CHAPTER 4.

SHANBHAG'S EXTENSION OF THE R-R CHARACTERIZATION. THE BIVARIATE AND MULTIVARIATE CASES.

4.0 Introduction

In this chapter we study the Bivariate and Multivariate aspect of Shanbhag's extension, as well as the respective truncated forms. Using these results, simpler proofs for some of the theorems of Chapter 2 are obtained and the bivariate and multivariate extensions of the characterizations established in Chapter 3 are derived. Making some changes in the conditions of the extension we arrive at some characterizations of truncated statistical distributions.

4.1 Shanbhag's Extension. The Bivariate Case.

Theorem 4.1.1 (Shanbhag 1976)

Let $\left\{\left(a_{n_1,n_2}, b_{n_1,n_2}\right): n_1, n_2 = 0,1,\ldots\right\}$ be a sequence of real vectors with

$$a_{n_1,n_2} > 0,\ b_{n_1,n_2} \geq 0 \text{ for every } n_1, n_2 \geq 0$$

and

$$b_{0,0} > 0,\ b_{0,1} > 0,\ b_{1,n_2} > 0 \text{ for some } n_2 \geq 0.$$

Define $\left\{c_{n_1,n_2}\right\}$ to be the convolution of $\left\{a_{n_1,n_2}\right\}$, $\left\{b_{n_1,n_2}\right\}$ given by

$$c_{n_1,n_2} = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} a_{r_1,r_2}\, b_{n_1-r_1,n_2-r_2} \qquad n_1, n_2 = 0,1,\ldots \qquad (4.1.1)$$

(Obviously we have $c_{n_1,n_2} > 0$ for all $n_1, n_2 \geq 0$.)

Consider a random vector (X_1, X_2, Y_1, Y_2) with non-negative, integer-valued components, such that

$$P\left\{X_1=n_1, X_2=n_2\right\} = P_{n_1,n_2} \qquad n_1, n_2=0,1,\ldots \tag{4.1.2}$$

with $P_{n_1,n_2} > 0$ for some $n_1 > 0$ and also for some $n_2 > 0$. Assume also, that whenever $P_{n_1,n_2} > 0$ we have

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=n_1, X_2=n_2) = \frac{a_{r_1,r_2}\, b_{n_1-r_1,n_2-r_2}}{c_{n_1,n_2}} \tag{4.1.3}$$

$$\left.\begin{array}{l} r_i=0,1,\ldots,n_i \\ n_i=0,1,\ldots \end{array}\right\} \; i=1,2.$$

Then,

$$\begin{aligned} P(Y_1=r_1, Y_2=r_2) &= P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2=Y_2) \\ &= P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2>Y_2) \end{aligned} \tag{4.1.4}$$

iff

$$\frac{P_{n_1,n_2}}{c_{n_1,n_2}} = \frac{P_{0,0}}{c_{0,0}}\, \theta_1^{n_1}\, \theta_2^{n_2} \quad \text{for some } \theta_1, \theta_2 > 0 \qquad n_1, n_2=0,1,\ldots \tag{4.1.5}$$

<u>Proof</u> (We will only outline the proof here, since this is a special case of the multivariate extension, which will be stated and examined fully in a later section.)

We first observe that (4.1.4) is equivalent to the following set of conditions. (This is a special case of the statement 1 in theorem 4.5.1 which appears later.)

$$P(Y_1=r_1,Y_2=r_2) = P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1,X_2=Y_2) \quad \text{(a)}$$
$$(4.1.6)$$
$$P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1) = P(Y_1=r_1,Y_2=r_2 \mid X_1=Y_1,X_2=Y_2)\ . \quad \text{(b)}$$

Define now the sequences

$$V_{n_2} = \frac{P_{r_1,n_2}}{c_{r_1,n_2}} \qquad \text{for every fixed } r_1 > 0 \text{ and } n_2 \geq 0 \qquad (4.1.7)$$

$$W_{n_2} = \frac{b_{0,n_2}\ P(X_1=Y_1,X_2=Y_2)}{P(X_1=Y_1)} \qquad n_2=0,1,\dots\ . \qquad (4.1.8)$$

Then, with the sequences V_n, W_n as given by (4.1.7) and (4.1.8) we see that (4.1.6b) is equivalent to

$$V_{m_2} = \sum_{n_2=0}^{\infty} V_{n_2+m_2} W_{n_2} \qquad m_2=0,1,\dots\ .$$

Hence, according to lemma 3.1.1, (4.1.6b) is valid iff

$$\frac{P_{r_1,n_2}}{c_{r_1,n_2}} = \frac{P_{r_1,0}}{c_{r_1,0}}\ \theta_2^{n_2} \qquad r_1,n_2=0,1,\dots \qquad (4.1.9)$$

(because r_1 was fixed but arbitrary).

[It is interesting to note, that if (4.1.9) is valid then, conditional on $X_1=Y_1$, the vector (Y_1,Y_2) and the variable X_2-Y_2 are independent.]

Define now the sequences

$$V_{n_1} = \frac{P_{n_1,0}}{c_{n_1,0}} \qquad n_1 \geq 0 \qquad (4.1.10)$$

and

$$W_{n_1} = \left\{ \sum_{n_2=0}^{\infty} b_{n_1,n_2}\theta_2^{n_2} \right\} \frac{P(X_1=Y_1,X_2=Y_2)}{b_{0,0}} \qquad (4.1.11)$$

Then,

$$V_{m_1} = \sum_{n_1=0}^{\infty} V_{n_1+m_1} W_{n_1} \qquad m_1=0,1,\ldots$$

is equivalent to (4.1.6a). Consequently, from lemma 3.1.1 again we find that (4.1.6a) holds iff

$$\frac{P_{n_1,0}}{c_{n_1,0}} = \frac{P_{0,0}}{c_{0,0}} \theta_1^{n_1} \qquad n_1=0,1,\ldots \tag{4.1.12}$$

Hence combining (4.1.9) and (4.1.12) we finally find that the pair of conditions (4.1.6) [and therefore the system (4.1.4)] holds iff (4.1.5) is true.

Remark If the vector (X_1, X_2, Y_1, Y_2) is such that

$$\frac{P_{n_1,n_2}}{c_{n_1,n_2}} = \frac{P_{0,0}}{c_{0,0}} \theta_1^{n_1} \theta_2^{n_2} \text{ for every } n_1, n_2 \geqslant 0$$

then (Y_1, Y_2) and (X_1-Y_1, X_2-Y_2) are independent.

Proof The proof is similar to the one given for the univariate case (Section 3.1 Note 3)

4.2 Characterizations of Bivariate Distributions based on the Extension

The result of theorem 4.1.1 can be employed in order to characterize bivariate statistical distributions. The corollaries that follow illustrate the method.

Corollary 4.2.1 (Shanbhag).

If (X_1, X_2, Y_1, Y_2) is a random vector of non-negative integer-valued components such that

$$P(Y_1 = r_1, Y_2 = r_2 \mid X_1 = n_1, X_2 = n_2) = \binom{n_1}{r_1} p_1^{r_1} q_1^{n_1 - r_1} \binom{n_2}{r_2} p_2^{r_2} q_2^{n_2 - r_2}$$

$$r_i = 0,1,\ldots,n_i$$
$$n_i = 0,1,\ldots \qquad (4.2.1)$$
$$0 < p_i < 1, \quad q_i = 1 - p_i$$
$$i = 1,2.$$

(i.e. double Binomial)

with $P_{n_1,n_2} = P(X_1 = n_1, X_2 = n_2)$ denoting a discrete probability distribution satisfying the conditions stated in (4.1.2) of theorem 4.1.1, then the condition (4.1.4) holds iff

$$P_{n_1,n_2} = e^{-\lambda_1 - \lambda_2} \frac{\lambda_1^{n_1}}{n_1!} \frac{\lambda_2^{n_2}}{n_2!} \quad n_1, n_2 = 0,1,\ldots \qquad (4.2.2)$$

for some $\lambda_1, \lambda_2 > 0$.

