CHAPTER 7.

THE EFFECT OF MIXING TO THE DAMAGE MODEL.

7.0 Introduction

Up to now, we have been examining characterizations of statistical distributions based on various properties of the random variables X, Y, $Y|X$ where $X \geq Y$. As it has already been said, Rao (1963) interpreted this mathematical model as damage model.

In this chapter we will examine changes which take place in the model when either X or $Y|X$ has a mixed distribution. Thus, we derive the p.g.f's of the "resulting" r.v. Y, and the resulting random variable when no damage has occurred (the r.v. $Y|X=Y$) in various cases. We also examine some properties of the model in this extended form.

7.1 Damage Model with Original Distribution Poisson and Survival Distribution Mixed Binomial.

Let us suppose that the conditional distribution of $Y|X$ is Binomial $b(r,n,p)$, $0 < p < 1$, with p following some particular distribution with distribution function $F(p)$.

Suppose also that the distribution of X is Poisson with parameter λ. Then, for the p.g.f. of Y we will have

$$G_Y(t) = \sum_{r=0}^{\infty} P(Y=r)t^r = \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} P_n s(r,n)t^r$$

$$= \sum_{r=0}^{\infty} \left\{ \sum_{n=r}^{\infty} P_n \int_0^1 b(r,n,p)\ dF(p) \right\} t^r$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left\{ \int_0^1 P_n\ b(r,n,p)\ dF(p) \right\} t^r$$

$$= \int_0^1 \sum_{n=0}^{\infty} P_n \left\{ \sum_{r=0}^{n} b(r,n,p)t^r \right\} dF(p)$$

(since we can change the order of integration and summation for $|t| \leqslant 1$.)

$$= \int_0^1 \sum_{n=0}^{\infty} P_n\ (pt+q)^n\ dF(p)\ .$$

So finally,

$$G_Y(t) = \int_0^1 G_X(pt+q)\ dF(p) \qquad (7.1.1)$$

where $G_X(t)$ denotes the p.g.f. of the r.v. X.

If the original distribution is Poisson with parameter λ,

$$G_Y(t) = \int_0^1 e^{\lambda p(t-1)}\ dF(p) = M_p(\lambda(t-1)) \qquad 0 < p < 1 \qquad (7.1.2)$$

For the p.g.f. of $Y|X=Y$ (the "undamaged situation" in terms of the damaged model) we have

$$G_{Y|X=Y}(t) = \frac{\sum_{r=0}^{\infty} P_r s(r,r) t^r}{\sum_{r=0}^{\infty} P_r s(r,r)} = \frac{\sum_{r=0}^{\infty} \left\{ P_r \int_0^1 b(r,r,p)\, dF(p) \right\} t^r}{\sum_{r=0}^{\infty} P_r \int_0^1 b(r,r,p)\, dF(p)}$$

$$= \frac{\int_0^1 G_X(pt)\, dF(p)}{\int_0^1 G_X(p)\, dF(p)}$$

i.e.

$$G_{Y|X=Y}(t) = \frac{\int_0^1 G_X(pt)\, dF(p)}{\int_0^1 G_X(p)\, dF(p)} \qquad (7.1.3)$$

and for P_n Poisson

$$G_{Y|X=Y}(t) = \frac{\int_0^1 e^{\lambda(pt-1)}\, dF(p)}{\int_0^1 e^{\lambda(p-1)}\, dF(p)}$$

$$= \frac{\int_0^1 e^{\lambda pt}\, dF(p)}{\int_0^1 e^{\lambda p}\, dF(p)} = \frac{M_p(\lambda t)}{M_p(\lambda)} \qquad (7.1.4)$$

$0 < p < 1.$

Various forms of F(p) lead to the following models (assuming always that $X \sim$ Poisson (λ)).

7.1.1 (Y|X) ~ Binomial ^ Beta (Negative Hypergeometric)

Suppose that the parameter p of the Binomial is Beta distributed with p.d.f.

$$f_1(p) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1} (1-p)^{\beta-1} \qquad a > 0,\ \beta > 0$$
$$0 < p < 1. \tag{7.1.5}$$

Then

$$M_p(\theta) = \frac{1}{B(\alpha,\beta)} \int_0^1 e^{p\theta} p^{\alpha-1} (1-p)^{\beta-1}\, dp \tag{7.1.6}$$

$$= {}_1F_1\{\alpha;\alpha+\beta;\theta\} \tag{7.1.7}$$

which is the m.g.f. of the Beta distribution.

