CHAPTER 6.

CHARACTERIZATIONS OF BIVARIATE AND MULTIVARIATE FINITE DISTRIBUTIONS

6.0 Introduction

In Chapter 5 we derived a general result characterizing a class of finite distributions, as well as an extension in the truncated case. In this Chapter we extend the results of Chapter 5 to the Bivariate and Multivariate cases.

6.1 The Bivariate Extension of Theorem 5.1.1

Theorem 6.1.1

Let $\{X_1, X_2, Y_1, Y_2\}$ be a random vector with non-negative, integer-valued components such that

$$P\{X_1=n_1, X_2=n_2\} = P_{n_1,n_2} \qquad \begin{array}{l} n_1=0,1,\ldots,N_1 \\ n_2=0,1,\ldots,N_2 \\ N_1,N_2=0,1,\ldots \end{array}$$

with $P_{n_1,n_2} > 0$ for $0 \leqslant n_i \leqslant \ell_i$, $i=1,2$; ℓ_i fixed, $1 \leqslant \ell_i \leqslant N_i - m_i$

Suppose that $\left\{\left(a_{n_1,n_2}; b_{n_1,n_2}\right) n_1,n_2=0,1,\ldots\right\}$ is a sequence of real vectors with

$$\begin{array}{lll} a_{n_1,n_2} > 0 \text{ if} & 0 \leqslant n_i \leqslant m_i & i=1,2 \\ a_{n_1,n_2} = 0 \text{ if} & m_i < n_i \leqslant N_i & \text{for some } i=1,2. \end{array} \tag{6.1.1}$$

and

$$b_{n_1,n_2} > 0 \quad \text{if} \quad 0 \leq n_i \leq \ell_i \qquad i=1,2. \tag{6.1.2}$$

Define by $\{c_{n_1,n_2}\}$ the following sequence

$$c_{n_1,n_2} = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} a_{r_1,r_2} b_{n_1-r_1,n_2-r_2} \qquad n_i=0,1,\ldots,N_i \quad i=1,2. \tag{6.1.3}$$

Assume also that the sequences (6.1.1), (6.1.2), (6.1.3) are such that whenever $P_{n_1,n_2} > 0$

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=n_1, X_2=n_2) = \frac{a_{r_1,r_2} b_{n_1-r_1,n_2-r_2}}{c_{n_1,n_2}}. \tag{6.1.4}$$

Then the condition

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2=Y_2) = P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1+i_1, X_2=Y_2+i_2)$$

$$i_j=1,2,\ldots,\ell_j \ \text{ for fixed } \ell_j : 1 \leq \ell_j \leq N_j - m_j \,;\ j=1,2 \tag{6.1.5}$$

(i.e. there are $\ell_1 \times \ell_2$ equations)

is necessary and sufficient for P_{n_1,n_2} to be such that

$$P_{n_1,n_2} = \begin{cases} P_{0,0} \dfrac{c_{n_1,n_2}}{c_{0,0}} \theta_1^{n_1} \theta_2^{n_2} & \text{for some } \theta_1 > 0,\ \theta_2 > 0 \\ & \text{if } n_i \leq m_i + \ell_i \text{ for } i=1,2 \\ d_{n_1,n_2} & \text{if } m_i + \ell_i < n_i \leq N_i \text{ for either } i=1 \text{ or } i=2 \end{cases} \tag{6.1.6}$$

where d_{n_1,n_2} are arbitrary constants independent of n such that

$$\sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} P_{n_1,n_2} = 1$$

Proof

Necessity is a straightforward extension to the bivariate case of the if part of Theorem 5.1.1. To prove sufficiency we introduce and prove two lemmata. Then using these we establish the main result of the theorem.

Firstly we note that (6.1.5) can be written as

$$\begin{aligned}
&P(Y_1=r_1,Y_2=r_2|X_1=Y_1,X_2=Y_2) = \dots = P(Y_1=r_1,Y_2=r_2|X_1=Y_1+\ell_1,X_2=Y_2) = \\
&P(Y_1=r_1,Y_2=r_2|X_1=Y_1,X_2=Y_2+1) = \dots = P(Y_1=r_1,Y_2=r_2|X_1=Y_1+\ell_1, X_2=Y_2+1) = \\
&\qquad\qquad = \dots = \qquad\qquad = \\
&\qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \\
&P(Y_1=r_1,Y_2=r_2|X_1=Y_1,X_2=Y_2+\ell_2) = \dots = P(Y_1=r_1,Y_2=r_2|X_1=Y_1+\ell_1,X_2=Y_2+\ell_2).
\end{aligned} \tag{6.1.7}$$

With the help of (6.1.4), each of the probabilities in (6.1.7) can be expressed in the following manner:

$$\left.\begin{aligned}
P(Y_1=r_1,Y_2=r_2|X_1=Y_1,X_2=Y_2) &= \frac{\frac{P_{r_1,r_2}}{c_{r_1,r_2}}\, a_{r_1,r_2}\, b_{0,0}}{P(X_1=Y_1,X_2=Y_2)} && (1)\\
P(Y_1=r_1,Y_2=r_2|X_1=Y_1+1,X_2=Y_2) &= \frac{\frac{P_{r_1+1,r_2}}{c_{r_1+1,r_2}}\, a_{r_1,r_2}\, b_{1,0}}{P(X_1=Y_1+1,X_2=Y_2)} && (2)\\
&\ \vdots && \vdots\\
P(Y_1=r_1,Y_2=r_2|X_1=Y_1+\ell_1,X_2=Y_2) &= \frac{\frac{P_{r_1+\ell_1,r_2}}{c_{r_1+\ell_1,r_2}}\, a_{r_1,r_2}\, b_{\ell_1,0}}{P(X_1=Y_1+\ell_1,X_2=Y_2)} && (\ell_1)
\end{aligned}\right\} \quad (6.1.8$$

(for the first set of probabilities (6.1.7))

and

$$\left.\begin{aligned}
P(Y_1=r_1,Y_2=r_2|X_1=Y_1,X_2=Y_2+1) &= \frac{\frac{P_{r_1,r_2+1}}{c_{r_1,r_2+1}}\, a_{r_1,r_2}\, b_{0,1}}{P(X_1=Y_1,X_2=Y_2+1)} && (1)\\
&\ \vdots && \vdots\\
P(Y_1=r_1,Y_2=r_2|X_1=Y_1+\ell_1,X_2=Y_2+1) &= \frac{\frac{P_{r_1+\ell_1,r_2+1}}{c_{r_1+\ell_1,r_2+1}}\, a_{r_1,r_2}\, b_{\ell_1,1}}{P(X_1=Y_1+\ell_1,X_2=Y_2+1)} && (\ell_1)
\end{aligned}\right\} \quad (6.1.9$$

(for the second set of probabilities in (6.1.7))

and so on. Finally,

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2=Y_2+\ell_2) = \frac{\frac{P_{r_1, r_2+\ell_2}}{c_{r_1, r_2+\ell_2}} a_{r_1,r_2} b_{0,\ell_2}}{P(X_1=Y_1, X_2=Y_2+\ell_2)} \quad (1)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad (6.1.10)$$

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1+\ell_1, X_2=Y_2+\ell_2) = \frac{\frac{P_{r_1+\ell_1, r_2+\ell_2}}{c_{r_1+\ell_1, r_2+\ell_2}} a_{r_1,r_2} b_{\ell_1,\ell_2}}{P(X_1=Y_1+\ell_1, X_2=Y_2+\ell_2)} \quad (\ell_1)$$

In other words, instead of (6.1.8), (6.1.9), (6.1.10) we can write in general,

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1+i_1, X_2=Y_2+i_2) = \frac{\frac{P_{r_1+i_1, r_2+i_2}}{c_{r_1+i_1, r_2+i_2}} a_{r_1,r_2} b_{i_1,i_2}}{P(X_1=Y_1+i_1, X_2=Y_2+i_2)}$$

$$i_1=0,1,\ldots,\ell_1 \qquad (6.1.11)$$
$$i_2=0,1,\ldots,\ell_2$$

Lemma 6.1.1

If (6.1.5) is true, then

$$\frac{P_{n_1,n_2}}{c_{n_1,n_2}} = \frac{P_{0,n_2}}{c_{0,n_2}} \theta_1^{n_1} \qquad n_1=0,1,\ldots,m_1+\ell_1, \quad n_2=0,1,\ldots,m_2 \qquad (6.1.12)$$

Proof of Lemma 6.1.1

Obviously (6.1.12) is true for $n_1=0$.