Proof

Consider the following sequences

$$a_{n_1,n_2} = \frac{p_1^{n_1}}{n_1!} \frac{p_2^{n_2}}{n_2!} \quad n_1, n_2 = 0,1,\ldots \qquad (4.2.3)$$

and

$$b_{n_1,n_2} = \frac{q_1^{n_1}}{n_1!} \frac{q_2^{n_2}}{n_2!} \quad n_1, n_2 = 0,1,\ldots \qquad (4.2.4)$$

Then the convolution of a_{n_1,n_2} and b_{n_1,n_2} will be

$$c_{n_1,n_2} = \frac{1}{n_1! n_2!} \quad n_1, n_2 = 0,1,\ldots \qquad (4.2.5)$$

But a_{n_1,n_2}, b_{n_1,n_2}, c_{n_1,n_2} as given by (4.2.3), (4.2.4) and (4.2.5) can clearly be used to express (4.2.1) in the form (4.1.3). Hence, applying theorem 4.1.1 the result follows.

Note 1 The previous corollary is an improved version of theorem 2.4.1. This is so because condition (4.1.4) is simpler than (2.4.2).

Corollary 4.2.2 (Characterization of the Double Binomial)

Let

$$P_{n_1,n_2} = e^{-\lambda-\mu} \frac{\lambda^{n_1}}{n_1!} \frac{\mu^{n_2}}{n_2!} \tag{4.2.6}$$

$$\lambda_1, \lambda_2 > 0, \quad n_1, n_2 = 0, 1, \ldots$$

Suppose that $P(Y_1=r_1, Y_2=r_2 | X_1=n_1, X_2=n_2)$ is of the form (4.1.3). Then, the condition (4.1.4) is satisfied iff $P(Y_1=r_1, Y_2=r_2 | X_1=n_1, X_2=n_2)$ is of the form (4.2.1), i.e. double binomial.

Proof

The "if" part of the proof is straightforward and is contained in Corollary 4.2.1.

For the "only if" part of the proof we have from Theorem 4.1.1 that the condition (4.1.4) holds iff

$$c_{n_1,n_2} = c_{0,0} \frac{P_{n_1,n_2}}{P_{0,0}} \theta_1^{n_1} \theta_2^{n_2} .$$

Because of (4.2.6) this is seen to be equivalent to

$$c_{n_1,n_2} = c_{0,0} \frac{\lambda^{n_1} \mu^{n_2}}{n_1! n_2!} \theta_1^{n_1} \theta_2^{n_2} . \tag{4.2.7}$$

Using Teicher's (1954) extension of Raikov Theorem we see that (4.2.7) is equivalent to

$$a_{r_1,r_2} = a_{0,0} \frac{\lambda_1^{r_1} \lambda_2^{r_2}}{r_1!r_2!} , \tag{4.2.8}$$

$$b_{n_1,n_2} = b_{0,0} \frac{\mu_1^{n_1} \mu_2^{n_2}}{n_1!n_2!} \tag{4.2.9}$$

where $0 < \lambda_1, \lambda_2 < \infty$ and $0 < \mu_1, \mu_2 < \infty$.

Hence it is immediate that the condition (4.1.4) holds iff (4.2.8) and (4.2.9) are satisfied.

Employing as a_{r_1,r_2}, b_{n_1,n_2}, c_{n_1,n_2} in (4.2.1) the expressions given by (4.2.8), (4.2.9) and (4.2.7) respectively, we arrive at the asserted result.

Note 1 Corollary 4.2.2 is a variant of a result obtained by Srivastava (1970) (Theorem 2.4.2). The difference is that in Theorem 2.4.2 λ, μ were variables while in (4.2.2) they are fixed. Once again, the additional condition (4.1.3) is necessary for our proof. It is also obvious that condition (4.1.4) is simpler than (2.4.4) employed in Theorem 2.4.2.

Corollary 4.2.3 (Characterization of the Double Negative Binomial)

Consider (X_1, X_2, Y_1, Y_2) as previously.

Suppose that

$$P(Y_1=r_1, Y_2=r_2 | X_1=n_1, X_2=n_2) = \frac{\binom{-m_1}{r_1}\binom{-\rho_1}{n_1-r_1}}{\binom{-m_1-\rho_1}{n_1}} \frac{\binom{-m_2}{r_2}\binom{-\rho_2}{n_2-r_2}}{\binom{-m_2-\rho_2}{n_2}} \tag{4.2.10}$$

$$r_i=0,1,\ldots,n_i, \quad m_i > 0, \quad \rho_i > 0 \quad i=1,2.$$

i.e. double Negative Hypergeometric.

Then the condition (4.1.4) holds iff

$$P(X_1=n_1, X_2=n_2) = \binom{-N_1}{n_1} p_1^{N_1} (-q_1)^{n_1} \binom{-N_2}{n_2} p_2^{N_2} (-q_2)^{n_2} \qquad (4.2.11)$$

with $N_i = m_i + \rho_i \quad i=1,2.$

i.e. Double Negative Binomial with parameters $(p_1, m_1+\rho_1, p_2, m_2+\rho_2)$.

Proof

This follows easily by applying the result of Theorem 4.1.1 with the sequences a_{n_1,n_2}, b_{n_1,n_2} defined as

$$a_{n_1,n_2} = \binom{m_1+n_1-1}{n_1} \binom{m_2+n_2-1}{n_2} q_1^{n_1} q_2^{n_2}$$

$$(4.2.12)$$

$$b_{n_1,n_2} = \binom{\rho_1+n_1-1}{n_1} \binom{\rho_2+n_2-1}{n_2} q_1^{n_1} q_2^{n_2} .$$

Note The remark in Note 2 of Corollary 3.3.2 (univariate case) applies again here.

4.3 The Truncated Bivariate Extension

Theorem 4.3.1 Define (X_1, Y_1, X_2, Y_2) as in theorem 4.1.1.

Suppose that k_1 and k_2 are some non-negative integers for which

$$P(X_1 \geqslant k_1, X_2 \geqslant k_2) = 1, \quad P(X_1 > k_1) > 0 \text{ and } P(X_2 > k_2) > 0 \qquad (4.3.1)$$

then

$$P(Y_1=r_1, Y_2=r_2 \mid Y_1 \geq k_1, Y_2 \geq k_2) = P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2=Y_2)$$

$$= P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2 > Y_2, Y_2 \geq k_2). \tag{4.3.2}$$

$$r_i = k_i, k_i+1, \ldots; \; i=1,2.$$

iff

$$\frac{P_{n_1,n_2}}{c_{n_1,n_2}} = \frac{P_{k_1,k_2}}{c_{k_1,k_2}} \theta_1^{n_1-k_1} \theta_2^{n_2-k_2} \tag{4.3.3}$$

for $n_i = k_i, k_i+1, k_i+2, \ldots; \; i=1,2,$

and some $\theta_1, \theta_2 > 0$.

<u>Proof</u>

The proof follows by combining the techniques used for the truncated extension of the univariate case (proof of Theorem 3.5.1) and the bivariate extension (Theorem 4.1.1).

The following corollaries of Theorem 4.3.1 can now be established.

<u>Corollary 4.3.1</u> (Characterization of the Truncated Double Poisson)

Suppose that (X_1, Y_1, X_2, Y_2) are defined as in (4.3.1). Assume that the conditional distribution of Y_1, Y_2 given X_1 and X_2, is double Binomial as in (4.2.1). Then, condition (4.3.2) holds iff

$$P_{n_1,n_2} = \frac{\dfrac{\lambda^{n_1}}{n_1!}}{\displaystyle\sum_{x_1=k_1}^{\infty} \frac{\lambda^{x_1}}{x_1!}} \; \frac{\dfrac{\mu^{n_2}}{n_2!}}{\displaystyle\sum_{x_2=k_2}^{\infty} \frac{\mu^{x_2}}{x_2!}} \qquad n_i = k_i, k_i+1, \ldots \quad i=1,2. \tag{4.3.4}$$

i.e. iff P_{n_1,n_2} is Double Poisson truncated at the points k_1-1 and k_2-1.