Then,

$$G_Y(t) = {}_1F_1\{\alpha;\alpha+\beta;\lambda(t-1)\}\ . \tag{7.1.8}$$

Also, from (7.1.4) we get

$$G_{Y|X=Y}(t) = \frac{{}_1F_1\{\alpha;\alpha+\beta;\lambda t\}}{{}_1F_1\{\alpha;\alpha+\beta;\lambda\}}\ . \tag{7.1.9}$$

7.1.2 (Y|X) ~ Binomial ^ Right Truncated Beta

If F(p) is Beta truncated to the right at the point x, $0 < x < 1$, with p.d.f.

$$f_x(p) = \frac{\alpha\, p^{\alpha-1}(1-p)^{\beta-1}}{x^{\alpha}\ {}_2F_1(\alpha,1-\beta;\alpha+1;x)} \quad , \qquad (7.1.10)$$

then

$$M_p(\theta) = \frac{\alpha}{x^{\alpha}\ {}_2F_1(\alpha,1-\beta;\alpha+1;x)} \int_0^x e^{p\theta}\, p^{\alpha-1}(1-p)^{\beta-1}\, dp.$$

Setting $P/x = \pi \quad 0 < \pi < 1$,

i.e. $p = \pi x$, we get,

$$M_p(t) = \frac{\alpha}{x^{\alpha}\ {}_2F_1(\alpha,1-\beta;\alpha+1;x)} \int_0^1 e^{\pi x\theta}\, (\pi x)^{\alpha-1}(1-\pi x)^{\beta-1}\, d(\pi x)$$

$$= \frac{\alpha}{{}_2F_1(\alpha,1-\beta;\alpha+1;x)} \int_0^1 e^{\pi x\theta}\, \pi^{\alpha-1}(1-\pi x)^{\beta-1}\, d\pi \quad .$$

But it is well-known

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}\, e^{uy}\, du$$

$$= \Phi_1[\alpha,\beta;\gamma;x,y] \ . \qquad (7.1.11)$$

So, since for our case $\alpha\to\alpha$, $\gamma\to\alpha+1$, $y\to x\theta$, $\beta\to 1-\beta$, we have

$$M_p(\theta) = \frac{\alpha\Gamma(\alpha)\Gamma(1)}{\Gamma(\alpha+1)} \quad \frac{\Phi_1[\alpha,1-\beta;\alpha+1;x,x\theta]}{{}_2F_1(\alpha,1-\beta;\alpha+1;x)} \quad ,$$

which finally gives

$$G_Y(t) = \frac{\Phi_1[\alpha,1-\beta;\alpha+1,x,\lambda x(t-1)]}{{}_2F_1(\alpha,1-\beta;\alpha+1;x)} \quad . \qquad (7.1.12)$$

For Y|X=Y we have, (from (7.1.4), (7.1.10))

$$G_{Y|X=Y}(t) = \frac{\int_0^x e^{\lambda pt} f(p)dp}{\int_0^x e^{\lambda p} f(p)dp}$$

$$= \frac{\int_0^x e^{\lambda pt}\, p^{\alpha-1}(1-p)^{\beta-1}\, dp}{\int_0^x e^{\lambda p}\, p^{\alpha-1}(1-p)^{\beta-1}\, dp}$$

and for $P/x = \pi \quad 0 < \pi < 1$ this becomes

$$G_{Y|X=Y}(t) = \frac{\int_0^1 e^{\lambda\pi xt}(\pi x)^{\alpha-1}(1-\pi x)^{\beta-1} d(\pi x)}{\int_0^1 e^{\lambda\pi x}(\pi x)^{\alpha-1}(1-\pi x)^{\beta-1} d(\pi x)}$$

$$= \frac{\int_0^1 e^{\lambda\pi xt}\,\pi^{\alpha-1}(1-\pi x)^{\beta-1} d\pi}{\int_0^1 e^{\lambda\pi x}\,\pi^{\alpha-1}(1-\pi x)^{\beta-1} d\pi} \quad .$$

Using (7.1.11) with $u\to\pi$, $\alpha\to\alpha$ $\gamma\to\alpha+1$, $\beta\to 1-\beta$, $x\to x$, $y\to\lambda xt$, we get

$$G_{Y|X=Y}(t) = \frac{\Phi_1[\alpha,1-\beta;\alpha+1;x,\lambda xt]}{\Phi_1[\alpha,1-\beta;\alpha+1;x,\lambda x]} \quad . \qquad (7.1.13)$$

Note 1

It can be verified that if we consider the case 7.1.2 for x=1 the p.g.f. (7.1.12) of Y becomes the same as in (7.1.8) of the case 7.1.1 as one would expect.