On the other hand, because of (6.1.7) all the probabilities given by the system (6.1.8) are equal. This implies that the first set of equations of (6.1.7) is equivalent to the following system

$$\left.\begin{aligned}
\frac{P_{r_1+1,r_2}}{c_{r_1+1,r_2}} &= \frac{P_{r_1,r_2}}{c_{r_1,r_2}}\,\theta_{1,1} && (1)\\
\frac{P_{r_1+2,r_2}}{c_{r_1+2,r_2}} &= \frac{P_{r_1+1,r_2}}{c_{r_1+1,r_2}}\,\theta_{1,2} && (2)\\
&\vdots && \vdots\\
\frac{P_{r_1+\ell_1,r_2}}{c_{r_1+\ell_1,r_2}} &= \frac{P_{r_1+\ell_1-1,r_2}}{c_{r_1+\ell_1-1,r_2}}\,\theta_{1,\ell_1} && (\ell_1)
\end{aligned}\right\} \qquad (6.1.13)$$

with

$$\theta_{1,j} = \frac{P(X_1=Y_1+j, X_2=Y_2)}{P(X_1=Y_1+j-1, X_2=Y_2)}\;\frac{b_{j-1,0}}{b_{j,0}} \qquad j=1,\ldots,\ell_1 .$$

It can be observed however that

$$\theta_{1,1} = \theta_{1,2} = \ldots = \theta_{1,\ell_1} \equiv \theta_1$$

(For example (6.1.13) (1) for $r_1=2$ becomes $\frac{P_{3,r_2}}{c_{3,r_2}} = \theta_{1,1}\,\frac{P_{2,r_2}}{c_{2,r_2}}$, and (6.1.13) (2) for $r_1=1$ becomes $\frac{P_{3,r_2}}{c_{3,r_2}} = \theta_{1,2}\,\frac{P_{2,r_2}}{c_{2,r_2}}$.

Hence $\theta_{1,1} = \theta_{1,2}$. The remaining equalities can be proved similarly.)

(6.1.13) (1) gives

for $r_1=0$, $$\frac{P_{1,r_2}}{c_{1,r_2}} = \theta_1 \frac{P_{0,r_2}}{c_{0,r_2}}$$

for $r_1=1$, $$\frac{P_{2,r_2}}{c_{2,r_2}} = \theta_1 \frac{P_{1,r_2}}{c_{1,r_2}}$$

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and for $r_1=r_1$, $$\frac{P_{r_1+1,r_2}}{c_{r_1+1,r_2}} = \theta_1 \frac{P_{r_1,r_2}}{c_{r_1,r_2}} \qquad r_1 \leq m_1 .$$

Consequently,

$$\frac{P_{r_1+1,r_2}}{c_{r_1+1,r_2}} = \frac{P_{0,r_2}}{c_{0,r_2}} \theta_1^{r_1+1} \qquad r_1=0,1,\ldots,m_1 . \tag{6.1.14}$$

(6.1.14) implies that (6.1.12) is valid for $n_1=0,1,\ldots,m_1+1$. In the same way we find from (6.1.13) (2) that

$$\frac{P_{r_1+2,r_2}}{c_{r_1+2,r_2}} = \theta_1 \frac{P_{r_1+1,r_2}}{c_{r_1+1,r_2}} = \theta_1^{r_1+2} \frac{P_{0,r_2}}{c_{0,r_2}} , \quad r_1=0,1,\ldots,m_1 . \tag{6.1.15}$$

(using (6.1.14)).

For $r_1=m_1$ (6.1.15) implies that (6.1.12) is also valid for $n_1=m_1+2$.

Continuing in the same way, we will finally get from (6.1.13) (ℓ_1) using the results of the previous relations that

$$\frac{P_{r_1+\ell_1,r_2}}{c_{r_1+\ell_1,r_2}} = \frac{P_{0,r_2}}{c_{0,r_2}} \theta_1^{r_1+\ell_1} \qquad r_1=0,\ldots,m_1 .$$

This for $r_1 = m_1$ becomes

$$\frac{P_{m_1+\ell_1, r_2}}{c_{m_1+\ell_1, r_2}} = \frac{P_{0,r_2}}{c_{0,r_2}} \theta_1^{m_1+\ell_1} \qquad r_2 = 0,1,\ldots,m_2 .$$

This completes the proof of Lemma 6.1.1.

Lemma 6.1.2

If (6.1.5) is true then

$$\frac{P_{n_1,n_2}}{c_{n_1,n_2}} = \frac{P_{0,n_2}}{c_{0,n_2}} \theta_1^{n_1} \qquad \begin{array}{l} n_1 = 0,1,\ldots,m_1+\ell_1 \\ n_2 = 0,1,\ldots,m_2+\ell_2 \end{array} \text{ for some } \theta_1 > 0 . \tag{6.1.16}$$

Proof of Lemma 6.1.2

By means of (6.1.8) the second set of equations of (6.1.7) gives

$$\left.\begin{array}{ll} \dfrac{P_{r_1+1,r_2+1}}{c_{r_1+1,r_2+1}} = \theta'_{1,1} \dfrac{P_{r_1,r_2+1}}{c_{r_1,r_2+1}} & (1) \\ \dfrac{P_{r_1+2,r_2+1}}{c_{r_1+2,r_2+1}} = \theta'_{1,2} \dfrac{P_{r_1+1,r_2+1}}{c_{r_1+1,r_2+1}} & (2) \\ \vdots & \vdots \\ \dfrac{P_{r_1+\ell_1,r_2+1}}{c_{r_1+\ell_1,r_2+1}} = \theta'_{1,\ell_1} \dfrac{P_{r_1+\ell_1-1,r_2+1}}{c_{r_1+\ell_1-1,r_2+1}} & (\ell_1) \end{array}\right\} \tag{6.1.17}$$

With the same kind of argument as before, it can be shown that

$$\theta'_{1,1} = \theta'_{1,2} = \ldots = \theta'_{1,\ell_1} \equiv \theta'_1 .$$

Considering also (6.1.13) (1) for $r_2=1$ and (6.1.17) (1) for $r_2=0$ we can see that $\theta_1' = \theta_1$. So, (6.1.17) gives,

$$\text{for } r_1=0, \quad \frac{P_{1,r_2+1}}{c_{1,r_2+1}} = \frac{P_{0,r_2+1}}{c_{0,r_2+1}}\,\theta_1,$$

$$\text{for } r_1=1, \quad \frac{P_{2,r_2+1}}{c_{2,r_2+1}} = \frac{P_{1,r_2+1}}{c_{1,r_2+1}}\,\theta_1$$

$$\vdots$$

$$\text{and for } r_1=r_1 \quad \frac{P_{r_1+1,r_2+1}}{c_{r_1+1,r_2+1}} = \frac{P_{r_1,r_2+1}}{c_{r_1,r_2+1}}\,\theta_1, \qquad r_1 \leqslant m_1.$$

The above equations give

$$\frac{P_{r_1+1,r_2+1}}{c_{r_1+1,r_2+1}} = \frac{P_{0,r_2+1}}{c_{0,r_2+1}}\,\theta_1^{r_1+1} \qquad \begin{array}{l} r_1=0,1,\ldots,m_1 \\ r_2=0,\ldots,m_2. \end{array} \tag{6.1.18}$$

Using the same technique, we can obtain relationships similar to (6.1.18) with r_1+1 replaced by $r_1+2,\ldots,r_1+\ell_1-1$.

Finally, the (ℓ_1) equation of (6.1.17) gives

$$\frac{P_{r_1+\ell_1,r_2+1}}{c_{r_1+\ell_1,r_2+1}} = \frac{P_{0,r_2+1}}{c_{0,r_2+1}}\,\theta_1^{r_1+\ell_1}\, r_1=0,\ldots,m_1.$$

This for $r_1=m_1$ becomes

$$\frac{P_{m_1+\ell_1,r_2+1}}{c_{m_1+\ell_1,r_2+1}} = \theta_1^{m_1+\ell_1}\,\frac{P_{0,r_2+1}}{c_{0,r_2+1}}. \tag{6.1.19}$$

Hence we come to the conclusion that

$$\frac{P_{n_1,r_2+1}}{c_{n_1,r_2+1}} = \frac{P_{0,r_2+1}}{c_{0,r_2+1}} \theta_1^{n_1} \qquad \begin{array}{l} r_2=0,\ldots,m_2 \\ n_1=0,\ldots,m_1+\ell_1 . \end{array} \tag{6.1.20}$$

(6.1.20) shows that Lemma 6.1.1 is also valid for $n_2=m_2+1$. Using the same approach, the last set of equations of (6.1.7) will give (using (6.1.10))

$$\frac{P_{n_1,r_2+\ell_2}}{c_{n_1,r_2+\ell_2}} = \frac{P_{0,r_2+\ell_2}}{c_{0,r_2+\ell_2}} \theta_1^{n_1} \qquad r_2=0,\ldots,m_2 .$$

This for $r_2=m_2$ becomes

$$\frac{P_{n_1,m_2+\ell_2}}{c_{n_1,m_2+\ell_2}} = \frac{P_{0,m_2+\ell_2}}{c_{0,m_2+\ell_2}} \theta_1^{n_1} .$$

Hence

$$\frac{P_{n_1,n_2}}{c_{n_1,n_2}} = \frac{P_{0,n_2}}{c_{0,n_2}} \theta_1^{n_1} \qquad \begin{array}{l} n_1=0,\ldots,m_1+\ell_1 \\ n_2=0,\ldots,m_2+\ell_2 . \end{array}$$

This completes the proof of Lemma (6.1.2).

<u>Proof of the Main Theorem 5.1.1</u>

If we look back at (6.1.7) we can see that we have not yet used the fact that

$$\begin{aligned} P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1,X_2=Y_2) &= P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1,X_2=Y_2+1) \\ = \ldots\ldots &= P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1,X_2=Y_2+\ell_2). \end{aligned} \tag{6.1.21}$$

From

$$P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1,X_2=Y_2) = P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1,X_2=Y_2+1)$$

we get

$$\frac{P_{r_1,r_2+1}}{c_{r_1,r_2+1}} = \frac{P_{r_1,r_2}}{c_{r_1,r_2}} \theta_2 \qquad (6.1.22)$$

where $\theta_2 = \dfrac{P(X_1=Y_1,X_2=Y_2+1)}{P(X_1=Y_1,X_2=Y_2)} \dfrac{b_{1,0}}{b_{0,1}}$ i.e. θ_2 is a constant independent of r_1,r_2.