Corollary 4.3.2 (Characterization of the Truncated Double Negative Binomial)

Suppose that the conditional distribution of Y_1, Y_2 given X_1 and X_2 is double Negative Hypergeometric as in (4.2.10). Then, (4.3.2) is true iff

$$P_{n_1,n_2} = \frac{\binom{-N_1}{n_1} p_1^{N_1} (-q_1)^{n_1}}{\sum_{x_1=k_1}^{\infty} \binom{-N_1}{x_1} p_1^{N_1} (-q_1)^{x_1}} \quad \frac{\binom{-N_2}{n_2} p_2^{N_2} (-q_2)^{n_2}}{\sum_{x_2=k_2}^{\infty} \binom{-N_2}{x_2} p_2^{N_2} (-q_2)^{x_2}} \tag{4.3.5}$$

$$n_i = k_i, k_i+1, \ldots \qquad i=1,2.$$
$$N_i = m_i + \rho_i$$

i.e. iff P_{n_1,n_2} is Double Negative Binomial truncated at $k_1 - 1$ and $k_2 - 1$.

The proofs of the above corollaries are similar to the ones given in the univariate case (Corollaries 3.5.1 and 3.5.2) and are direct applications of Theorem 4.3.1, where as sequences a_{n_1,n_2}, b_{n_1,n_2} we consider those defined in (4.2.3), (4.2.4) (for Corollary 4.3.1) and in (4.2.12) (for Corollary 4.3.2).

4.4 A Remark on the Truncated Bivariate Extension

A situation similar to the one examined in Section 3.6 concerning the sequence $\{a_{n_1,n_2}\}$ arises again in the truncated Bivariate Extension. It can be checked as before that the theorem 4.3.1 remains valid, if the sequence $\{a_{n_2,n_2}\}$ is defined for $n_1 = k_1, k_1+1, \ldots$ and $n_2 = k_2, k_2+1, \ldots$ with k_1 and k_2 positive integers. (Conditions for b_{n_1,n_2} remain the same.)

This fact enables us to arrive at some characterizations based on the assumption that the conditional distribution of Y_1, Y_2 given X_1 and X_2 is truncated at the points k_1-1, k_2-1. The variant of Theorem 4.3.1 can be stated in the following way.

Theorem 4.4.1

Suppose that (X_1, X_2, Y_1, Y_2), k_1, k_2 are defined as in Theorem 4.3.1 and that (4.3.1) is satisfied.

Let $\left\{a_{n_1,n_2}\right\}$ $n_i=k_i, k_i+1, \ldots;\ i=1,2$ and $\left\{b_{n_1,n_2}\right\}$ $n_i=0,1,\ldots;\ i=1,2$ be two sequences of real numbers such that

$$a_{n_1,n_2} > 0 \text{ for all } n_i \geq k_i \qquad i=1,2. \tag{4.4.1}$$

and b_{n_1,n_2} as defined in Theorem 4.1.1.

Also assume that whenever $P_{n_1,n_2} > 0$ $n_i \geq k_i$ $i=1,2$, we have

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=n_1, X_2=n_2) = \frac{a_{r_1,r_2}\, b_{n_1-r_1,n_2-r_2}}{c_{n_1,n_2}} \qquad r_i=k_i, k_i+1, \ldots \quad i=1,2. \tag{4.4.2}$$

Then (4.3.2) holds iff (4.3.3) is true.

Theorem 4.4.1 provides characterizations of the distributions which are products of two independent r.v's each of which is the convolution of a Double Poisson with a truncated Double Poisson (the distribution of $Y_1, Y_2 \mid X_1, X_2$ being truncated Double Binomial), and a Double Negative Binomial with a truncated Double Negative Binomial (the conditional distribution being truncated Double Negative Hypergeometric). These characterizations are given in the following corollaries.

Corollary 4.4.1

Suppose that

$$P(Y_1=r_1, Y_2=r_2 | X_1=n_1, X_2=n_2) = \frac{\binom{n_1}{r_1} p_1^{r_1} q_1^{n_1-r_1}}{\sum_{x_1=k_1}^{n_1} \binom{n_1}{x_1} p_1^{x_1} q_1^{n_1-x_1}} \; \frac{\binom{n_2}{r_2} p_2^{r_2} q_2^{n_2-r_2}}{\sum_{x_2=k_2}^{n_2} p_2^{x_2} q_2^{n_2-x_2}} \tag{4.4.3}$$

$r_i=k_i, k_i+1, \ldots;\ i=1,2.$

i.e. truncated Double Binomial.

Then condition (4.3.2) holds iff

$$P_{n_1,n_2} = \frac{e^{-\mu_1-\mu_2} \left\{ \sum_{r_1=k_1}^{n_1} \binom{n_1}{r_1} \lambda_1^{r_1} \mu_1^{n_1-r_1} \right\} \left\{ \sum_{r_2=k_2}^{n_2} \binom{n_2}{r_2} \lambda_2^{r_2} \mu_2^{n_2-r_2} \right\}}{n_1!\,n_2! \sum_{n_1=k_1}^{\infty} \frac{\lambda_1^{n_1}}{n_1!} \sum_{n_2=k_2}^{\infty} \frac{\lambda_2^{n_2}}{n_2!}} \tag{4.4.4}$$

$n_i=k_i, k_i+1, \ldots;\ i=1,2.$

$\lambda_1, \lambda_2, \mu_1, \mu_2 > 0.$

i.e. iff P_{n_1,n_2} is the convolution of a Double Poisson (μ_1, μ_2) with a Truncated Double Poisson (λ_1, λ_2)

Corollary 4.4.2

Suppose that

$$P(Y_1=r_1, Y_2=r_2 | X_1=n_1, X_2=n_2) = \frac{\binom{-m_1}{r_1}\binom{-\rho_1}{n_1-r_1}}{\sum_{x_1=k_1}^{n_1} \binom{-m_1}{x_1}\binom{-\rho_1}{n_1-x_1}} \; \frac{\binom{-m_2}{r_2}\binom{-\rho_2}{n_2-r_2}}{\sum_{x_2=k_2}^{n_2} \binom{-m_2}{x_2}\binom{-\rho_2}{n_2-x_2}} \tag{4.4.5}$$

$r_i=k_i, k_i+1, \ldots;\quad i=1,2.$

$r_i \leq n_i$

i.e. truncated Double Negative Hypergeometric.

Then, condition (4.3.2) holds iff

$$P_{n_1,n_2} = \frac{\sum_{r_1=k_1}^{n_1} \binom{-m_1}{r_1}\binom{-\rho_1}{n_1-r_1}(-q_1)^{n_1}(1-q_1)^{\rho_1}}{\sum_{r_1=0}^{\infty}\binom{-m_1}{r_1}(-q_1)^{r_1}} \frac{\sum_{r_2=k_2}^{n_2} \binom{-m_2}{r_2}\binom{-\rho_2}{n_2-r_2}(-q_2)^{n_2}(1-q_2)^{\rho_2}}{\sum_{r_2=0}^{\infty}\binom{-m_2}{r_2}(-q_2)^{r_2}} \tag{4.4.6}$$

$n_i = k_i, k_i+1, \ldots$
$i=1,2.$

(Observe that in (4.4.6) P_{n_1,n_2} is the joint distribution of two independent random variables X_i (i=1,2) such that X_i is distributed as the convolution of a Negative Binomial (p_i, ρ_i) truncated at k-1, with a Negative Binomial (p_i, ρ_i) i=1,2.