Actually we have by definition

$$\Phi_1(\alpha,1-\beta;\alpha+1;1,\lambda(t-1)) = \sum_m \sum_n \frac{\alpha_{(m+n)}(1-\beta)_{(m)}}{(\alpha+1)_{(m+n)}} \frac{1^m\{\lambda(t-1)\}^n}{m!n!}$$

$$= \sum_n \frac{\alpha_{(n)}}{(\alpha+1)_{(n)}} \frac{\{\lambda(t-1)\}^n}{n!} \sum_m \frac{(\alpha+n)_{(m)}(1-\beta)_{(m)}}{(\alpha+m+1)_{(m)}} \frac{1^m}{m!}$$

$$= \sum_n \frac{\alpha_{(n)}}{(\alpha+1)_{(n)}} \frac{\{\lambda(t-1)\}^n}{n!} \; {}_2F_1\,(\alpha+n,1-\beta;\alpha+n+1;1)$$

$$= \sum_n \frac{\alpha_{(n)}}{(\alpha+1)_{(n)}} \frac{\{\lambda(t-1)\}^n}{n!} \frac{\Gamma(\alpha+n+1)\Gamma(\beta)}{\Gamma(1)\Gamma(\alpha+\beta+n)}$$

$$= \Gamma(\beta) \sum_n \frac{\alpha_{(n)}}{(\alpha+1)_{(n)}} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+1+\beta-1)} \frac{\{\lambda(t-1)\}^n}{n!}$$

$$= \Gamma(\beta) \sum_n \frac{\alpha_{(n)}}{(\alpha+1)_{(n)}(\alpha+n+1)_{(\beta-1)}} \frac{\{\lambda(t-1)\}^n}{n!}$$

$$= \Gamma(\beta) \sum_n \frac{\alpha_{(n)}}{(\alpha+1)_{(n+\beta-1)}} \frac{\{\lambda(t-1)\}^n}{n!} = \frac{\Gamma(\beta)}{(\alpha+1)_{(\beta-1)}} \sum_n \frac{\alpha_{(n)}}{(\alpha+\beta)_{(n)}} \frac{\{\lambda(t-1)\}^n}{n!}$$

$$= \frac{\Gamma(\beta)}{(\alpha+1)_{(\beta-1)}} \; {}_1F_1\{\alpha;\alpha+\beta;\lambda(t-1)\} \; , \qquad (7.1.14)$$

Taking into account (7.1.14), (7.1.12) at x=1 can be written as

$$G_Y(t)_{(x=1)} = \frac{\Phi_1(\alpha,1-\beta;\alpha+1;1,\lambda(t-1))}{{}_2F_1(\alpha,1-\beta;\alpha+1;1)}$$

$$= \frac{\Gamma(1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} \frac{\Gamma(\beta)}{(\alpha+1)_{(\beta-1)}} \; {}_1F_1\{\alpha;\alpha+\beta;\lambda(t-1)\}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta)} \; {}_1F_1\{\alpha;\alpha+\beta;\lambda(t-1)\}$$

$$= {}_1F_1(\alpha,\alpha+\beta;\lambda(t-1)).$$

Similarly we can show that (7.1.13) for x=1 is identical to (7.1.9).

7.1.3 (Y|X) ~ Binomial ^ Right Truncated Exponential

The p.d.f. of the exponential truncated to the right at the point 1 is

$$f(p) = \frac{1/\mu \, e^{-p/\mu}}{1 - e^{-1/\mu}} \qquad 0 < p < 1, \quad 0 < \mu . \tag{7.1.15}$$

Hence we have,

$$M_p(\theta) = \int_0^1 e^{p\theta} f(p)\, dp$$

$$= \frac{1/\mu}{1 - e^{-1/\mu}} \int_0^1 e^{p\theta} \; e^{-p/\mu}\, dp$$

$$= \frac{1/\mu}{1 - e^{-1/\mu}} \int_0^1 e^{p(\theta - 1/\mu)}\, dp = \frac{1/\mu \, [e^{\theta - 1/\mu} - 1]}{(\theta - 1/\mu)(1 - e^{-1/\mu})} .$$

Hence

$$G_Y(t) = \frac{1/\mu\left[e^{\lambda t-\frac{1}{\mu}-\lambda} - 1\right]}{\left(\lambda t-\frac{1}{\mu}-\lambda\right)\left(1 - e^{-\frac{1}{\mu}}\right)} . \tag{7.1.16}$$

On the other hand,

$$G_{Y|X=Y}(t) = \frac{M_p(\lambda t)}{M_p(\lambda)}$$

i.e.

$$G_{Y|X=Y}(t) = \frac{\left(\lambda - \frac{1}{\mu}\right)\left(e^{\lambda t-\frac{1}{\mu}} - 1\right)}{\left(\lambda t - \frac{1}{\mu}\right)\left(e^{\lambda-\frac{1}{\mu}} - 1\right)} . \tag{7.1.17}$$

7.1.4 (Y|X) ~ Binomial ^ Right Truncated Gamma.