(6.1.22) gives the following equations for the different values of r_2 and $r_1=0$.

for $r_2=0$, $\dfrac{P_{0,1}}{c_{0,1}} = \dfrac{P_{0,0}}{c_{0,0}} \theta_2$,

for $r_2=1$, $\dfrac{P_{0,2}}{c_{0,2}} = \dfrac{P_{0,1}}{c_{0,1}} \theta_2$,

$\vdots$

and for $r_2=r_2$, $\dfrac{P_{0,r_2+1}}{c_{0,r_2+1}} = \dfrac{P_{0,r_2}}{c_{0,r_2}} \theta_2$, $r_2 \leqslant m_2$.

Hence

$$\frac{P_{0,r_2+1}}{c_{0,r_2+1}} = \frac{P_{0,0}}{c_{0,0}} \theta_2^{r_2+1} \qquad r_2=0,1,\ldots,m_2. \qquad (6.1.23)$$

This implies that

$$\frac{P_{0,n_2}}{c_{0,n_2}} = \frac{P_{0,0}}{c_{0,0}} \theta_2^{n_2} \quad \text{for } n_2=0,1,\ldots,m_2+1. \qquad (6.1.24)$$

Also, from

$$P(Y_1=r_1,Y_2=r_2|X_1=Y_1,X_2=Y_2+2) = P(Y_1=r_1,Y_2=r_2|X_1=Y_1,X_2=Y_2)$$

(which is the second equation of (6.1.21)) we find that

$$\frac{P_{r_1,r_2+2}}{c_{r_1,r_2+2}} = \frac{P_{r_1,r_2+1}}{c_{r_1,r_2+1}}\ \theta_2 \qquad \begin{matrix} r_1=0,\dots,m_1 \\ r_2=0,\dots,m_2. \end{matrix} \tag{6.1.25}$$

(It can be checked as before, that the constant in (6.1.25) is the same as that for (6.1.22)).

(6.1.25) gives for $r_1=0$ and for the different values of r_2.

$$\text{for } r_2=0, \quad \frac{P_{0,2}}{c_{0,2}} = \frac{P_{0,1}}{c_{0,1}}\ \theta_2,$$

$$\text{for } r_2=1, \quad \frac{P_{0,3}}{c_{0,3}} = \frac{P_{0,2}}{c_{0,2}}\ \theta_2,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\text{and for } r_2=r_2, \ \frac{P_{0,r_2+2}}{c_{0,r_2+2}} = \frac{P_{0,r_2+1}}{c_{0,r_2+1}}\ \theta_2, \ r_2 \leqslant m_2.$$

Consequently

$$\frac{P_{0,r_2+2}}{c_{0,r_2+2}} = \frac{P_{0,1}}{c_{0,1}}\ \theta_2^{r_2+1} = \frac{P_{0,0}}{c_{0,0}}\ \theta_2^{r_2+2}$$

$$r_2=0,1,\dots,m_2.$$

(We have taken into account (6.1.23).)

So, (6.1.24) is also true for $n_2=m_2+2$. From the last of the equations (6.1.21) (using all the previous ones) we find that

$$\frac{P_{0,r_2+\ell_2}}{c_{0,r_2+\ell_2}} = \frac{P_{0,0}}{c_{0,0}}\ \theta_2^{r_2+\ell_2} \qquad r_2=0,\dots,m_2.$$

Consequently (6.1.21) implies that

$$\frac{P_{0,n_2}}{c_{0,n_2}} = \frac{P_{0,0}}{c_{0,0}} \theta_2^{n_2} \qquad n_2=0,1,\ldots,m_2+\ell_2 \tag{6.1.26}$$

Combining (6.1.26) with the result of Lemma 6.1.2 we conclude that

$$\frac{P_{n_1,n_2}}{c_{n_1,n_2}} = \frac{P_{0,0}}{c_{0,0}} \theta_1^{n_1} \theta_2^{n_2} \qquad \begin{array}{l} n_1=0,\ldots,m_1+\ell_1 \\ n_2=0,\ldots,m_2+\ell_2 \end{array} \tag{6.1.27}$$

(6.1.27) gives the "sufficient" part of the theorem. Hence Theorem 6.1.1 is established.

6.2 Characterizations of Some Known Bivariate Distributions.

Theorem 6.1.1 provides the appropriate theory for the following characterizations.

Corollary 6.2.1 (Characterization of the Double Binomial)

Let (X_1, X_2, Y_1, Y_2) satisfy the conditions for Theorem 6.1.1. Suppose also that the conditional distribution of Y_1, Y_2 on X_1, X_2 is double Hypergeometric, i.e. suppose that

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=n_1, X_2=n_2) = \frac{\binom{m_1}{r_1}\binom{N_1-m_1}{n_1-r_1}}{\binom{N_1}{n_1}} \frac{\binom{m_2}{r_2}\binom{N_2-m_2}{n_2-r_2}}{\binom{N_2}{n_2}} \tag{6.2.1}$$

$$n_1, n_2 \; m_1, m_2, N_1, N_2 > 0. \qquad \begin{array}{l} r_1=0,1,\ldots,n_1 \\ r_2=0,1,\ldots,n_2. \end{array}$$

Then, the condition (6.1.5) for $\ell_j = N_j - m_j$; j=1,2 is necessary and sufficient for P_{n_1,n_2} to be double Binomial with probabilties

$$P_{n_1,n_2} = \binom{N_1}{n_1} p_1^{n_1} q_1^{N_1-n_1} \binom{N_2}{n_2} p_2^{n_2} q_2^{N_2-n_2} . \tag{6.2.2}$$

<u>Proof</u> Let a_{n_1,n_2} and b_{n_1,n_2} be the sequences

$$a_{n_1,n_2} = \binom{m_1}{n_1}\binom{m_2}{n_2} \text{ and } b_{n_1,n_2} = \binom{N_1-m_1}{n_1}\binom{N_2-m_2}{n_2} \tag{6.2.3}$$

$$n_i = 0,1,\dots,N_i \quad i=1,2$$

Then,

$$c_{n_1,n_2} = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} a_{r_1,r_2} b_{n_1-r_1,n_2-r_2} = \binom{N_1}{n_1}\binom{N_2}{n_2}. \tag{6.2.4}$$

Clearly a_{n_1,n_2}, b_{n_1,n_2}, c_{n_1,n_2} as given by (6.2.3), (6.2.4) meet all the conditions set by Theorem 6.1.1. Hence (6.2.1) can be expressed in the form (6.1.4). So, according to the conclusion of Theorem 6.1.1, the condition (6.1.5) with $\ell_j = N_j - m_j$; j=1,2 holds iff (6.1.6) is true. Moreover in this particular case (6.1.6) becomes

$$P_{n_1,n_2} = P_{0,0} \binom{N_1}{n_1}\binom{N_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} .$$

Clearly $P_{0,0}^{-1} = (1+\theta_1)^{N_1} (1+\theta_2)^{N_2}$.

Hence, P_{n_1,n_2} comes out to be double Binomial, as in (6.2.2), with θ_1, θ_2 such that $p_i = \frac{\theta_i}{1+\theta_i}$ i=1,2.

Corollary 6.2.2 (Characterization of the Double Hypergeometric)

Consider (X_1, X_2, Y_1, Y_2) as in Corollary (6.2.1).

Suppose that P_{n_1,n_2} is a double Binomial as given by (6.2.2). Let the conditional distribution of Y_1, Y_2 on X_1, X_2 be of the form (6.1.4).

Then, the condition (6.1.5) for $\ell_j = N_j - m_j$; $j=1,2$ is necessary and sufficient for $P(Y_1=r_1, Y_2=r_2 | X_1=n_1, X_2=n_2)$ to be double Hypergeometric as in (6.2.1).

To show the validity of Corollary (6.2.2) one needs the following Lemma.

Lemma 6.2.1 Let $G_1(t_1,t_2)$, $G_2(t_1,t_2)$ be the p.g.f.'s of two independent bivariate r.v.'s and let $G(t_1,t_2)$ denote their convolution, i.e.

$$G(t_1,t_2) = G_1(t_1,t_2) \times G_2(t_1,t_2) . \tag{6.2.5}$$

Suppose also that $G(t_1,t_2)$ is the p.g.f. of the bivariate distribution which is the product of two independent Binomials with parameters (p_1,m), (p_2,n), respectively, i.e.

$$G(t_1,t_2) = (p_1t_1+q_1)^m (p_2t_2+q_2)^n . \tag{6.2.6}$$

Then, $G_1(t_1,t_2)$, $G_2(t_1,t_2)$ are also p.g.f.'s of bivariate distributions of independent binomial variables, with the same set of probabilities (p_1,p_2), i.e.

$$G_1(t_1,t_2) = (p_1t_1+q_1)^{m_1} (p_2t_2+q_2)^{n_1}$$

$$G_2(t_1,t_2) = (p_1t_1+q_1)^{m_2} (p_2t_2+q_2)^{n_2}$$

$$m_1+m_2=m, \; n_1+n_2=n.$$

Proof of the lemma

From (6.2.5) and (6.2.6)

$$G_1(t_1,t_2)\, G_2(t_1,t_2) = (p_1t_1+q_1)^m (p_2t_2+q_2)^n \qquad (6.2.7)$$

which for $t_1=1$ becomes,

$$G_1(1,t_2)\, G_2(1,t_2) = (p_2t_2+q_2)^n. \qquad (6.2.8)$$

Dividing (6.2.7) by (6.2.8), we get

$$\frac{G_1(t_1,t_2)}{G_1(1,t_2)} \frac{G_2(t_1,t_2)}{G_2(1,t_2)} = (p_1t_1+q_1)^m. \qquad (6.2.9)$$

However $\frac{G_1(t_1,t_2)}{G_1(1,t_2)}$ and $\frac{G_2(t_1,t_2)}{G_2(1,t_2)}$ are valid p.g.f.'s in t_1.