Note 1 It is worth pointing out here that everything that was said in the note of Corollary 3.6.1, problem 3.6.1, and the note of Corollary 3.6.2 is valid in the bivariate case also.

Note 2 Corollaries 4.4.1 and 4.4.2 can also be derived from the following theorem which is the bivariate extension of Theorem 3.6.2, and hence is in fact a generalization of Theorem 4.1.1 and a variant of Theorem 4.4.1.

Theorem 4.4.2

Let $\left\{\left(a_{n_1,n_2}, b_{n_1,n_2}\right) : n_1,n_2=0,1,\ldots\right\}$ be a sequence of vectors of non-negative real numbers such that

$$a_{n_1,n_2} > 0 \text{ for } n_1 \geq k_1,\ n_2 \geq k_2,\ k_1,k_2$$

non-negative integers, and $b_{n_1,n_2} \geq 0$ for all $n_1,n_2 > 0$, $b_{0,0} > 0$,

$b_{0,1} > 0$, $b_{1,n_2} > 0$ for some $n_2 \geqslant 0$.

Let $\{c_{n_1,n_2}\}$ be the convolution of $\{a_{n_1,n_2}\}$ and $\{b_{n_1,n_2}\}$. (Observe that $c_{n_1,n_2} > 0$, $n_1 \geqslant k_1$, $n_2 \geqslant k_2$).

Let also (X_1, X_2, Y_1, Y_2) be a random vector of non-negative integer-valued r.v's such that conditions (4.3.1) are met. Suppose that whenever $P_{n_1,n_2} > 0$

$$P(Y_1 = r_1, Y_2 = r_2 \mid X_1 = n_1, X_2 = n_2) = \frac{a_{r_1,r_2} b_{n_1-r_1,n_2-r_2}}{c_{n_1,n_2}} \qquad (4.4.7)$$

$$r_i = 0,1,\ldots,n_i$$
$$n_i = k_i, k_i+1,\ldots$$
$$i=1,2.$$

Then, condition (4.3.2) is true iff (4.3.3) holds.

Proof The proof of this theorem is the immediate extension to the bivariate case of the proof of Theorem 3.6.2, which was stated in the previous chapter.

4.5 The Multivariate Extension

Theorem 4.5.1

Let $\left\{\left(a_{\underset{\sim}{n}}, b_{\underset{\sim}{n}}\right): \underset{\sim}{n} = (n_1, n_2, \ldots, n_s)\ n_i = 0,1,2,\ldots; i=1,2,\ldots,s;\ s=1,2,\ldots\right\}$ be a sequence of real vectors such that

$$a_{\underset{\sim}{n}} > 0,\ b_{\underset{\sim}{n}} \geqslant 0 \text{ for every } n_i \geqslant 0 \quad i=1,2,\ldots,s \qquad (4.5.1)$$

with

$$b_{\underset{\sim}{0}} > 0,\ b_{0,0,\ldots,0,1} > 0 \qquad (4.5.2)$$

and

$$\left.\begin{array}{ll} \text{some } b_{0,0,\ldots,0,1,n_s} & > 0 \\ \text{some } b_{0,0,\ldots,1,n_{s-1},n_s} & > 0 \\ \vdots \quad \vdots & \vdots \\ \text{some } b_{1,n_2,n_3,\ldots,n_s} & > 0 \end{array}\right\} \qquad (4.5.3)$$

Define $\{c_{\underset{\sim}{n}}\}$ to be the convolution of $\{a_{\underset{\sim}{n}}\}$ and $\{b_{\underset{\sim}{n}}\}$ given by

$$c_{\underset{\sim}{n}} = \sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}} \qquad (4.5.4)$$

where $a_{\underset{\sim}{r}} = a_{r_1,\ldots,r_s}$ and $\sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}}$ denoting

$$\sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \cdots \sum_{r_s=0}^{n_s} .$$

Consider a random vector $(\underset{\sim}{X}, \underset{\sim}{Y})$ where $\underset{\sim}{X} = (X_1,\ldots,X_s)$, $\underset{\sim}{Y} = (Y_1,\ldots,Y_s)$ with non-negative integer-valued components such that

$$P(\underset{\sim}{X}=\underset{\sim}{n}) = P_{\underset{\sim}{n}} \qquad (4.5.5)$$

i.e. $P(X_1=n_1,\ldots,X_s=n_s) = P_{n_1,\ldots,n_s}$

with

$$P_{n_1,n_2,\ldots,n_s} > 0 \text{ for some } n_i \text{ for every } i=1,\ldots,s \qquad (4.5.6)$$

and whenever $P_{\underset{\sim}{n}} > 0$ we have

$$P(\underset{\sim}{Y}=\underset{\sim}{r} \mid \underset{\sim}{X}=\underset{\sim}{n}) = \frac{a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}}}{c_{\underset{\sim}{n}}} \qquad r_i=0,1,\ldots,n_i \quad i=1,\ldots,s. \qquad (4.5.7)$$

Also define $X^{(j)} = (X_1,\ldots,X_j)$, $Y^{(j)} = (Y_1,\ldots,Y_j)$ $j=2,3,\ldots,s$ and let $X^{(j)} > Y^{(j)}$ denote that $(X_k=Y_k,\ k=1,\ldots,j-1$ and $X_j > Y_j)$.

Then

$$P(\underset{\sim}{Y}=\underset{\sim}{r}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(j)} > Y^{(j)}) \qquad j=2,3,\ldots,s \tag{4.5.8}$$

(i.e. 4.5.8 consists of s equations)

iff

$$\frac{P_{\underset{\sim}{n}}}{c_{\underset{\sim}{n}}} = \frac{P_{\underset{\sim}{0}}}{c_{\underset{\sim}{0}}} \prod_{i=1}^{s} \theta_i^{n_i} \quad \text{for some } \theta_1,\ldots,\theta_s > 0. \tag{4.5.9}$$

Also if (4.5.9) is true then

$$\underset{\sim}{Y} \text{ and } \underset{\sim}{X}-\underset{\sim}{Y} \text{ are independent.} \tag{4.5.10}$$

<u>Proof</u> This is given in the following four statements.

<u>Statement 1</u>

The system of equations (4.5.8) is equivalent to the following set of conditions

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}) \qquad \text{(a)}$$

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(\ell-1)} = Y^{(\ell-1)}) \qquad \text{(b)} \tag{4.5.11}$$

$$\ell=2,\ldots,s.$$

(4.5.11) consists again of s equations)
where by $X^{(\ell)} = Y^{(\ell)}$ we mean $X_k=Y_k$ $k=1,..,\ell$. (4.5.12)

(Note that if we use the notation $X^{(0)} = Y^{(0)}$ to denote

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(0)}=Y^{(0)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}) \tag{4.5.13}$$

we can combine the equations of system (4.5.11) to one, namely

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(\ell-1)}=Y^{(\ell-1)}) \tag{4.5.14}$$

$$\ell=1,2,\ldots,s.)$$

Proof of Statement 1

We will first show that (4.5.8) gives (4.5.11). Obviously (4.5.8) gives (4.5.11a). On the other hand for ℓ=s we have (using 4.5.8)

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s-1)}=Y^{(s-1)})$$

$$= P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ P(X_s=Y_s|X^{(s-1)}=Y^{(s-1)}) + P(X_s>Y_s|X^{(s-1)}=Y^{(s-1)})\ P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s)}>Y^{(s)})$$

$$= P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ P(X_s=Y_s|X^{(s-1)}=Y^{(s-1)}) + P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ P(X_s>Y_s|X^{(s-1)}=Y^{(s-1)}).$$

Hence

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s-1)}=Y^{(s-1)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}),$$

and so (4.5.11b) is valid for ℓ=s.