The p.d.f. of the Gamma distribution, truncated to the right at the point 1 is

$$f(p) = \frac{\alpha p^{\alpha-1} \quad e^{-\frac{p}{\beta}}}{{}_1F_1\left(\alpha;\alpha+1; -\frac{1}{\beta}\right)} , \qquad 0 < p < 1 . \tag{7.1.18}$$

Thus, following (7.1.2), the p.g.f. of Y becomes

$$G_Y(t) = \int_0^1 e^{\lambda p(t-1)} f(p)dp = \frac{\alpha}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}\right)} \int_0^1 p^{\alpha-1} e^{p\left[\lambda(t-1)-\frac{1}{\beta}\right]} dp$$

$$= \frac{\alpha}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}\right)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} {}_1F_1\left(\alpha;\alpha+1;\lambda(t-1)-\frac{1}{\beta}\right) .$$

Hence, finally,

$$G_Y(t) = \frac{{}_1F_1\left(\alpha;\alpha+1;\lambda(t-1)-\frac{1}{\beta}\right)}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}\right)} . \tag{7.1.19}$$

[We have made use of the definition of the Confluent Hypergeometric function

$${}_1F_1(\alpha;c;x) = \frac{\Gamma(c)}{\Gamma(\alpha)\Gamma(c-\alpha)} \int_0^1 e^{xu} u^{\alpha-1} (1-u)^{c-\alpha-1} du$$

which is applied in our case with u=p, $x = \lambda(t-1)-\frac{1}{\beta}$ and $c = \alpha+1$].

The p.g.f. of the r.v. $Y|X=Y$ is

$$G_{Y|X=Y}(t) = \frac{\int_0^1 e^{\lambda pt} f(p)dp}{\int_0^1 e^{\lambda p} f(p)dp} = \frac{\int_0^1 p^{\alpha-1} e^{p\left(\lambda t-\frac{1}{\beta}\right)} dp}{\int_0^1 p^{\alpha-1} e^{p\left(\lambda-\frac{1}{\beta}\right)} dp}$$

i.e.

$$G_{Y|X=Y}(t) = \frac{{}_1F_1\left(\alpha;\alpha+1;\lambda t-\frac{1}{\beta}\right)}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}+\lambda\right)} . \tag{7.1.20}$$

(The distribution with p.g.f. (7.1.9) has been studied by Kemp (1968b).)

7.1.5 An Interesting Relation Between $G_Y(t)$ and $G_{Y|X=Y}(t)$ in the Case where $(Y|X=Y) \sim$ Binomial $\wedge$ Right Truncated Gamma.

A relation between the p.g.f's of Y and $Y|X=Y$ can be obtained by observing that $G_Y(t+1)$ can be written (from 7.1.19) as

$$G_Y(t+1) = \frac{{}_1F_1\left(\alpha;\alpha+1;\lambda t-\frac{1}{\beta}\right)}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}\right)}$$

$$= \frac{{}_1F_1\left(\alpha;\alpha+1;\lambda t-\frac{1}{\beta}\right)}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}\right)} \quad \frac{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}+\lambda\right)}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}+\lambda\right)}$$

$$= G_{Y|X=Y}(t) \frac{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}+\lambda\right)}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}\right)} \text{, (see (7.1.20))} . \qquad (7.1.21)$$

But from (7.1.20) we also have,

$$\frac{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}\right)}{{}_1F_1\left(\alpha;\alpha+1;-\frac{1}{\beta}+\lambda\right)} = G_{Y|X=Y}(0) . \qquad (7.1.22)$$

Combining (7.1.21) and (7.1.22) gives

$$G_Y(t+1) = \frac{G_{Y|X=Y}(t)}{G_{Y|X=Y}(0)} \qquad (7.1.23)$$

i.e. by adopting the idea of the factorial moment generating function,

$$M_{[Y]}(t) = C\, G_{Y|X=Y}(t) \qquad (7.1.24)$$

with $C^{-1} = G_{Y|X=Y}(0) = \text{constant}$.