Hence, since the lemma is true in the univariate case, (6.2.9) is equivalent to the following two relations

$$\frac{G_1(t_1,t_2)}{G_1(1,t_2)} = (p_1t_1+q_1)^{m_1} \text{ and } \frac{G_2(t_1,t_2)}{G_2(1,t_2)} = (p_2t_2+q_2)^{m_2} \qquad (6.2.10)$$

$$m_1+m_2=m.$$

Similarly, by considering (6.2.7) for $t_2=1$ we can prove that

$$\frac{G_1(t_1,t_2)}{G_1(t_1,1)} = (p_2t_2+q_2)^{n_1},\ \frac{G_2(t_1,t_2)}{G_2(t_1,1)} = (p_2t_2+q_2)^{n_2}. \qquad (6.2.11)$$

However, one can observe that the R.H.S. of the first of the two relations of (6.2.10) is independent of t_2. Hence, since the relation is valid for $t_2=1$ we have

$$\frac{G_1(t_1,1)}{G_1(1,1)} = (p_1 t_1 + q_1)^{m_1},$$

which gives

$$G_1(t_1,1) = (p_1 t_1 + q_1)^{m_1}. \tag{6.2.12}$$

Consequently, the first parts of relations (6.2.11) and (6.2.12) give

$$G_1(t_1,t_2) = (p_1 t_1 + q_1)^{m_1}(p_2 t_2 + q_2)^{n_1}. \tag{6.2.13}$$

In the same way, we find that

$$G_2(t_1,t_2) = (p_1 t_1 + q_1)^{m_2}(p_2 t_2 + q_2)^{n_2} \tag{6.2.14}$$

where $m_1 + m_2 = m$, $n_1 + n_2 = n$.

(6.2.13) and (6.2.14) complete the proof of the lemma.

Proof of Corollary 6.2.2

Going back to the proof of Corollary 6.2.2., one can see that "necessity" is a side result of Corollary 6.1.1. For "sufficiency" we make use of the fact that, as in Theorem 6.1.1, the condition (6.1.5) for $\ell_j = N_j - m_j$ is equivalent to (6.1.6); in this instance

$$c_{n_1,n_2} = c \binom{N_1}{n_1} \pi_1^{n_1} \phi_1^{N_1-n_1} \binom{N_2}{n_2} \pi_2^{n_2} \phi_2^{N_2-n_2} \tag{6.2.15}$$

where c is a constant.

Hence, applying the result of lemma 6.2.1, we can determine uniquely sequences a_{n_1,n_2}; b_{n_1,n_2}, which are of the same form as (6.2.15) with the same set of probabilities as (6.2.15) and which have as their convolution the sequence c_{n_1,n_2}. Since $P(Y_1, Y_2 | X_1, X_2)$ is of the form (6.1.4), the desired result can be derived by a straight substitution.

6.3 The Extension of Theorem 6.1.1 to the Truncated Case

The following theorem, provides a combined extension to the bivariate case of Theorem 5.3.1 and to what was said in Note 1 of Theorem 5.4.1.

Theorem 6.3.1

Let $\{X_1, X_1, Y_2, Y_2\}$ be a random vector with non-negative integer-valued components such that

$$P\{X_1=n_1, X_2=n_2\} = P_{n_1,n_2} \qquad n_i=k_i, k_i+1, \ldots, N_i \quad k_i > 0, \quad i=1,2 \tag{6.3.1}$$

and

$$P(X_1 \geq k_1, X_2 \geq k_2) = 1, \; P_{n_1,n_2} > 0 \text{ for } k_i \leq n_i \leq \ell_i + k_i, \quad \ell_i \text{ fixed}, \; 1 \leq \ell_i \leq N_i - m_i . \tag{6.3.2}$$

Suppose that $\left\{(a_{n_1,n_2}; b_{n_1,n_2}) \; n_i=0,1,\ldots,N_i \; ; \; i=1,2\right\}$ is a sequence of real, non-negative vectors with

$$\begin{aligned} a_{n_1,n_2} &> 0 \quad \text{if} \quad n_i=k_i, k_i+1, \ldots, m_i , \\ a_{n_1,n_2} &= 0 \quad \text{if} \quad m_i < n_i \leq N_i \\ & \qquad k_i \leq m_i \leq N_i \quad i=1,2 \end{aligned} \tag{6.3.3}$$

and

$$b_{n_1,n_2} > 0 \quad \text{if} \quad n_i=0,1,\ldots,\ell_i , \qquad i=1,2. \tag{6.3.4}$$

Let $\{c_{n_1,n_2}\}$ be a sequence defined as follows

$$c_{n_1,n_2} = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} a_{r_1,r_2} \, b_{n_1-r_1,\, n_2-r_2} .$$

(Obviously $c_{n_1,n_2} > 0$ at least for $n_1 \geq k_1$, $n_2 \geq k_2$.)

Suppose also, that whenever $P_{n_1,n_2} > 0$ we have

$$P(Y_1=r_1, Y_2=r_2 | X_1=n_1, X_2=n_2) = \frac{a_{r_1,r_2},\ b_{n_1-r_1,n_2-r_2}}{c_{n_1,n_2}} \tag{6.3.5}$$

$$r_1=0,1,\ldots,n_1,\ r_2=0,1,\ldots,n_2,\ n_1=k_1,k_1+1,\ldots,N_1,\ n_2=k_2,k_2+1,\ldots,N_2.$$

Then

$$P_{n_1,n_2} = \begin{cases} P_{k_1,k_2} \dfrac{c_{n_1,n_2}}{c_{k_1,k_2}} \theta_1^{n_1-k_1} \theta_2^{n_2-k_2} & \text{for some } \theta_i > 0 \quad i=1,2 \\ & \text{if } n_i=k_i, k_i+1,\ldots,m_i+\ell_i \\ d_{n_1,n_2} & \text{if } m_i+\ell_i < n_i \leq N_i \text{ for some } i,\ i=1,2. \end{cases} \tag{6.3.6}$$

(where d_{n_1,n_2} are arbitrary constants such that

$$\sum_{n_1=k_1}^{N_1} \sum_{n_2=k_2}^{N_2} P_{n_1,n_2} = 1,)$$

iff

$$P(Y_1=r_1, Y_2=r_2 | X_1=Y_1, X_2=Y_2) = P(Y_1=r_1, Y_2=r_2 | X_1=Y_1+i_1, X_2=Y_2+i_2, Y_1 \geq k_1, Y_2 \geq k_2)$$

$$i_j=1,2,\ldots,\ell_j \quad j=1,2 \tag{6.3.7}$$

for any fixed $\ell_j : 1 \leq \ell_j \leq N_j - m_j$; j=1,2.

<u>Proof</u>

A proof can be obtained by combining the techniques used in Theorem *6.1.1 (bivariate extension of Theorem 5.1.1)* and in *Theorem 5.3.1* (Truncated Univariate Case).

Various forms of the sequence $\{(a_n, b_n)\ n=0,1,\ldots\}$ yield the following corollaries.

Corollary 6.3.1 (Characterization of the Truncated Double Binomial)

Suppose that the conditional distribution of Y_1, Y_2 on X_1, X_2 is double Hypergeometric as in (6.2.1). Then the condition (6.3.7) for $\ell_i = N_i - m_i$ holds iff P_{n_1,n_2} have the truncated double Binomial form

$$P_{n_1,n_2} = \frac{\binom{n_1}{r_1} p_1^{r_1} q_1^{n_1-r_1}}{\sum_{x_1=k_1}^{n_1} \binom{n_1}{x_1} p_1^{x_1} q_1^{n_1-x_1}} \; \frac{\binom{n_2}{r_2} p_2^{r_2} q_2^{n_2-r_2}}{\sum_{x_2=k_2}^{n_2} \binom{n_2}{x_2} p_2^{x_2} q_2^{n_2-x_2}} \tag{6.3.8}$$

$$n_1 = r_1, \ldots, N_1$$
$$n_2 = r_2, \ldots, N_2 \ .$$

Proof

If we define the sequences $\{a_{n_1,n_2}\}$, $\{b_{n_1,n_2}\}$ as in (6.2.3) we can see that all the requirements of Theorem 6.3.1 are met. Hence, the result follows.

Corollary 6.3.2 (Characterization of the distribution which is the product of two independent r.v's each of which is the convolution of a Binomial with a truncated Binomial.)

Suppose that the distribution of $Y_1, Y_2 | X_1, X_2$ is truncated double Hypergeometric, i.e.