Suppose now that it is valid for ℓ=k+1,k=2,...,s-1.

i.e. suppose that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k)}=Y^{(k)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) \quad k=2,\dots,s-1. \qquad (4.5.1$$

We will show that it is also valid for ℓ=k.

i.e. that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-1)}=Y^{(k-1)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) \quad k=2,\dots,s-1. \qquad (4.5.1$$

For the L.H.S. of (4.5.16) we have

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-1)}=Y^{(k-1)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k)}=Y^{(k)})\ P(X_k=Y_k|X^{(k-1)}=Y^{(k-1)})$$
$$+ P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k)}>Y^{(k)})\ P(X_k>Y_k|X^{(k-1)}=Y^{(k-1)}).$$

Hence (by making use of (4.5.8) and (4.5.15))

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-1)}=Y^{(k-1)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ P(X_k=Y_k|X^{(k-1)}=Y^{(k-1)})$$
$$+ P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ P(X_k>Y_k|X^{(k-1)}=Y^{(k-1)})$$

which results in (4.5.16).

Converse (4.5.11a) is the first equation of (4.5.8).

On the other hand (4.5.11b) for ℓ=s gives

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s-1)}=Y^{(s-1)})$$

$$= P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ P(X_s=Y_s|X^{(s-1)}=Y^{(s-1)}) + P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s)}>Y^{(s)})\ P(X_s>Y_s|X^{(s-1)}=Y^{(s-1)})$$

so,

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ [1-P(X_s=Y_s|X^{(s-1)}=Y^{(s-1)})] = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s)}>Y^{(s)})\ P(X_s>Y_s|X^{(s-1)}=Y^{(s-$$

i.e.

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s)}>Y^{(s)}) \qquad (4.5.17$$

which is (4.5.8) with j=s.

Suppose now that (4.5.11b) gives (4.5.8) for j=k, k=3,...,s, i.e. that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k)}>Y^{(k)}) \quad k=3,\dots,s-1. \qquad (4.5.18$$

We will show that it also gives (4.5.8) for j=k-1. But using (4.5.11b) for ℓ=k-1 we obtain

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-2)}=Y^{(k-2)})$$

$$= P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-1)}=Y^{(k-1)})P(X_{k-1}=Y_{k-1}|X^{(k-2)}=Y^{(k-2)})$$
$$+P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-1)}>Y^{(k-1)})\ P(X_{k-1}>Y_{k-1}|X^{(k-2)}=Y^{(k-2)}).$$

So, taking into consideration (4.5.18)

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})\ [1-P(X_{k-1}=Y_{k-1}|X^{(k-2)}=Y^{(k-2)})]$$
$$= P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-1)}>Y^{(k-1)})\ P(X_{k-1}>Y_{k-1}|X^{(k-2)}=Y^{(k-2)})$$

which is (4.5.8) for j=k-1.

That completes the proof of Statement 1.

We now define the sequences

$$\left.\begin{aligned} V_{n_s} &= \frac{P_{r_1,r_2,\ldots,r_{s-1},n_s}}{c_{r_1,r_2,\ldots,r_{s-1},n_s}} \quad \text{for every } r_i > 0 \quad i=1,\ldots,s-1 \\ &\qquad r_i \text{ fixed and } n_s \geq 0 \qquad (a) \\ W_{n_s} &= \frac{b_{0,0,\ldots,0,n_s} P(\underset{\sim}{X}=\underset{\sim}{Y})}{b_{\underset{\sim}{0}} P(X^{(s-1)}=Y^{(s-1)})} \qquad (b) \end{aligned}\right\} \quad (4.5.19)$$

Statement 2

Equation (4.5.11b) for $\ell=s$ is equivalent to

$$\sum_{n_s=0}^{\infty} V_{n_s+r_s} W_{n_s} = V_{r_s} \qquad (4.5.20)$$

with V_{n_s} and W_{n_s} as in (4.5.19).

Proof of Statement 2

Substituting for V_{n_s} and W_{n_s} the expression (4.5.19) we find that (4.5.20) is equivalent to

$$\sum_{n_s=0}^{\infty} \frac{P_{r_1,\ldots,r_{s-1},n_s+r_s}}{c_{r_1,\ldots,r_{s-1},n_s+r_s}} \quad \frac{b_{0,\ldots,0,n_s} \; P(\underset{\sim}{X}=\underset{\sim}{Y})}{b_{\underset{\sim}{0}} \; P(X^{(s-1)}=Y^{(s-1)})} = \frac{P_{\underset{\sim}{r}}}{c_{\underset{\sim}{r}}} . \qquad (4.5.21)$$

On the other hand,

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s-1)}=Y^{(s-1)})$$

is equivalent to

$$\frac{P(\underset{\sim}{Y}=\underset{\sim}{r},\underset{\sim}{X}=\underset{\sim}{Y})}{P(\underset{\sim}{X}=\underset{\sim}{Y})} = \frac{P(\underset{\sim}{Y}=\underset{\sim}{r},X^{(s-1)}=Y^{(s-1)})}{P(X^{(s-1)}=Y^{(s-1)})}$$

i.e. to

$$\frac{P_{\underset{\sim}{r}} \dfrac{a_{\underset{\sim}{r}} b_{\underset{\sim}{0}}}{c_{\underset{\sim}{r}}}}{P(\underset{\sim}{X}=\underset{\sim}{Y})} = \frac{\sum_{n_s=r_s}^{\infty} P_{r_1,\ldots,r_{s-1},n_s} \dfrac{a_{\underset{\sim}{r}} b_{0,\ldots,0,n_s-r_s}}{c_{r_1,\ldots,r_{s-1},n_s-r_s}}}{P(X^{(s-1)}=Y^{(s-1)})}$$

i.e. to

$$\sum_{n_s=0}^{\infty} \frac{P_{r_1,\ldots,r_{s-1},n_s+r_s}}{c_{r_1,\ldots,r_{s-1},n_s}} \quad \frac{b_{0,\ldots,0,n_s} \; P(\underset{\sim}{X}=\underset{\sim}{Y})}{b_{\underset{\sim}{0}} \; P(X^{(s-1)}=Y^{(s-1)})} = \frac{P_{\underset{\sim}{r}}}{c_{\underset{\sim}{r}}} . \tag{4.5.22}$$

Comparison of (4.5.21) and (4.5.22) completes the proof of Statement 2.

As a result of Statement 2 and by making use of lemma 3.1.1, we come to the conclusion that (4.5.11b) is true for $\ell=s$ iff

$$\frac{P_{r_1,\ldots,r_{s-1},n_s}}{c_{r_1,\ldots,r_{s-1},n_s}} = \frac{P_{r_1,\ldots,r_{s-1},0}}{c_{r_1,\ldots,r_{s-1},0}} \; \theta_s^{n_s}$$

for some $\theta_s > 0$ and every $r_i > 0$ $i=1,\ldots,s-1$, $n_s > 0$ (since r_i, $i=1,2,\ldots,s-1$ were fixed but arbitrary).

Hence (4.5.11b) holds iff

$$\frac{P_{n_1,\ldots,n_{s-1},n_s}}{c_{n_1,\ldots,n_{s-1},n_s}} = \frac{P_{n_1,\ldots,n_{s-1},0}}{c_{n_1,\ldots,n_{s-1},0}} \; \theta_s^{n_s} \tag{4.5.23}$$

for some $\theta_s > 0$ and every $n_i > 0$, $i=1,2,\ldots,s$.

<u>Note 1</u> In (4.5.23) θ_s is independent of r_i $i=1,\ldots,s-1$. This is so, because as we saw in lemma 3.1.1 if (4.5.20) holds then $\Sigma W_{n_s} \theta_s^{n_s} = 1$ and hence θ_s is unique, i.e. independent of r_i.

<u>Statement 3</u>

Whenever (4.5.23) is valid, we have the conditional on $X^{(s-1)}=Y^{(s-1)}$ the vector $\underset{\sim}{Y}$ and the random variable X_s-Y_s are independent.

Proof of Statement 3.