7.1.6 Some Examples in the Case where the Distribution of Y|X is Binomial Mixed with a Discrete Distribution.

(1) Let us suppose that $P(Y=r|X=n) = \binom{n}{r} p^r q^{n-r}$ where p can take two values: p_1 with probability α and p_2 with probability (1-a). (0 < a < 1)

Then we have

$$M_p(\theta) = \alpha e^{p_1\theta} + (1-\alpha)\, e^{p_2\theta} . \qquad (7.1.25)$$

Hence, from (7.1.2) and (7.1.4)

$$G_Y(t) = \alpha e^{\lambda p_1 (t-1)} + (1-\alpha)\, e^{\lambda p_2 (t-1)} \qquad (7.1.26)$$

and

$$G_{Y|X=Y}(t) = \frac{\alpha\, e^{\lambda p_1 t} + (1-\alpha) e^{\lambda p_2 t}}{\alpha\, e^{\lambda p_1} + (1-\alpha) e^{\lambda p_2}} . \qquad (7.1.27)$$

(2) Let us now assume that p is distributed in the following way

$$P\left(p = \frac{k}{n}\right) = \binom{n}{k} \alpha^k\, (1-\alpha)^{n-k} \qquad 0 < \alpha < 1. \qquad (7.1.28)$$

Then

$$M_p(\theta) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k\, (1-\alpha)^{n-k} e^{\theta k/n}$$

i.e.

$$M_p(\theta) = (1-\alpha+\alpha e^{\theta/n})^n .$$

Consequently,

$$G_Y(t) = \left[1-\alpha+\alpha e^{\lambda/n\,(t-1)}\right]^n \qquad (7.1.29)$$

and

$$G_{Y|X=Y} = \frac{(1-\alpha+\alpha e^{\lambda/nt})^n}{(1-\alpha+\alpha e^{\lambda/n})^n} . \qquad (7.1.30)$$

7.2 General Relations Between $G_Y(t)$ and $G_{Y|X=Y}(t)$ when X is Poisson and Y|X is Mixed Binomial.

In part 7.1.5 of the previous section we found a relation between $G_Y(t)$ and $G_{Y|X=Y}(t)$ that exists when X is Poisson and Y|X is Binomial ^ Right truncated Gamma. It can however be seen that in the case where the distribution of X is Poisson and Y|X is Mixed Binomial, $G_Y(t)$ can always be expressed in terms of $G_{Y|X=Y}(t)$ in a way that remains unchanged whatever the mixing distribution is. This is established in the following theorem.

Theorem 7.2.1

If X is Poisson and Y|X is Mixed Binomial, then

$$G_Y(t) = \frac{G_{Y|X=Y}(t-1)}{G_{Y|X=0}(0)} \tag{7.2.1}$$

and

$$G_{Y|X=Y}(t) = \frac{G_Y(t+1)}{G_Y(2)} \tag{7.2.2}$$

Proof

This is straightforward, if we consider the general forms that $G_Y(t)$, $G_{Y|X=Y}(t)$ have in the case under study; these are given in Section 7.1 ((7.1.2), (7.1.4)).

Remark

Rao's result (1963) is a special case of theorem 7.2.1 for F degenerate.

7.3 Damage Model with Original Distribution Mixed Poisson and Survival Distribution Binomial.

Let us now turn to the situation in which the distribution of X is Poisson with parameter λ, where λ is a variable taking values in an interval $(0,x)$, with $0 < x < \infty$. Let $F(\lambda)$ be the d.f. of λ.

Suppose also that $Y|X$ follows the Binomial probability law with parameters n,p. Denote by $G_Y^*(t)$ the p.g.f. of the resulting r.v., and by $G_{Y|X=Y}^*(t)$ the p.g.f. of the resulting r.v. in the case where no damage has occurred. Then following the same steps as in Section 7.1 we find that for any P_n

$$G_Y^*(t) = \int_0^\infty G_X(pt+q)dF(\lambda) \qquad (7.3.1)$$

and

$$G_{Y|X=Y}^*(t) = \frac{\int_0^\infty G_X(pt)dF(\lambda)}{\int_0^\infty G_X(p)dF(\lambda)} \qquad (7.3.2)$$

and in particular for P_n Poisson,

$$G_Y^*(t) = \int_0^\infty e^{\lambda p(t-1)}\, dF(\lambda) = M_\lambda\{p(t-1)\}, \quad 0 < \lambda < x, \quad 0 < x < \infty, \qquad (7.3.3)$$

and,

$$G_{Y|X=Y}^*(t) = \frac{\int_0^\infty e^{\lambda(pt-1)}\,dF(\lambda)}{\int_0^\infty e^{\lambda(p-1)}\,dF(\lambda)} = \frac{M_\lambda\{pt-1\}}{M_\lambda\{p-1\}} \quad 0 < \lambda < \infty. \qquad (7.3.4)$$

Comparing (7.3.3) with (7.1.2) and (7.3.4) with (7.1.4), one can make the following observations.