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=n_1, X_2=n_2) = \frac{\binom{m_1}{r_1}\binom{N_1-m_1}{n_1-r_1}}{\sum_{x_1=k_1}^{m_1}\binom{m_1}{x_1}\binom{N_1-m_1}{n_1-x_1}} \; \frac{\binom{m_2}{r_2}\binom{N_2-m_2}{n_2-r_2}}{\sum_{x_2=k_2}^{m_2}\binom{m_2}{x_2}\binom{N_2-m_2}{n_2-x_2}} \qquad (6.3.9)$$

$r_1=k_1, k_1+1, \ldots, m_1$, $r_2=k_2, k_2+1, \ldots, m_2$, $n_1=k_1, k_1+1, \ldots, N_1$, $n_2=k_2, k_2+1, \ldots, N_2$.

Then, condition (6.3.7) for $\ell_i=N_i-m_i$ $i=1,2$ holds iff

$$P_{n_1,n_2} = \frac{p_1^{n_1} q_1^{N_1-n_1} \sum_{r_1=k_1}^{n_1}\binom{m_1}{r_1}\binom{N_1-m_1}{n_1-r_1}}{I_{p_1}(k_1, m_1-k_1+1)} \; \frac{p_2^{n_2} q_2^{N_2-n_2} \sum_{r_2=k_2}^{n_2}\binom{m_2}{r_2}\binom{N_2-m_2}{n_2-r_2}}{I_{p_2}(k_2, m_2-k_2+1)} \;. \qquad (6.3.10)$$

<u>Proof</u>

This is a direct consequence of Theorem 6.3.1 with the sequences a_{n_1,n_2}, b_{n_1,n_2} defined as follows

$$a_{n_1,n_2} = \begin{cases} 0 \quad \text{if } n_i < k_i \text{ for some } i,\ i=1,2 \\[2ex] \dfrac{\binom{m_1}{r_1}\pi_1^{r_1}\phi_1^{m_1-r_1}}{\sum_{x_1=k_1}^{m_1}\binom{m_1}{x_1}\pi_1^{x_1}\phi_1^{m_1-x_1}} \; \dfrac{\binom{m_2}{r_2}\pi_2^{r_2}\phi_2^{m_2-r_2}}{\sum_{x_2=k_2}^{m_2}\binom{m_2}{x_2}\pi_2^{x_2}\phi_2^{m_2-x_2}} \end{cases} \qquad (6.3.11)$$

$n_1=k_1, k_1+1, \ldots, m_1$; $n_2=k_2, k_2+1, \ldots, m_2$.

and b_{n_1,n_2} as in (6.2.3).

<u>Note 1</u> An extension to the bivariate case of Theorem 5.4.1, based on a sequence $\{a_{n_1,n_2}\}$ defined for $n_1=k_1, k_1+1, \ldots$; $n_2=k_2, k_2+1, \ldots$, can also be derived, thus providing another way of arriving at Corollary 6.3.2.

6.4 The Multivariate Extension

(In this section we use the same notation as in Chapter 4.)

Theorem 6.4.1

Consider a random vector $(\underset{\sim}{X},\underset{\sim}{Y})$ where $\underset{\sim}{X} = (X_1, X_2, \ldots, X_s)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_s)$ having non-negative integer-valued components such that

$$P(\underset{\sim}{X}=\underset{\sim}{n}) = P_{\underset{\sim}{n}} \qquad \underset{\sim}{n} = (n_1, \ldots, n_s), \quad n_i = 0,1,\ldots,N_i,\ N_i > 0,\ i=1,2,\ldots,s, \tag{6.4.1}$$

(i.e. $P(X_1=n_1, \ldots, X_s=n_s) = P_{n_1,n_2,\ldots,n_s}$), with $P_{\underset{\sim}{n}} > 0$ for some $n_1 \leq \ell_1,\ n_2 \leq \ell_2, \ldots, n_s \leq \ell_s$, ℓ_i fixed, $1 \leq \ell_i \leq N_i - m_i$ $i=1,2,\ldots,s$.

Let $\{(a_{\underset{\sim}{n}}, b_{\underset{\sim}{n}}) : \underset{\sim}{n}=(n_1,\ldots,n_s);\ n_i=0,1,\ldots;\ i=1,2,\ldots,s\}$ be a sequence of real vectors with

$$\begin{aligned} a_{\underset{\sim}{n}} &> 0 \quad \text{if } n_i \leq m_i \quad i=1,2,\ldots,s \\ a_{\underset{\sim}{n}} &= 0 \quad \text{if } m_i < n_i \leq N_i \text{ for some } i=1,2,\ldots,s \\ b_{\underset{\sim}{n}} &> 0 \quad \text{if } n_i \leq \ell_i \quad i=1,2,\ldots,s. \end{aligned} \tag{6.4.2}$$

Define a sequence $\{c_{\underset{\sim}{n}}\}$ as follows

$$c_{\underset{\sim}{n}} = \sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} a_{\underset{\sim}{r}}\, b_{\underset{\sim}{n}-\underset{\sim}{r}}, \tag{6.4.3}$$

where $a_{\underset{\sim}{r}} = (a_{r_1,\ldots,r_s})$ and $\sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}}$ meaning $\sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \cdots \sum_{r_s=0}^{n_s}$.

Let us also assume that whenever $P_{\underset{\sim}{n}} > 0$ we have

$$P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,\underset{\sim}{X}=\underset{\sim}{n}) = \frac{a_{\underset{\sim}{r}}\, b_{\underset{\sim}{n}-\underset{\sim}{r}}}{c_{\underset{\sim}{n}}} \qquad r_i=0,1,\ldots,n_i \quad i=1,\ldots,s. \qquad (6.4.4)$$

Then, the condition

$$P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,X_1=Y_1+R_1, X_2=Y_2+R_2,\ldots,X_s=Y_s+R_s) \qquad (6.4.5)$$

for $R_i=0,1,\ldots,\ell_i$ with $\sum_{i=0}^{s} R_i \neq 0$ at any time and fixed $\ell_i : 1 \leq \ell_i \leq N_i - m_i$

is necessary and sufficient for $P_{\underset{\sim}{n}}$ to be such that

$$\left.\begin{array}{l} \dfrac{P_{\underset{\sim}{n}}}{c_{\underset{\sim}{n}}} = \dfrac{P_{\underset{\sim}{0}}}{c_{\underset{\sim}{0}}} \prod_{i=1}^{s} \theta_i^{n_i} \quad \text{for some } \theta_i > 0 \quad i=1,\ldots,s \\ \qquad\qquad\qquad \text{if } n_i \leq m_i + \ell_i \\ \text{and} \\ P_{\underset{\sim}{n}} = d_{\underset{\sim}{n}} \text{ an arbitrary constant if } m_i + \ell_i < n_i \leq N_i \text{ for} \\ \text{at least one } i=1,2,\ldots,s. \\ \left(\text{Of course these } d_{\underset{\sim}{n}} \text{ should satisfy } \sum_{\underset{\sim}{n}=\underset{\sim}{0}}^{\underset{\sim}{N}} P_{\underset{\sim}{n}} = 1\right) \end{array}\right\} \qquad (6.4.6)$$

It can be observed that (6.4.5) represents a system of $\ell_1 \times \ell_2 \times \ldots \times \ell_s$ equations.

<u>Proof</u>

Sufficiency: From

$$P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,X_1=Y_1+R_1, X_2=Y_2,\ldots,X_s=Y_s) \qquad (6.4.7)$$

$$R_1=1,\ldots,\ell_1$$

we find (as in the bivariate case) that

$$\frac{P_{n_1,r_2,\ldots,r_s}}{c_{n_1,r_2,\ldots,r_s}} = \frac{P_{0,r_2,\ldots,r_s}}{c_{0,r_2,\ldots,r_s}} \theta_1^{n_1} \quad \text{for } n_1=0,1,\ldots,m_1+\ell_1, \text{ all } r_i > 0, \text{ and some } \theta_1 > 0. \tag{6.4.8}$$

Let us now consider

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X_1=Y_1, X_2=Y_2+R_2, X_3=Y_3,\ldots,X_s=Y_s)$$
$$= P(\underset{\sim}{Y}=\underset{\sim}{r}|X_1=Y_1+R_1, X_2=Y_2+R_2, X_3=Y_3,\ldots,X_s=Y_s) \tag{6.4.9}$$

$R_1=1,\ldots,\ell_1$; $R_2=1,\ldots,\ell_2$.

for the values of $R_2 > 0$, (the case $R_2=0$ has already been used).

For

$$R_2 = 1, \frac{P_{n_1,r_2+1,r_3,\ldots,r_s}}{c_{n_1,r_2+1,r_3,\ldots,r_s}} = \frac{P_{0,r_2+1,r_3,\ldots,r_s}}{c_{0,r_2+1,r_3,\ldots,r_s}} \theta_1^{n_1}$$

$$n_1=0,1,\ldots,m_1+\ell_1$$

For $R_2=2,\ldots,\ell_2-1$ we have similar expressions.