From (4.5.7) we have,

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s-1)}=Y^{(s-1)},X_s=n_s) = \frac{a_{\underset{\sim}{r}}\, b_{0,\ldots,0,n_s-r_s}}{c_{r_1,\ldots,r_{s-1},n_s}} .$$

So, (from(4.5.23))

$$P(\underset{\sim}{Y}=\underset{\sim}{r},X^{(s-1)}=Y^{(s-1)},X_s-Y_s=n_s-r_s)$$

$$= \frac{P_{r_1,\ldots,r_{s-1},0}}{c_{r_1,\ldots,r_{s-1},0}} \left(a_{\underset{\sim}{r}}\theta_s^{r_s}\right)\left(b_{0,\ldots,0,n_s-r_s}\;\theta_s^{n_s-r_s}\right)$$

i.e. (by replacing n_s-r_s by n_s) to

$$P(\underset{\sim}{Y}=\underset{\sim}{r},X^{(s-1)}=Y^{(s-1)},X_s-Y_s=n_s)$$

$$= \frac{P_{r_1,\ldots,r_{s-1},0}}{c_{r_1,\ldots,r_{s-1},0}} \left(a_{\underset{\sim}{r}}\theta_s^{r_s}\right) b_{0,\ldots,0,n_s}\;\theta_s^{n_s} \qquad (4.5.24)$$

But it is

$$P(\underset{\sim}{Y}=\underset{\sim}{r},X_s-Y_s=n_s\,|X^{(s-1)}=Y^{(s-1)}) = \frac{P(\underset{\sim}{Y}=\underset{\sim}{r},X^{(s-1)}=Y^{(s-1)},X_s-Y_s=n_s)}{P(X^{(s-1)}=Y^{(s-1)})}$$

or taking into account (4.5.24)

$$P(\underset{\sim}{Y}=\underset{\sim}{r},X_s-Y_s=n_s\,|X^{(s-1)}=Y^{(s-1)}) = R\,\frac{P_{r_1,\ldots,r_{s-1},0}}{c_{r_1,\ldots,r_{s-1},0}} \left(a_{\underset{\sim}{r}}\;\theta_s^{r_s}\right)\left(b_{0,\ldots,0,n_s}\theta_s^{n_s}\right)$$

(4.5.25)

where R is a constant $R^{-1} = P(X^{(s-1)}=Y^{(s-1)})$.

So by summing over n_s and $\underset{\sim}{r}$ in succession on both sides of (4.5.25) we get

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s-1)}=Y^{(s-1)}) = R_1 \frac{P_{r_1,\ldots,r_{s-1},0}}{c_{r_1,\ldots,r_{s-1},0}} a_{\underset{\sim}{r}} \theta_s^{r_s} \qquad (4.5.26)$$

and also

$$P(X_s-Y_s=n_s|X^{(s-1)}=Y^{(s-1)}) = R_2\, b_{0,\ldots,0,n_s} \theta_s^{n_s} \qquad (4.5.27)$$

where R_1, R_2 are constants, $R_1R_2 = R$.

(4.5.25), (4.5.26) and (4.5.27) give

$$P(\underset{\sim}{Y}=\underset{\sim}{r},X_s-Y_s=n_s|X^{(s-1)}=Y^{(s-1)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(s-1)}=Y^{(s-1)})\, P(X_s-Y_s=n_s|X^{(s-1)}=Y^{(s-1)}).$$

Hence Statement 3 follows.

Now, define the sequences V_{n_ℓ} and W_{n_ℓ} where

$$V_{n_\ell} = \frac{P_{r_1,\ldots,r_{\ell-1},n_\ell,0,\ldots,0}}{c_{r_1,\ldots,r_{\ell-1},n_\ell,0,\ldots,0}}, \quad r_i > 0 \text{ fixed},\ i=1,2,\ldots,\ell-1 \text{ and every } n_\ell > 0. \qquad \text{(a)}\ (4.5.28)$$

and

$$W_{n_\ell} = \sum_{n_{\ell+1}=0}^{\infty} \cdots \sum_{n_s=0}^{\infty} b_{0,\ldots,0,n_\ell,n_{\ell+1},\ldots,n_s} \theta_{\ell+1}^{n_{\ell+1}} \cdots \theta_s^{n_s} \frac{P(\underset{\sim}{X}=\underset{\sim}{Y})}{b_{\underset{\sim}{0}} P(X^{(\ell-1)}=Y^{(\ell-1)})} \qquad \text{(b)}$$

for $\theta_{\ell+1},\ldots,\theta_s > 0$, $\ell=1,\ldots,s-1$.

<u>Statement 4</u>

Assume that (4.5.14) holds for $\ell=k,k+1,\ldots,s$, $2 \leqslant k \leqslant s$ and is equivalent to

$$\frac{P_{n_1,\ldots,n_{k-1},n_k,\ldots,n_s}}{c_{n_1,\ldots,n_{k-1},n_k,\ldots,n_s}} = \frac{P_{n_1,\ldots,n_{k-1},0,\ldots,0}}{c_{n_1,\ldots,n_{k-1},0,\ldots,0}} \theta_k^{n_k}\ldots\theta_s^{n_s} \qquad (4.5.29)$$

for some $\theta_k,\ldots,\theta_s > 0$ and every $n_i > 0$ $i=1,\ldots,s$.

(Note that if (4.5.29) is valid then conditional on $X^{(k-1)}=Y^{(k-1)}$, $\underset{\sim}{Y}$ and $(X_k-Y_k, X_{k+1}-Y_{k+1},\ldots,X_s-Y_s)$ are independent.)

Then we have that

(a) (4.5.14) for $\ell=k-1,k,\ldots,s$ is equivalent to

$$\frac{P_{n_1,\ldots,n_{k-1},\ldots,n_s}}{c_{n_1,\ldots,n_{k-1},\ldots,n_s}} = \frac{P_{n_1,\ldots,n_{k-2},0,\ldots,0}}{c_{n_1,\ldots,n_{k-2},0,\ldots,0}} \theta_{k-1}^{n_{k-1}}\theta_k^{n_k}\ldots\theta_s^{n_s} \qquad (4.5.30)$$

for some suitable $\theta_{k-1},\ldots,\theta_s > 0$ and for every $n_i > 0$ $i=1,\ldots,s$ $2 \leqslant k \leqslant s$.

(b) If (4.5.30) is true, then conditional on $X^{(k-2)}=Y^{(k-2)}$, $\underset{\sim}{Y}$ and $(X_{k-1} - Y_{k-1}, X_k-Y_k,\ldots,X_s-Y_s)$ will be independent ($2 \leqslant k \leqslant s$).

<u>Proof of Statement 4</u>

<u>Part (a)</u> Considering (4.5.28) for $\ell=k-1$ the relation

$$\sum_{n_{k-1}=0}^{\infty} V_{n_{k-1}+r_{k-1}} W_{n_{k-1}} = V_{r_{k-1}}$$

is equivalent to

$$\sum_{n_{k-1}=0}^{\infty} \frac{P_{r_1,\ldots,r_{k-2},n_{k-1}+r_{k-1},0,\ldots,0}}{c_{r_1,\ldots,r_{k-2},n_{k-1}+r_{k-1},0,\ldots,0}} \sum_{n_k=0}^{\infty}\cdots\sum_{n_s=0}^{\infty} b_{0,\ldots,0,n_{k-1},\ldots,n_s}\theta_k^{n_k}\ldots\theta_s^{n_s}$$

$$\times \frac{P(\underset{\sim}{X}=\underset{\sim}{Y})}{b_{\underset{\sim}{0}}P(X^{(k-2)}=Y^{(k-2)})} = \frac{P_{r_1,\ldots,r_{k-1},0,\ldots,0}}{c_{r_1,\ldots,r_{k-1},0,\ldots,0}} \qquad (4.5.31)$$