For those $F(\lambda)$, for which $0 < \lambda < x$ with $0 < x < 1$ one can arrive at $G_Y^*(t)$ just by interchanging λ and p in the corresponding expressions of $G_Y(t)$. As for $G_{Y|X=Y}^*(t)$, this can be derived from $G_{Y|X=Y}(t)$ by replacing λt with pt-1 and λ by p-1. This is so, because in the case where X is mixed, we integrate with respect to λ (the parameter of the distribution of X). So, while for $G_Y^*(t)$ we just have an interchange of the parameters λ and p, for $G_{Y|X=Y}^*(t), e^{-\lambda}$ is not cancelled from the nominator and the denominator of (7.1.4). The consequence is that the integration now gives pt-1 instead of λt in the nominator, and p-1 instead of λ in the denominator. By making use of this result one can obtain $G_Y^*(t)$ and $G_{Y|X=Y}^*(t)$ for the following distributions.

7.3.1 X ~ Poisson ^ Beta

Suppose that λ is defined in (0,1) and follows Beta distribution, as in (7.1.5).

Then, from (7.1.5) and (7.1.9) we get

$$G_Y^*(t) \quad = \quad {}_1F_1\{\alpha;\alpha+\beta;p(t-1)\} \tag{7.3.5}$$

and

$$G_{Y|X=Y}^*(t) \quad = \quad \frac{{}_1F_1\{\alpha;\alpha+\beta;pt-1\}}{{}_1F_1\{\alpha;\alpha+\beta;p-1\}} \,. \tag{7.3.6}$$

7.3.2 X ~ Poisson ^ Right Truncated Beta.

Let $\lambda \in (0,x)$ with $0 < x < 1$, and let λ be distributed according to Beta distribution truncated to the right at x as in (7.1.10).

Then, from (7.1.2) and (7.1.13) we get

$$G^*_Y(t) = \frac{\Phi_1\{\alpha,1-\beta;\alpha+1;x,px(t-1)\}}{{}_2F_1\{\alpha,1-\beta;\alpha+1;x\}} \tag{7.3.7}$$

and

$$G^*_{Y|X=Y}(t) = \frac{\Phi_1\{\alpha,1-\beta;\alpha+1;x,x(pt-1)\}}{\Phi_1\{\alpha,1-\beta;\alpha+1;x,x(p-1)\}} \tag{7.3.8}$$

7.3.3 X ~ Poisson ^ Exponential Truncated to the Right.

Let $0 < \lambda < 1$, and let $f(\lambda)$ be given by (7.1.15) (for $\mu=\theta$).

Then, (7.1.16) and (7.1.17) give

$$G^*_Y(t) = \frac{1/\theta \left[e^{pt-\frac{1}{\theta}-p} - 1\right]}{\left(pt-\frac{1}{\theta}-p\right)\left(1 - e^{-\frac{1}{\theta}}\right)} \tag{7.3.9}$$

and

$$G^*_{Y|X=Y}(t) = \frac{\left(p-\frac{1}{\theta}-1\right)\left[e^{pt-\frac{1}{\theta}-1} - 1\right]}{\left(pt-\frac{1}{\theta}-1\right)\left(e^{p-\frac{1}{\theta}-1} - 1\right)}\,. \tag{7.3.10}$$

7.3.4 X ~ Poisson ^ Gamma Truncated at 1.

Now assume that λ takes values in (0,1) and that it is distributed as Gamma Truncated at 1 (see (7.1.18)).

Then, from (7.1.19) and (7.1.20)

$$G_Y^*(t) = \frac{{}_1F_1\{\alpha;\alpha+1;p(t-1)-\frac{1}{\beta}\}}{{}_1F_1\{\alpha;\alpha+1;-\frac{1}{\beta}\}} \tag{7.3.11}$$

and

$$G_{Y|X=Y}^*(t) = \frac{{}_1F_1\{\alpha;\alpha+1;pt-\frac{1}{\beta}-1\}}{{}_1F_1\{\alpha;\alpha+1;p-\frac{1}{\beta}-1\}} \;. \tag{7.3.12}$$

We next examine two other interesting cases using (7.3.3) and (7.3.4).