Finally for

$$R_2 = \ell_2, \frac{P_{n_1,r_2+\ell_2,r_3,\ldots,r_s}}{c_{n_1,r_2+\ell_2,r_3,\ldots,r_s}} = \frac{P_{0,r_2+\ell_2,r_3,\ldots,r_s}}{c_{0,r_2+\ell_2,r_3,\ldots,r_s}} \theta_1^{n_1}$$

$$n_1=0,\ldots,m_1+\ell_1 \quad r_2=0,1,\ldots,m_2.$$

Combining the expressions for all the different values of R_2 enables us to show that (6.4.9) implies that

$$\frac{P_{n_1,n_2,r_3,\ldots,r_s}}{c_{n_1,n_2,r_3,\ldots,r_s}} = \frac{P_{0,n_2,r_3,\ldots,r_s}}{c_{0,n_2,r_3,\ldots,r_s}}\,\theta_1^{n_1} \qquad \begin{array}{l} n_1=0,1,\ldots,m_1+\ell_1 \\ n_2=0,1,\ldots,m_2+\ell_2 \\ r_i=0,\ldots,m_i \\ i=3,\ldots,s. \end{array} \tag{6.4.10}$$

Now using the fact that (6.4.7) is equivalent to (6.4.9) i.e. that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,X_1=Y_1, X_2=Y_2+R_2, X_3=Y_3, \ldots, X_s=Y_s) \tag{6.4.11}$$

we derive

$$\frac{P_{0,n_2,r_3,\ldots,r_s}}{c_{0,n_2,r_3,\ldots,r_s}} = \frac{P_{0,0,r_3,\ldots,r_s}}{c_{0,0,r_3,\ldots,r_s}}\,\theta_2^{n_2} \quad \text{for } n_2=0,1,\ldots,m_2+\ell_2 \text{ and some } \theta_2 > 0. \tag{6.4.12}$$

Substituting (6.4.12) in (6.4.10) gives

$$\frac{P_{n_1,n_2,r_3,\ldots,r_s}}{c_{n_1,n_2,r_3,\ldots,r_s}} = \frac{P_{0,0,r_3,\ldots,r_s}}{c_{0,0,r_3,\ldots,r_s}}\,\theta_1^{n_1}\,\theta_2^{n_2} \qquad \begin{array}{l} n_i=0,\ldots,m_i+\ell_i \\ i=1,2 \\ r_j=0,\ldots,m_j \\ j=3,\ldots,s. \end{array} \tag{6.4.13}$$

Now from

$$\begin{aligned} &P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,X_1=Y_1, X_2=Y_2, X_3=Y_3+R_3, X_4=Y_4, \ldots, X_s=Y_s) \\ &= P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,X_1=Y_1+R_1, X_2=Y_2+R_2, X_3=Y_3+R_3, X_4=Y_4, \ldots, X_s=Y_s) \end{aligned} \tag{6.4.14}$$

$R_1=0,1,\ldots,\ell_1$; $R_2=0,1,\ldots,\ell_2$; $R_3=1,\ldots,\ell_3$; $R_1+R_2\neq 0$ ($R_3=0$ was considered previously) we find that for $R_3=t$, $t=1,2,\ldots,\ell_3$,

$$\frac{P_{n_1,n_2,r_3+t,r_4,\ldots,r_s}}{c_{n_1,n_2,r_3+t,r_4,\ldots,r_s}} = \frac{P_{0,0,r_3+t,r_4,\ldots,r_s}}{c_{0,0,r_3+t,r_4,\ldots,r_s}} \theta_1^{n_1} \theta_2^{n_2} \qquad (6.4.15)$$

$$r_3=0,1,\ldots,m_3.$$

As a result of (6.4.15)

$$\frac{P_{n_1,n_2,n_3,r_4,\ldots,r_s}}{c_{n_1,n_2,n_3,r_4,\ldots,r_s}} = \frac{P_{0,0,n_3,r_4,\ldots,r_s}}{c_{0,0,n_3,r_4,\ldots,r_s}} \theta_1^{n_1} \theta_2^{n_2} \qquad (6.4.16)$$

for $n_i=0,1,\ldots,m_i+\ell_i$, $i=1,2,3$; $r_j=0,\ldots,m_j$, $j=4,\ldots,s$.

Also because (6.4.14) is equivalent to (6.4.7), i.e. because

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X_1=Y_1, X_2=Y_2, X_3=Y_3+R_3, X_4=Y_4,\ldots,X_s=Y_s)$$

we find similarly that

$$\frac{P_{0,0,n_3,r_4,\ldots,r_s}}{c_{0,0,n_3,r_4,\ldots,r_s}} = \frac{P_{0,0,0,r_4,\ldots,r_s}}{c_{0,0,0,r_4,\ldots,r_s}} \theta_3^{n_3} \quad \text{for some } \theta_3 > 0. \qquad (6.4.17)$$

So, (6.4.16) and (6.4.17) give

$$\frac{P_{n_1,n_2,n_3,r_4,\ldots,r_s}}{c_{n_1,n_2,n_3,r_4,\ldots,r_s}} = \frac{P_{0,0,0,r_4,\ldots,r_s}}{c_{0,0,0,r_4,\ldots,r_s}} \theta_1^{n_1} \theta_2^{n_2} \theta_3^{n_3} \qquad (6.4.18)$$

where $n_i=0,1,\ldots,m_i+\ell_i$; $i=1,2,3$, $r_j=0,\ldots,m_j$; $j=4,\ldots,s$.

Continuing in the same way we will finally get from the system s (using the previous s-1) i.e. from the system

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X_1=Y_1,\ldots,X_{s-1}=Y_{s-1},X_s=Y_s+R_s) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X_i=Y_i+R_i \quad i=1,\ldots,s)$$

that

$$\frac{P_{n_1,\ldots,n_{s-1},n_s}}{c_{n_1,\ldots,n_{s-1},n_s}} = \frac{P_{0,0,\ldots,0,n_s}}{c_{0,0,\ldots,0,n_s}} \theta_1^{n_1}\ldots\theta_{s-1}^{n_{s-1}} \tag{6.4.19}$$

$$n_i=0,1,\ldots,m_i+\ell_i, \quad i=1,\ldots,s.$$

Also from

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X_1=Y_1,\ldots,X_{s-1}=Y_{s-1},X_s=Y_s+R_s)$$

we have that

$$\frac{P_{0,\ldots,0,n_s}}{c_{0,\ldots,0,n_s}} = \frac{P_{\underset{\sim}{0}}}{c_{\underset{\sim}{0}}} \theta_s^{n_s} \quad n_s=0,1,\ldots,m_s+\ell_s. \tag{6.4.20}$$

(6.4.19) and (6.4.20) prove that if (6.4.5) holds then

$$\frac{P_{\underset{\sim}{n}}}{c_{\underset{\sim}{n}}} = \frac{P_{\underset{\sim}{0}}}{c_{\underset{\sim}{0}}} \prod_{i=1}^{s} \theta_i^{n_i} \quad n_i=0,1,\ldots,m_i+\ell_i, \quad i=1,2,\ldots,s.$$

This completes the "sufficient" part of the proof.

The "necessary" part of the proof is a direct extension to the multivariate case, of the necessary part of Theorem 5.1.1.

6.5 Characterization of Some Multivariate Distributions

A straightforward application of Theorem 6.4.1 yields the following extensions in the multivariate case to the characterizations made in

Section 6.2 (bivariate case).

Corollary 6.5.1 (Characterizations of the Multiple Binomial Distribution)

Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be defined as in Theorem 6.4.1, and suppose that $\underset{\sim}{Y}|\underset{\sim}{X}$ is Multiple Hypergeometric, i.e. that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{n}) = \prod_{i=1}^{s} \frac{\binom{m_i}{r_i}\binom{N_i-m_i}{n_i-r_i}}{\binom{N_i}{n_i}} \qquad \begin{array}{l} r_i, n_i, m_i, N_i > 0 \\ \text{and } r_i \leq m_i \\ i=1,\ldots,s. \end{array} \tag{6.5.1}$$

Then, condition (6.4.5) for $\ell_i = N_i - m_i$; i=1,...,s holds iff $P_{\underset{\sim}{n}}$ has the multiple Binomial form

$$P_{\underset{\sim}{n}} = \prod_{i=1}^{s} \binom{N_i}{n_i} p_i^{n_i} q_i^{N_i-n_i} . \tag{6.5.2}$$

Corollary 6.5.2 (Characterization of the Multiple Hypergeometric Distribution)

Suppose that $\underset{\sim}{Y}|\underset{\sim}{X}$ is of the form (6.4.4), where $(\underset{\sim}{X},\underset{\sim}{Y})$ are defined as in Theorem 6.4.1.

Also, assume that $\underset{\sim}{X}$ is multiple Binomial as in (6.5.2). Then $\underset{\sim}{Y}|\underset{\sim}{X}$ is Multiple Hypergeometric iff the condition (6.4.5) with $\ell_i = N_i - m_i$, i=1,2,...,s holds.

Proof

The proof is the immediate extension of the one given for the Bivariate case (Corollary 6.2.2). In the course of the proof, a multivariate extension of Lemma 6.2.1 will be required. This extension can be obtained very easily.