On the other hand (4.5.14) for $\ell=k-1$ becomes

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-2)}=Y^{(k-2)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y})$$

which is equivalent to

$$\frac{\sum_{n_s=r_s}^{\infty}\cdots\sum_{n_{k-1}=r_{k-1}}^{\infty} P_{r_1,\ldots,r_{k-2},n_{k-1},\ldots,n_s} \dfrac{a_{\underset{\sim}{r}}\, b_{0,\ldots,0,n_{k-1}-r_{k-1},\ldots,n_s-r_s}}{c_{r_1,\ldots,r_{k-2},n_{k-1}-r_{k-1},\ldots,n_s-r_s}}}{P(X^{(s-2)}=Y^{(s-2)})}$$

$$= \frac{P_{\underset{\sim}{r}}}{c_{\underset{\sim}{r}}} \frac{a_{\underset{\sim}{r}}\, b_{\underset{\sim}{0}}}{P(\underset{\sim}{X}=\underset{\sim}{Y})}$$

i.e. (by replacing $n_j - r_j$ by n_j ; $j=k-1,\ldots,s$) to

$$\sum_{n_s=0}^{\infty}\cdots\sum_{n_{k-1}=0}^{\infty} \frac{P_{r_1,\ldots,r_{k-2},n_{k-1}+r_{k-1},\ldots,n_s+r_s}}{c_{r_1,\ldots,r_{k-2},n_{k-1}+r_{k-1},\ldots,n_s+r_s}} \frac{b_{0,\ldots,0,n_{k-1},\ldots,n_s}}{b_{\underset{\sim}{0}}} \frac{P(\underset{\sim}{X}=\underset{\sim}{Y})}{P(X^{(s-2)}=Y^{(s-2)})}$$

$$= \frac{P_{\underset{\sim}{r}}}{c_{\underset{\sim}{r}}}$$

and by making use of (4.5.29) to

$$\sum_{n_s=0}^{\infty}\cdots\sum_{n_{k-1}=0}^{\infty} \frac{P_{r_1,\ldots,r_{k-2},n_{k-1}+r_{k-1},0,\ldots,0}}{c_{r_1,\ldots,r_{k-2},n_{k-1}+r_{k-1}\,0,\ldots,0}} \theta_k^{n_k+r_k}\ldots\theta_s^{n_s+r_s}$$

$$\times \frac{b_{0,\ldots,0,n_{k-1},\ldots,n_s}}{b_{\underset{\sim}{0}}} \frac{P(\underset{\sim}{X}=\underset{\sim}{Y})}{P(X^{(k-2)}=Y^{(k-2)})} = \frac{P_{r_1,\ldots,r_{k-1},0,\ldots,0}}{c_{r_1,\ldots,r_{k-1},0,\ldots,0}} \theta_k^{r_k}\ldots\theta_s^{r_s}. \tag{4.5.32}$$

Since (4.5.31) and (4.5.32) are identical we have, (using lemma 3.1.1) that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(k-2)}=Y^{(k-2)}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) \qquad \text{(i.e. (4.5.14) for } \ell=k-1\text{)}$$

iff

$$\frac{P_{r_1,\ldots,r_{k-2},n_{k-1},0,\ldots,0}}{c_{r_1,\ldots,r_{k-2},n_{k-1},0,\ldots,0}} = \frac{P_{r_1,\ldots,r_{k-2},0,\ldots,0}}{c_{r_1,\ldots,r_{k-2},0,\ldots,0}} \theta_{k-1}^{n_{k-1}}$$

for some $\theta_{k-1} > 0$ and for every $r_1,\dots,r_{k-2},n_{k-1} > 0$ (since r_i, $i=1,\dots,k-1$ were fixed but arbitrary), in other words iff

$$\frac{P_{n_1,\dots,n_{k-1},0,\dots,0}}{c_{n_1,\dots,n_{k-1},0,\dots,0}} = \frac{P_{n_1,\dots,n_{k-2},0,\dots,0}}{c_{n_1,\dots,n_{k-2},0,\dots,0}} \theta_{k-1}^{n_{k-1}} \qquad 2 \leqslant k \leqslant s \tag{4.5.33}$$

Hence, taking into account the assumption of Statement 4 and combining (4.5.29) and (4.5.33) we come to the conclusion that (4.5.14) for $\ell=k-1,k,\dots,s$ is equivalent to (4.5.30). That completes the proof of Part (a) of Statement 4.

Part b can be derived by the same method used to prove Statement 3.

The statement we have just proved shows that the system of equations (4.5.14) is equivalent to (4.5.9). Thus, since (4.5.14) is equivalent to (4.5.8), the first part of the theorem follows. On the other hand, as a direct consequence of Statement 4 Part (b), we have that if (4.5.9) is valid then $\underset{\sim}{Y}$ and $\underset{\sim}{X}-\underset{\sim}{Y}$ are independent; this is the second part of Theorem 4.5.1. Hence the whole theorem is established.

<u>Corollary 4.5.1</u> (Characterization of the Multiple Poisson distribution)

Suppose that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}\,|\,\underset{\sim}{X}=\underset{\sim}{n}) = \prod_{i=1}^{s} \binom{n_i}{r_i} p_i^{r_i} q_i^{n_i-r_i} \qquad r_i=1,\dots,n_i,\quad i=1,2,\dots,s \tag{4.5.34}$$

i.e. multiple binomial.

Then, condition (4.5.8) holds, iff

$$P_{\underset{\sim}{n}} = e^{-\lambda} \prod_{i=1}^{s} \frac{\lambda_i^{n_i}}{n_i!} \qquad i=1,2,\dots,s,\quad \lambda = \sum_{i=1}^{s} \lambda_i,\quad \lambda_i > 0 \tag{4.5.35}$$

i.e. multiple Poisson.

Corollary 4.5.2 (Characterization of the Multiple Binomial distribution)

Suppose that $P_{\underset{\sim}{n}}$ is multiple Poisson of the form (4.5.35) and that the conditional probability of $\underset{\sim}{Y}|\underset{\sim}{X}$ can be written in the form (4.5.7). Then, condition (4.5.8) is true iff $P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{n})$ is multiple Binomial, given by (4.5.34).

Corollary 4.5.3 (Characterization of the Multiple Negative Binomial Distribution)

Suppose that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{n}) = \prod_{i=1}^{s} \frac{\binom{-m_i}{r_i}\binom{-\rho_i}{n_i-r_i}}{\binom{-m_i-\rho_i}{n_i}} \qquad \begin{array}{l} r_i=0,\dots,n_i \\ m_i > 0 \\ \rho_i > 0 \\ i=1,2,\dots,s \end{array} \qquad (4.5.3\ldots)$$

i.e. multiple Negative Hypergeometric.

Then, condition (4.5.8) holds iff

$$P_{\underset{\sim}{n}} = \prod_{i=1}^{s} \binom{-N_i}{n_i} p_i^{N_i} (-q_i)^{n_i} \qquad N_i=m_i+\rho_i \qquad (4.5.3\ldots)$$

i.e. multiple Negative Binomial ($p_i, m_i+\rho_i$ $i=1,\dots,s$).

We omit the proofs of these three corollaries since they are the immediate extension of the corresponding bivariate ones, given in Section 4.2. Corollary 4.5.1 is an improved version of a result given by Talwalker (1970).

Corollary 4.5.4 (Characterization of the Negative Multinomial Distribution)

Suppose that

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{n}) = \frac{B(m+r_1+\dots+r_s,\ \rho+(n_1-r_1)+\dots(n_s-r_s))}{B(m,\rho)} \prod_{i=1}^{s} \binom{n_i}{r_i} \qquad (4.5.3\ldots)$$

$$r_i=0,1,\dots,n_i,\ m>0,\ \rho>0,\ i=1,2,\dots,s$$

(Multivariate Inverse Hypergeometric with parameters m,ρ).