7.3.5 X ~ Geometric (Poisson ^ Exponential)

For the p.g.f. of X it is known that

$$G_X^*(t) = \frac{1}{\theta}\int_0^\infty e^{\lambda(t-1)}\, e^{-\frac{\lambda}{\theta}}\, d\lambda \qquad 0 < \lambda < \infty, \quad \theta > 0$$

$$\left[\text{for} \quad f(\lambda) = \frac{1}{\theta}\, e^{-\frac{\lambda}{\theta}} \quad 0 < \lambda < \infty\right].$$

Hence,

$$G_X^*(t) = \frac{\frac{1}{1+\theta}}{1 - \frac{\theta}{1+\theta}\, t} \;. \tag{7.3.13}$$

Using (7.3.3) and (7.3.4) we can obtain

$$G_Y^*(t) = \frac{1}{\theta}\int_0^\infty e^{\lambda p(t-1)}\, e^{-\frac{\lambda}{\theta}}\, d\lambda = \frac{1}{1+\theta p-\theta pt} = \frac{\frac{1}{1+\theta p}}{1-\frac{\theta p}{1+\theta p}\, t} \tag{7.3.14}$$

and

$$G^*_{Y|X=Y}(t) = \frac{1 - \frac{\theta p}{1+\theta}}{1 - \frac{\theta p}{1+\theta} t} . \tag{7.3.15}$$

From (7.3.13), (7.3.14) and (7.3.15) it is obvious that $G^*_Y(t)$ and $G^*_{Y|X=Y}(t)$ are also geometric distributions, with a change in the parameter.

7.3.6 X ~ Negative Binomial (Poisson ^ Gamma)

Here we have

$$G^*_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{\lambda(t-1)} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}} d\lambda$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\frac{\lambda[1-\beta(t-1)]}{\beta}} \lambda^{\alpha-1} d\lambda$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)[1-\beta(t-1)]^\alpha}$$

$$\times \int_0^\infty e^{-\lambda[1-\beta(t-1)]} \{\lambda[1-\beta(t-1)]\}^{\alpha-1} d\{\lambda[1-\beta(t-1)]\}$$

$$= \{1-\beta(t-1)\}^{-\alpha} = \left\{\frac{1}{1+\beta-\beta t}\right\}^\alpha ,$$

i.e.

$$G^*_X(t) = \left\{\frac{\frac{1}{1+\beta}}{1 - \frac{\beta t}{1+\beta}}\right\}^\alpha . \tag{7.3.16}$$

Similarly from (7.3.3)

$$G_Y^*(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{\lambda p(t-1)} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}} d\lambda$$

$$= \left\{ \frac{\frac{1}{1+p\beta}}{1 - \frac{p\beta t}{1+p\beta}} \right\}^\alpha , \tag{7.3.17}$$

and from (7.3.4)

$$G^*_{Y|X=Y}(t) = \frac{\int_0^\infty e^{\lambda(pt-1)} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}} d\lambda}{\int_0^\infty e^{\lambda(p-1)} \cdot \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}} d\lambda}$$

which eventually becomes

$$G^*_{Y|X=Y}(t) = \left(1 - \frac{\beta p}{\beta+1}\right)^\alpha \left(1 - \frac{\beta p}{1+\beta} t\right)^{-\alpha} . \tag{7.3.18}$$

It can be observed again that $G_Y^*(t)$ and $G^*_{Y|X=Y}(t)$ are also negative binomials with the same shape parameter α, but different p and with β and p confounded.

This is a particular case of the following more general property which is possessed by this particular form of the damage model.

<u>Theorem 7.3.1</u>

If the distribution of X is mixed Poisson and that of $Y|X$ is Binomial, then Y and $Y|X=Y$ are also mixed Poisson.