Remark Let us now make the following changes in the conditions of Theorem 6.4.1. Firstly, suppose that $P_{\underset{\sim}{n}} > 0$ for $\underset{\sim}{n}=(n_1,\dots,n_s)$ such that $\Sigma n_i \leqslant \ell^*$, ℓ^* fixed, $1 \leqslant \ell^* \leqslant \Sigma(N_i - m_i)$. Define $a_{\underset{\sim}{n}}>0$ for $\Sigma n_i \leqslant \Sigma m_i$ and $a_{\underset{\sim}{n}}=0$ otherwise. Also define $b_{\underset{\sim}{n}}>0$ for $\Sigma n_i \leqslant \ell^*$. Suppose that conditions (6.4.3) and (6.4.4) of Theorem 6.4.1 are valid. Then it can be shown, by an argument similar to the one used in Theorem (6.4.1) that, for $R_i : 0 < \Sigma R_i \leqslant \ell^*$, relation (6.4.5) holds iff

$$\frac{P_{\underset{\sim}{n}}}{P_{\underset{\sim}{0}}} = \frac{c_{\underset{\sim}{n}}}{c_{\underset{\sim}{0}}} \prod_i \theta_i^{n_i}, \quad \theta_i > 0, \ i=1,2,\dots,s \text{ for } \underset{\sim}{n}:\ \Sigma n_i \leqslant \Sigma m_i + \ell^*.$$

Using this remark we can prove the following corollaries.

Corollary 6.5.3 (Characterization of the Multinomial Distribution)

Suppose that $\underset{\sim}{Y}|\underset{\sim}{X} = \underset{\sim}{n} \sim$ Multivariate Hypergeometric i.e.,

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{n}) = \binom{n_0}{r_0}\binom{n_1}{r_1}\cdots\binom{n_s}{r_s}\Big/\binom{n}{r} \tag{6.5.3}$$

$$r,n,r_i,n_i \geqslant 0 \text{ for all } i,\ \Sigma n_i \leqslant n,\ \Sigma r_i \leqslant r,\ n_0=n-\Sigma n_i,\ \ r_0=r-\Sigma r_i .$$

Then $\underset{\sim}{X} = (X_1,\dots,X_s)$ follows a Multinomial distribution, i.e.

$$P_{\underset{\sim}{n}} = \frac{n!}{n_0!n_1!\dots n_s!} p_0^{n_0} p_1^{n_1}\cdots p_s^{n_s} \quad 0 < p_i < 1 \tag{6.5.4}$$

$$\Sigma n_i \leqslant n,\ \ \Sigma p_i < 1,\ \ n_0 = n - \Sigma n_i,\ \ p_0 = 1 - \Sigma p_i$$

iff

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X_1=Y_1+R_1, X_2=Y_2+R_2,\dots,X_s=Y_s+R_s)$$

where $0 < \Sigma R_i \leqslant n - r$.

Proof

Let us define the following sequences

$$a_{\underset{\sim}{n}} = \begin{cases} \dfrac{r!}{n_0!n_1!\ldots n_s!} & n_i \geq 0 \quad \sum_{i=1}^{s} n_i \leq r, \quad n_0 = r - \sum_{i=1}^{s} n_i \\ 0 & \text{otherwise.} \end{cases} \tag{6.5.6}$$

and

$$b_{\underset{\sim}{n}} = \frac{(n-r)!}{n_0!n_1!n_2!\ldots n_s!} \qquad \sum_{i=1}^{s} n_i \leq n-r, \quad n_0 = n - r - \sum_{i=1}^{s} n_i . \tag{6.5.7}$$

For the convolution of $a_{\underset{\sim}{n}}$ and $b_{\underset{\sim}{n}}$ we have

$$c_{\underset{\sim}{n}} = \sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}} = \sum_{r_1=0}^{n_1} \cdots \sum_{r_s=0}^{n_s} \frac{r!}{r_0!r_1!\ldots r_s!} \frac{(n-r)!}{(n_0-r_0)!(n_1-r_1)!\ldots(n_s-r_s)!}$$

$$= \frac{n!}{n_0!n_1!\ldots n_s!} \sum_{r_1=0}^{n_1} \cdots \sum_{r_s=0}^{n_s} \frac{\binom{n_0}{r_0}\binom{n_1}{r_1}\cdots\binom{n_s}{r_s}}{\binom{n}{r}}$$

with $r-r_0 = \sum_{i=1}^{s} r_i$, $n-n_0 = \sum_{i=1}^{s} n_i$, $r_i \geq 0$, $n_i \geq 0$.

Hence,

$$c_{\underset{\sim}{n}} = \frac{n!}{n_0!n_1!\ldots n_s!}, \quad \sum_{i=1}^{s} n_i = n-n_0, \quad n_i \geq 0. \tag{6.5.8}$$

Clearly, the sequences $a_{\underset{\sim}{n}}$, $b_{\underset{\sim}{n}}$, $c_{\underset{\sim}{n}}$, as given by (6.5.6), (6.5.7) and (6.5.8), can be used to express (6.5.3) in the form $a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}} / c_{\underset{\sim}{n}}$. Consequently, the main Theorem 6.4.1 as revised by the Remark is applicable to this situation and implies that (6.5.5) holds iff

$$P_{\underset{\sim}{n}} = P_{\underset{\sim}{0}} \frac{c_{\underset{\sim}{n}}}{c_{\underset{\sim}{0}}} \prod_{i=1}^{s} \theta_i^{n_i}$$

i.e. (from (6.5.8)) iff

$$P_{\underset{\sim}{n}} = P_{\underset{\sim}{0}} \frac{n!}{n_0!n_1!\dots n_s!} \theta_1^{n_1}\dots\theta_s^{n_s}. \qquad (6.5.9)$$

But $\sum_{\underset{\sim}{n}} P_{\underset{\sim}{n}} = 1$. In other words

$$P_{\underset{\sim}{0}}^{-1} = \frac{1}{\theta_0^{n_0}} \sum_{n_1=0} \cdots \sum_{n_s=0} \frac{n!}{n_0!n_1!\dots n_s!} \theta_0^{n_0}\theta_1^{n_1}\dots\theta_s^{n_s}$$

$$= \frac{\left(\sum_{i=0}^{s}\theta_i\right)^{n_0+n_1+\dots+n_s}}{\theta_0^{n_0}} \sum_{n_1}\cdots\sum_{n_s} \frac{n!}{n_0!n_1!\dots n_s!} \frac{\theta_0^{n_0}\theta_1^{n_1}\dots\theta_s^{n_s}}{\left(\sum_{i=0}^{s}\theta_i\right)^{n_0+\dots+n_s}}$$

$$= \frac{\left(\sum_{i=0}^{s}\theta_i\right)^{n}}{\theta_0^{n_0}} \sum_{n_1}\cdots\sum_{n_s} \frac{n!}{n_0!n_1!\dots n_s!} \left(\frac{\theta_0}{\sum_{i=0}^{s}\theta_i}\right)^{n_0}\cdots\left(\frac{\theta_s}{\sum_{i=0}^{s}\theta_i}\right)^{n_s}.$$

Hence,

$$P_{\underset{\sim}{0}}^{-1} = \left(\sum_{i=0}^{s}\theta_i\right)^{n} \Big/ \theta_0^{n_0} \qquad (6.5.10)$$

Finally (6.5.9) and (6.5.10) give

$$P_{\underset{\sim}{n}} = \frac{n!}{n_0!n_1!\dots n_s!} p_0^{n_0} p_1^{n_1} \dots p_s^{n_s} \text{ with } p_i = \frac{\theta_i}{\sum_{i=0}^{s} \theta_i} . \qquad (6.5.11)$$

<u>Corollary 6.5.4</u> (Characterization of the Multivariate Hypergeometric Distribution)

Let us assume now that the random vector $(\underset{\sim}{X},\underset{\sim}{Y})$ defined in the Remark of Theorem 6.4.1 is such that $P(\underset{\sim}{X}=\underset{\sim}{n}) \sim$ Multinomial as in (6.5.4). Also assume that we can find sequences $a_{\underset{\sim}{n}}$ and $b_{\underset{\sim}{n}}$ such that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{n}) = \frac{a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}}}{c_{\underset{\sim}{n}}}$$

where $c_{\underset{\sim}{n}}$ is the convolution of $a_{\underset{\sim}{n}}$ and $b_{\underset{\sim}{n}}$. Then the condition (6.5.5) holds iff the conditional distribution of $\underset{\sim}{Y}$ on $\underset{\sim}{X}$ is Multivariate Hypergeometric as in (6.5.3).

<u>Proof</u>

From Theorem 6.4.1 and its Remark we know that $P_{\underset{\sim}{n}}$ is Multinomial iff

$$c_{\underset{\sim}{n}} = c_{\underset{\sim}{o}} \frac{P_{\underset{\sim}{n}}}{P_{\underset{\sim}{o}}} \theta_1^{-n_1} \dots \theta_s^{-n_s}, \quad \theta_i > 0,$$

i.e. (taking into consideration (6.5.4))

$$c_{\underset{\sim}{n}} = c_{\underset{\sim}{o}} \frac{\frac{n!}{n_0!n_1!\dots n_s!} p_0^{n_0} \left(\frac{p_1}{\theta_1}\right)^{n_1} \dots \left(\frac{p_s}{\theta_s}\right)^{n_s}}{p_0^{n}}$$

(because for $P_{\underset{\sim}{0}}, n_0=n$.)