Then, condition (4.5.8) holds iff

$$P(\underset{\sim}{X}=\underset{\sim}{n}) = \frac{\Gamma(m+\rho+n_1+\ldots+n_s)}{\Gamma(m+\rho)} p_0^{m+\rho} \prod_{i=1}^{s} \frac{p_i^{n_i}}{n_i!} \tag{4.5.39}$$

$$n_i=0,1,\ldots;\ 0<p_i<1,\ \sum_{i=1}^{s} p_i < 1,\ i=1,2,\ldots,s,\ p_0 = 1-\sum_{i=1}^{s} p_i$$

(Negative Multinomial with parameters $m+\rho$, $p_1,\ldots,p_s$).

Proof Let us consider the following sequences

$$a_{\underset{\sim}{r}} = \frac{\Gamma(m+r_1+\ldots+r_s)}{\Gamma(m)\prod_{i=1}^{s} r_i!}, \qquad b_{\underset{\sim}{n}} = \frac{\Gamma(\rho+n_1+\ldots+n_s)}{\Gamma(\rho)\prod_{i=1}^{s} n_i!} \tag{4.5.40}$$

$$r_i=0,1,\ldots,n_i,\quad n_i=0,1,\ldots$$

The convolution $\{c_n\}n=0,1,\ldots$ of these sequences is

$$c_{\underset{\sim}{n}} = \frac{\Gamma(m+\rho+n_1+\ldots+n_s)}{\Gamma(m+\rho)\prod_{i=1}^{s} n_i} \sum_{\underset{\sim}{r}=\underset{\sim}{0}}^{\underset{\sim}{n}} \frac{B(m+r_1+\ldots+r_s,\rho+(n_1-r_1)+\ldots+(n_s-r_s))}{B(m,\rho)} \prod_{i=1}^{s} \binom{n_i}{r_i}$$

i.e.

$$c_{\underset{\sim}{n}} = \frac{\Gamma(m+\rho+n_1+\ldots+n_s)}{\Gamma(m+\rho)\prod_{i=1}^{s} n_i} \qquad n_i=1,2,\ldots,\quad i=1,2,\ldots,s\ . \tag{4.5.41}$$

It can be checked that the conditional distribution (4.5.38) can be expressed in the form $a_{\underset{\sim}{r}} b_{\underset{\sim}{n}-\underset{\sim}{r}}/c_{\underset{\sim}{n}}$ with $a_{\underset{\sim}{r}}$, $b_{\underset{\sim}{n}}$, $c_{\underset{\sim}{n}}$ given by (4.5.40) and (4.5.41). Hence from Theorem 4.5.1 we have that condition (4.5.8) is equivalent to

$$\frac{P_{\underset{\sim}{n}}}{c_{\underset{\sim}{n}}} = \frac{P_{\underset{\sim}{0}}}{c_{\underset{\sim}{0}}} \prod_{i=1}^{s} \theta_i \quad \text{for some } \theta_i > 0 \quad i=1,2,\ldots,s$$

i.e. to

$$P_{\underset{\sim}{n}} = P_{\underset{\sim}{0}} \frac{\Gamma(m+\rho+n_1+\ldots+n_s)}{\Gamma(m+\rho)} = \frac{\prod_{i=1}^{s} \theta_i^{n_i}}{n_i\,!} . \tag{4.5.42}$$

Since $P_{\underset{\sim}{n}}$ is a probability distribution

$$P_{\underset{\sim}{0}}^{-1} = \frac{\left(\sum_{i=1}^{s} \theta_i\right)^n}{P_0^{m+\rho}} \sum_{\underset{\sim}{n}=\underset{\sim}{0}}^{\infty} \frac{\Gamma(m+\rho+n_1+\ldots+n_s)}{\Gamma(m+\rho)} P_0^{m+\rho} \prod_{i=1}^{s} \frac{\left(\frac{\theta_i}{\sum_{i=1}^{s}\theta_i}\right)^{n_i}}{n_i\,!}$$

where $n = \sum_{i=1}^{s} n_i$

i.e.

$$P_{\underset{\sim}{0}}^{-1} = \frac{\left(\sum_{i=1}^{s} \theta_i\right)^n}{P_0^{m+\rho}} . \tag{4.5.43}$$

Substituting (4.5.43) in (4.5.42) gives the required result with

$$p_i = \frac{\theta_i}{\sum_{i=1}^{s} \theta_i} .$$

<u>Note 1</u> A similar result in the bivariate case has been proved by Patil and Ratnaparkhi but with the additional condition that

$\dfrac{\partial^{r+\ell}\, G(t_1, t_2)}{\partial t_1^r \;\partial t_2^\ell}$ exists, for r,ℓ positive integers and

$G(t_1, t_2)$ the p.g.f. of (X_1, X_2).

4.6 The Multivariate Truncated Extension and its Variant

Theorem 4.6.1

Consider the random vector $(\underset{\sim}{X},\underset{\sim}{Y})$ and the sequence

$$\{(a_{\underset{\sim}{n}},b_{\underset{\sim}{n}}): \underset{\sim}{n}=(n_1,\ldots,n_s),\ n_i=0,1,\ldots \quad i=1,2,\ldots,s \ \ s=1,2\ldots\} \text{ as}$$

in theorem 4.5.1.

Suppose that k_i, $i=1,\ldots,s$ are s positive integers such that

$$\begin{aligned} &P(X_i \geqslant k_i) > 0 \qquad i=1,\ldots,s \\ &P(\underset{\sim}{X} \geqslant \underset{\sim}{k}) = 1 \end{aligned} \tag{4.6.1}$$

where $\underset{\sim}{k} = (k_1,\ldots,k_s)$ and by $\underset{\sim}{X} \geqslant \underset{\sim}{k}$ we mean $X_i \geqslant k_i$ for every $i=1,\ldots,s$.

Then,

$$P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{Y} \geqslant \underset{\sim}{k}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|\underset{\sim}{X}=\underset{\sim}{Y}) = P(\underset{\sim}{Y}=\underset{\sim}{r}|X^{(j)} > Y^{(j)},\ Y_j \geqslant k_j) \qquad j=2,3,\ldots,s \tag{4.6.2}$$

iff

$$\frac{P_{\underset{\sim}{n}}}{c_{\underset{\sim}{n}}} = \frac{P_{\underset{\sim}{k}}}{c_{\underset{\sim}{k}}} \prod_{i=1}^{s} \theta_i^{n_i-k_i} \quad \text{for some } \theta_1,\ldots,\theta_s > 0 \qquad n_i=k_i,k_i+1,\ldots;\ \ i=1,2,\ldots,s. \tag{4.6.3}$$

Proof This follows straightforwardly.

Note 1 Theorem 4.6.1 can be used to characterize the truncated Multiple Poisson, (with $\underset{\sim}{Y}|\underset{\sim}{X}$ ~ Multiple Binomial) and the truncated Multiple Negative Binomial (with $\underset{\sim}{Y}|\underset{\sim}{X}$ ~ Multiple Negative Hypergeometric) through the condition (4.6.2).

Note 2 The Remark made in Section (4.4) about the sequences $\{a_{\underset{\sim}{n}},b_{\underset{\sim}{n}}\}$ can be extended to the Multivariate case, and thus provide the Multivariate extensions of theorem 4.4.1 and Corollaries 4.4.1 and 4.4.2. These corollaries can also be obtained as a result of the multivariate extension of theorem 4.4.2.

Note 3 Using the Multivariate Extension of theorem 4.4.2 a characterization of the truncated Negative Multinomial can be obtained through the condition (4.6.2) and under the assumption that the conditional distribution of $\underset{\sim}{Y}|\underset{\sim}{X}$ is Multivariate Inverse Hypergeometric as in (4.5.38). It can also be observed that if the distribution of $\underset{\sim}{Y}|\underset{\sim}{X}$ is truncated Multivariate Inverse Hypergeometric, then the condition (4.6.2) characterizes the distribution of $\underset{\sim}{X}$ as being the convolution of a Negative Multinomial with a truncated Negative Multinomial.