Proof Let

$$X \sim \text{Poisson}(\lambda) \wedge F(\lambda)$$

Then from (7.3.3) it follows that

$$Y \sim \text{Poisson}(\lambda p) \wedge F(\lambda).$$

Also from (7.3.4) the distribution of $Y|X=Y$ can be viewed as Poisson $(\lambda p) \wedge F^*(\lambda)$, where

$$F^*(\lambda) = \frac{\int_0^{\lambda} e^{\lambda'(P-1)}\, dF(\lambda')}{\int_0^{\infty} e^{\lambda'(P-1)}\, dF(\lambda')} \tag{7.3.19}$$

7.3.7 An Example with a Discrete Mixing Distribution

Suppose that $X \sim$ Poisson with parameter λ, and λ takes the values $k,k+1,\ldots;k=1,2,\ldots$ with probabilities

$$g_\lambda = \binom{\lambda-1}{k-1} \alpha^k (1-\alpha)^{\lambda-k}, \quad \lambda=k,k+1,\ldots,\ k=1,2,\ldots \quad 0<\alpha<1, \tag{7.3.20}$$

i.e. Pascal with parameters α and k.

Then,

$$M_\lambda(\theta) = \sum_{\lambda=k}^{\infty} e^{\lambda\theta} \binom{\lambda-1}{k-1} \alpha^k (1-\alpha)^{\lambda-k}$$

$$= \sum_{\lambda=k}^{\infty} \binom{\lambda-1}{k-1} \{\alpha\, e^{\theta}\}^k \{(1-\alpha)\, e^{\theta}\}^{\lambda-k}$$

$$= \{\alpha\, e^{\theta}\}^k \sum_{\lambda=k}^{\infty} \binom{\lambda-1}{k-1} \{(1-\alpha)\, e^{\theta}\}^{\lambda-k}$$

$$= \{\alpha\, e^{\theta}\}^k \{1-(1-\alpha)\, e^{\theta}\}^{-k} = \left\{\frac{\alpha\, e^{\theta}}{1-(1-\alpha)\, e^{\theta}}\right\}^k. \tag{7.3.21}$$

Hence

$$G_Y^*(t) = M_\lambda(p(t-1)) = \left\{\frac{\alpha\ e^{p(t-1)}}{1-(1-\alpha)\ e^{p(t-1)}}\right\}^k . \tag{7.3.22}$$

Relation (7.3.22) shows that the p.g.f. of the resulting random variable is Pascal (α,k), generalized with Poisson (p).

Note. The result obtained in the previous section can be viewed as a particular application of a more general result, which can be stated as follows.

Theorem 7.3.2

Suppose that the original r.v. X follows a Poisson distribution with parameter λ, with λ itself having a distribution with p.g.f. of the form $\{g(t)\}^k$. Suppose also that the conditional distribution of $Y|X$ is Binomial (p). Then, the resulting r.v. Y will have the distribution of λ, generalized with a Poisson distribution with parameter p, i.e.

$$G_Y^*(t) = \{g(e^{p(t-1)})\}^k .$$

Proof

The result follows immediately from (7.3.3) and the well-known fact that $M_\lambda(t) = G_\lambda(e^t)$.

7.4 A General Relation Between $G_Y^*(t)$ and $G_{Y|X=Y}^*(t)$ when X is Mixed Poisson and $Y|X$ is Binomial.

In Section 7.2 we found a relation which enabled us to obtain the p.g.f. of Y in terms of $Y|X=Y$ and vice versa, whenever X was Poisson and $Y|X$ was mixed Binomial.

In fact, we can give similar relations for the case where X is mixed Poisson and $Y|X$ is Binomial.

Theorem 7.4.1

If X is Mixed Poisson and $Y|X$ is Binomial, then

$$G^*_Y(t) = \frac{G^*_{Y|X=Y}\left(t + \frac{q}{p}\right)}{G^*_{Y|X=Y}\left(\frac{1}{p}\right)} \tag{7.4.1}$$

and

$$G^*_{Y|X=Y}(t) = \frac{G^*_Y\left(t - \frac{q}{p}\right)}{G^*_Y\left(1 - \frac{q}{p}\right)} \tag{7.4.2}$$

Proof

The proof follows immediately from (7.3.3) and (7.3.4).

In the special cases, examined in Sections 7.3.5 and 7.3.6, where the distribution of X is Geometric and Negative Binomial, respectively, the following theorem can be established.

Theorem 7.4.2

If we denote by $G^*(t,\theta)$ the p.g.f. of the distribution of X, when X is Negative Binomial, then

$$G^*_Y(t,\theta) = G^*(t,p\theta) \tag{7.4.3}$$

$$G^*_{Y|X=Y}(t,\theta) = G^*\left(t, \frac{\theta p}{1+p(1-\theta)}\right) \tag{7.4.4}$$

$$G^*_Y(t,\theta) = G^*_{Y|X=Y}\left(t, \frac{\theta}{1-\theta p}\right) \tag{7.4.5}$$

Proof

This is straightforward.