Hence,

$$c_{\underset{\sim}{n}} = c_{\underset{\sim}{o}} \frac{n!}{n_0!n_1!\ldots n_s!} \left(\frac{p_1}{p_0\theta_1}\right)^{n_1} \cdots \left(\frac{p_s}{p_0\theta_s}\right)^{n_s}, \tag{6.5.12}$$

i.e. (by summing over $\underset{\sim}{n}$ in both sides of (6.5.12)).

$$\sum_{\underset{\sim}{n}=\underset{\sim}{0}}^{\underset{\sim}{N}} \sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}} = c_{\underset{\sim}{o}} \theta_o^{n_0} \sum_{n_1=0}^{N_1} \cdots \sum_{n_s=0}^{N_s} \frac{n!}{n_0!\ldots n_s!} \left(\frac{p_0}{p_0\theta_0}\right)^{n_0} \left(\frac{p_1}{p_0\theta_1}\right)^{n_1} \cdots \left(\frac{p_s}{p_0\theta_s}\right)^{n_s}.$$

So,

$$c_{\underset{\sim}{o}} = \theta_0^{-n_0} \sum_{\underset{\sim}{n}=\underset{\sim}{0}}^{\underset{\sim}{N}} \sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}}. \tag{6.5.13}$$

Substituting (6.5.13) in (6.5.12) gives

$$\sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} \frac{a_{\underset{\sim}{r}}}{\sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{N}} a_{\underset{\sim}{r}}} \frac{b_{\underset{\sim}{n}-\underset{\sim}{r}}}{\sum_{\underset{\sim}{n}=\underset{\sim}{r}}^{\underset{\sim}{N}-\underset{\sim}{m}} b_{\underset{\sim}{n}-\underset{\sim}{r}}} = \frac{n!}{n_0!n_1!\ldots n_s!} \left(\frac{p_0}{p_0\theta_0}\right)^{n_0} \left(\frac{p_1}{p_0\theta_1}\right)^{n_1} \cdots \left(\frac{p_s}{p_0\theta_s}\right)^{n_s} \tag{6.5.14}$$

Hence, the convolution of the distributions $\frac{a_{\underset{\sim}{r}}}{\Sigma_{\underset{\sim}{r}} a_{\underset{\sim}{r}}}$ and $\frac{b_{\underset{\sim}{n}}}{\Sigma_{\underset{\sim}{n}} b_{\underset{\sim}{n}}}$ is multinomial.

But is has been proved by Shanbhag and Basawa (1974) that if the convolution of two s-dimensional non-negative independent r.v.'s is multinomial, then each of the components is multinomial with the same set of probabilities as their convolution. Hence,

$$\frac{a_{\underset{\sim}{r}}}{\Sigma_{\underset{\sim}{r}} a_{\underset{\sim}{r}}} = \frac{r!}{r_0!\ldots r_s!} \left(\frac{p_0}{p_0\theta_0}\right)^{n_0} \cdots \left(\frac{p_s}{p_0\theta_s}\right)^{n_s} \tag{6.5.15}$$

and

$$\frac{b_{\underset{\sim}{n}-\underset{\sim}{r}}}{\sum_{\underset{\sim}{n}=\underset{\sim}{r}}^{\underset{\sim}{N}-\underset{\sim}{m}} b_{\underset{\sim}{n}-\underset{\sim}{r}}} = \frac{(n-r)!}{(n_0-r_0)!\ldots(n_s-r_s)!}\left(\frac{p_0}{p_0\theta_0}\right)^{n_0}\cdots\left(\frac{p_s}{p_0\theta_s}\right)^{n_s} \qquad (6.5.16)$$

Moreover because we have assumed that $\underset{\sim}{Y}|\underset{\sim}{X}$ has the form $a_{\underset{\sim}{r}}b_{\underset{\sim}{n}-\underset{\sim}{r}}/c_{\underset{\sim}{n}}$ we find from (6.5.14), (6.5.15) and (6.5.16) that $\underset{\sim}{Y}|\underset{\sim}{X}$ is Multivariate Hypergeometric as in (6.5.3).

Note Extensions to Corollaries 6.5.3 and 6.5.4, providing characterizations for the Multiple Multinomial and the Multiple Multivariate Hypergeometric, are easily obtainable.

6.6 The Extension of Theorem 6.4.1 to the Truncated Case.

The theorem that follows is a direct extension of the Theorem 6.3.1, which we established previously for the Bivariate Case.

Theorem 6.6.1

Let $\{\underset{\sim}{X},\underset{\sim}{Y}\}$ $\underset{\sim}{X} = (X_1, X_2, \ldots, X_s)$, $\underset{\sim}{Y} = (Y_1, \ldots, Y_s)$ be a random vector with non-negative, integer-valued components such that

$$P(\underset{\sim}{X}=\underset{\sim}{n}) = P_{\underset{\sim}{n}} \qquad \begin{array}{l}\underset{\sim}{n}=(n_1,\ldots,n_s)\\ n_i=k_i, k_i+1,\ldots,N_i\\ k_i > 0,\ i=1,\ldots,s\end{array} \qquad (6.6.1)$$

and

$$P(X \geqslant k) = 1, \quad P_{\underset{\sim}{n}} > 0 \text{ for } k_i \leqslant n_i \leqslant k_i+\ell_i, \quad \ell_i \text{ fixed, } 1 \leqslant \ell_i \leqslant N_i - m_i \qquad (6.6.2)$$

Suppose now, that $\{(a_{\underset{\sim}{n}}, b_{\underset{\sim}{n}}): n_i=0,1,\ldots\}$ is a sequence of non-negative real vectors with

$$a_{\underset{\sim}{n}} > 0 \quad \text{if} \qquad n_i = k_i, k_i+1, \ldots, m_i$$
$$i=1,\ldots,s$$

$$a_{\underset{\sim}{n}} = 0 \quad \text{if} \qquad m_i < n_i \leqslant N_i \tag{6.6.3}$$

and

$$b_{\underset{\sim}{n}} > 0 \quad \text{if} \qquad n_i = 0,1,\ldots,\ell_i$$
$$i=1,\ldots,s$$

Furthermore let $c_{\underset{\sim}{n}}$ be defined as follows

$$c_{\underset{\sim}{n}} = \sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} a_{\underset{\sim}{r}}\, b_{\underset{\sim}{n}-\underset{\sim}{r}}$$

(Obviously $c_{\underset{\sim}{n}} > 0$ at least for $n_i \geqslant k_i$, $i=1,2,\ldots,s$.)

Finally assume that, whenever $P_{\underset{\sim}{n}} > 0$,

$$P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,\underset{\sim}{X}=\underset{\sim}{n}) = \frac{a_{\underset{\sim}{r}}\, b_{\underset{\sim}{n}-\underset{\sim}{r}}}{c_{\underset{\sim}{n}}} \qquad \begin{array}{l} r_i = 0,1,\ldots,n_i \\ n_i = k_i, k_i+1,\ldots,N_i \\ i=1,2,\ldots,s. \end{array} \tag{6.6.4}$$

Then, the condition

$$P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,X_1=Y_1+R_1,\ldots,X_s=Y_s+R_s\,;Y_1 \geqslant k_1,\ldots,Y_s \geqslant k_s)$$

with $\sum_{i=1}^{s} R_i \neq 0$ at any time, $R_i = 0,1,\ldots,\ell_i$; $i=1,2,\ldots,s$. (6.6.5)

is necessary and sufficient for $P_{\underset{\sim}{n}}$ to be such that

$$\left.\begin{array}{l} \dfrac{P_{\underset{\sim}{n}}}{c_{\underset{\sim}{n}}} = \dfrac{P_{\underset{\sim}{k}}}{c_{\underset{\sim}{k}}} \prod_{i=1}^{s} \theta_i^{n_i - k_i} \quad \text{if all } n_i \leqslant m_i + \ell_i \\ \text{for some } \theta_i > 0 \ i=1,\ldots,s;\ \ell_i \text{ fixed, } \ell_i \leqslant N_i - m_i \\ \text{and } P_{\underset{\sim}{n}} = d_{\underset{\sim}{n}},\ \text{if } m_i + \ell_i < n_i \leqslant N_i \text{ for at least} \\ \text{one } i=1,2,\ldots,s.\ \left(d_{\underset{\sim}{n}} \text{ is an arbitrary constant} \right. \\ \left. \text{such that } \sum_{\underset{\sim}{n}=\underset{\sim}{k}}^{\underset{\sim}{N}} P_{\underset{\sim}{n}} = 1.\right) \end{array}\right\} \qquad (6.6.6)$$

(Here again, (6.6.5) represents a system of $\ell_1 \times \ell_2 \times \ldots \times \ell_s$ equations.) We omit the proof, since it follows on from the proof of the Bivariate extension (Section 6.3).

As a result of Theorem 6.6.1, it can be seen that the relation (6.6.5) with $\ell_i = N_i - m_i$, i=1,2,...,s characterizes the truncated Multiple Binomial as distribution $\underset{\sim}{X}$, in the case where $\underset{\sim}{Y}|\underset{\sim}{X}$ is Multiple Hypergeometric.

Furthermore, if the distribution of $Y|X$ is Truncated Multiple Hypergeometric, relation (6.6.5), for $\ell_i = N_i - m_i$, is a characterizing property for the distribution of X to be the distribution of the product of s independent r.v.'s each of which is the convolution of a Binomial with a Truncated Binomial.

Making changes in the conditions of Theorem 6.6.1 analogous to those in the Remark of Theorem 6.4.1 we can prove that relation (6.6.5) for $R_i : \sum_i R_i \leqslant n-r$ characterizes the Truncated Multinomial (when $Y|X$ is Multivariate Hypergeometric). It also characterizes the convolution of a Multinomial with a Truncated Multinomial (when $Y|X$ is Truncated Multivariate Hypergeometric).