

<u>Theorem 8.3.2</u>

Let us suppose that the distribution of X is Poisson $(\lambda) \hat{} F_1(\lambda)$ where $F_1(\lambda)$ is absolutely continuous with density

$$f_1(\lambda|\theta) = e^{-\theta\lambda} \frac{\phi(\lambda)}{\psi(\theta)} \text{ where } \lambda \in (a,b) \text{ with } \phi(\lambda),\psi(\theta),\theta > 0. \tag{8.3.11}$$

Let us also assume that the distribution of $Y|X$ is independent of θ. Then the distribution of Y is mixed Poisson of the form Poisson $(\lambda p) \wedge F_1(\lambda) \wedge F_2(p)$ iff the distribution of $Y|X$ is mixed Binomial with mixing distribution $F_2(p)$.

Proof The "if" part is straightforward.

To show that $s(r,n)$ = Binomial $\wedge F_2(p)$ is the only solution, suppose that there is another one $s^*(r,n)$. Then we would have

$$\int_0^\infty \sum_{n=r}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \frac{e^{-\theta\lambda}}{\psi(\theta)} \phi(\lambda) \{s(r,n)-s^*(r,n)\}d\lambda = 0$$

which implies that

$$\sum_{n=r}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \phi(\lambda) \{s(r,n)-s^*(r,n)\} = 0 \text{ for } \lambda \in (a,b).$$

i.e., that

$$s^*(r,n) = s(r,n).$$

8.4 Another Extension of the R-R Characterization

In Chapter 2 we examined the R-R characterization of the Poisson distribution. As we saw, if $Y|X$ is Binomial, then X is Poisson iff

$$G(q+pt) = \frac{G(pt)}{G(p)} .$$

We are now going to give a more general characterization of the Poisson distribution, which includes the characterization by Rao and Rubin.

This characterization is based on the general relation (7.2.2) that holds between the p.g.f.'s of Y and $Y|X=Y$ in the case where $Y|X$ is Mixed Binomial; it can be stated as follows.

Theorem 8.4.1

Let us consider the random vector (X,Y) with non-negative, real components such that $P(X=n) = P_n$, n=0,1,..., with $P_0 \neq 0$ and

$$P(Y=r|X=n) = b(r,n,p) \wedge F(p) \qquad 0 < p < 1. \tag{8.4.1}$$

Then, P_n is Poisson iff

$$G_Y(t+1) = C^* G_{Y|X=Y}(t) \tag{8.4.2}$$

where

$$(C^*)^{-1} = G_{Y|X=Y}(0). \tag{8.4.3}$$

Proof

The "only if" part has been proved in Theorem 7.2.1. On the other hand, as it was shown in the previous chapter (Section 7.1), the p.g.f. of Y is given by

$$G_Y(t) = \int_0^1 G(pt+q)\, dF(p) .$$

Hence,

$$G_Y(t+1) = \int_0^1 G(pt+1)\, dF(p). \tag{8.4.4}$$

Also as in (7.1.3) we have

$$G_{Y|X=Y}(t) = \frac{\int_0^1 G(pt)\, dF(p)}{\int_0^1 G(p)\, dF(p)} \tag{8.4.5}$$

Substituting (8.4.4) and (8.4.5) in (8.4.2) gives

$$\int_0^1 G(pt+1)\, dF(p) \;=\; C\int_0^1 G(pt)\, dF(p) \qquad (8.4.6)$$

with C constant.

(8.4.6) implies that

$$\int_0^1 \sum_{n=0}^{\infty} P_n\, (pt+1)^n\, dF(p) \;=\; C\int_0^1 \sum_{n=0}^{\infty} P_n\, (pt)^n\, dF(p)$$

$$\sum_{n=0}^{\infty} \left\{ \sum_{r=0}^{n} P_n \int_0^1 \binom{n}{r} p^r\, dF(p) \right\} t^r \;=\; C \sum_{n=0}^{\infty} P_n \left\{ \int_0^1 p^n\, dF(p) \right\} t^n$$

$$\sum_{r=0}^{\infty} \left\{ \sum_{n=r}^{\infty} P_n \int_0^1 \binom{n}{r} p^r\, dF(p) \right\} t^r \;=\; C \sum_{r=0}^{\infty} P_r \left\{ \int_0^1 p^r\, dF(p) \right\} t^r \;.$$

Hence

$$\sum_{n=r}^{\infty} P_n \binom{n}{r} \int_0^1 p^r\, dF(p) \;=\; C\, P_r \int_0^1 p^r\, dF(p),$$

i.e.

$$\sum_{n=r}^{\infty} P_n \binom{n}{r} \;=\; C\, P_r \;. \qquad (8.4.7)$$

Taking the p.g.f.'s for both sides of (8.4.7) we find that

$$G(z+1) \;=\; C\, G(z) \qquad 0 \leqslant z \leqslant 1 \;. \qquad (8.4.8)$$

Now, following exactly the same method that we used in Chapter 2 to prove the R-R theorem (see Theorem 2.1.1 relation (2.1.5)), we arrive at the conclusion that

$$G(t) \;=\; e^{\lambda(t-1)}$$

i.e. X ~ Poisson (λ).

Note 1 Clearly, if $Y|X \sim$ Binomial, i.e. if $F(p)$ is degenerate, then (8.4.2) reduces to

$$G(q+pt) = \frac{G(pt)}{G(p)}$$

and hence, Theorem 8.4.1 reduces to the R-R characterization examined in Theorem 2.1.1.

Note 2 We arrive to the same characterization if we replace (8.4.2) with (7.2.1).