CHAPTER 2.

THE RAO-RUBIN CHARACTERIZATION.

) Introduction

The work of Rao (1963) and Rao and Rubin (1964) has been mentioned already. Rao and Rubin used damage model theory to characterize the Poisson distribution as the original distribution given binomial-type damage. They also used the same type of damage to characterize the truncated Poisson as an original distribution. The same characterization of the truncated Poisson appeared later in Kagan Linnik and Rao (K.L.R.) (1973). However, there seems to be some confusion in the two places as far as the definition and the role of the resulting r.v. is concerned. On the other hand, some workers have found it difficult to follow the original proof given by Rao and Rubin (1964), and also the subsequent proof of an elementary nature given by Shanbhag (1974). In order to give a complete and clear picture of the literature in this thesis, we give a full version of Shanbhag's elementary proof. We then point out the differences of meaning and notation in the characterization of the truncated Poisson. We also refer to the difficulties in characterizing the survival distribution using the R-R condition. Finally, we refer to the bivariate and multivariate extensions of the R-R theorem.

The Rao-Rubin Theorem. An Elementary Proof.

Theorem 2.1.1 (Rao-Rubin (1964)).

Let X be a non-negative, integer-valued r.v. with distribution $P(X=n) = P_n$, and Y be another r.v. such that for every n with $P_n > 0$

$$P(Y=r|X=n) = \binom{n}{r} p^r q^{n-r} \qquad r = 0,1,\dots,n, \qquad (2.1.1)$$

where p is some number lying in (0,1) and q = 1-p. Then

$$P(Y=r) = P(Y=r|X=Y) = P(Y=r|X>Y) \qquad (2.1.2)$$

iff

$$P_n = P(X=n) = e^{-\lambda} \frac{\lambda^n}{n!} \quad n = 0,1,\dots \qquad (2.1.3)$$

for some $\lambda>0$.

Proof

We follow Shanbhag (1974).

Let $G(t)$ denote the p.g.f. of X. Then (2.1.2) is equivalent to

$P(Y=r) = P(Y=r|X=Y)$, $r = 0,1,\dots$ and hence to

$$\frac{P_r p^r}{G(p)} = \sum_{n=r}^{\infty} P_n \binom{n}{r} p^r q^{n-r} \quad r = 0,1,\dots . \qquad (2.1.4)$$

Note that (2.1.4) is equivalent to

$$G(q+t) = C\, G(t) \quad |t| \leq 1 \qquad (2.1.5)$$

where $C = \{G(p)\}^{-1}$. If now $\{P_j\}$ is a distribution satisfying (2.1.4), or equivalently (2.1.5) and $\sum_{j=0}^{\infty} P_j t^j < \infty$ for $0 \leq t \leq k\,q + 1$, then using (2.1.4) we see that

$$\begin{aligned} \infty > \frac{1}{G(p)} \sum_{r=0}^{\infty} P_r t^r &= \sum_{r=0}^{\infty} t^r \sum_{n=r}^{\infty} P_n \binom{n}{r} q^{n-r} \\ &= \sum_{n=0}^{\infty} P_n \sum_{r=0}^{n} t^r \binom{n}{r} q^{n-r} \\ &= \sum_{n=0}^{\infty} P_n (t+q)^n \quad 0 \leq t \leq 1+kq,\ k \in I^+ \qquad (2.1.6) \end{aligned}$$

(The change in the order of summation is justified because the sums are non-negative).

Relation (2.1.6) implies that

$$\sum_{n=0}^{\infty} P_n \left(1 + (k+1)q\right)^n < \infty \qquad (2.1.7)$$

and hence from the monotonic increasing nature of the p.g.f. we have

$$\sum_{n=0}^{\infty} P_n t^n < \infty \quad \text{for } 0 \leqslant t \leqslant 1+(k+1)q \ . \tag{2.1.8}$$

Consequently by induction we get that if (2.1.5) is satisfied then

$$\sum_{n=0}^{\infty} P_n t^n < \infty \quad \text{for all} \quad t \in [0,\infty). \tag{2.1.9}$$

In that case G is defined for all $t \in (-\infty,\infty)$ and so, (2.1.5) reduces to

$$G(q+t) = C\,G(t), \ -\infty < t < \infty. \tag{2.1.10}$$

Since (2.1.10) holds for every t we have

$$\begin{aligned} G(t+kq) = G\big(t+(k-1)q+q\big) &= C\,G(t+(k-1)q\big) = C^2 G\big(t+(k-2)q\big) \\ &= \dots = C^k G(t). \end{aligned}$$

In other words

$$G(t+kq) = C^k G(t) \qquad -\infty < t < +\infty \tag{2.1.11}$$

which implies that

$$G(t) > 0 \text{ for all } t \in (-\infty,\infty). \tag{2.1.12}$$

(Obviously this is true for $t>0$. For $t<0$ we can always find an integer k such that $t+kq > 0$. For that k, $G(t+kq) > 0$, hence from (2.1.11) $G(t) > 0$.) Further, the fact that $\sum_{j=0}^{\infty} P_j t^j$ is absolutely convergent for all $t \in (-\infty,\infty)$ implies that G(t) is differentiable any number of times at all t.

Let us now denote by $G'(t)$ and $G''(t)$ the first and second derivatives of G(t). Restrict, first, for convenience to $t>0$. From (2.1.11) we get

$$\frac{G'(t)}{G(t)} = \frac{G'(t+kq)}{G(t+kq)}$$

and since this is true for every k, it will also be true for k going to infinity, i.e.

$$\frac{G'(t)}{G(t)} = \lim_{k\to\infty} \left\{ \frac{\sum_{j=1}^{\infty} j \, P_j \, (t+kq)^{j-1}}{\sum_{j=0}^{\infty} P_j \, (t+kq)^j} \right\} \tag{2.1.13}$$

For the same reason,

$$\frac{G''(t)}{G(t)} = \frac{G''(t+kq)}{G(t+kq)} = \lim_{k\to\infty} \left\{ \frac{\sum_{j=1}^{\infty} j \, (j-1) \, P_j \, (t+kq)^{j-2}}{\sum_{j=0}^{\infty} P_j \, (t+kq)^j} \right\}$$

$$= \lim_{k\to\infty} \left\{ \frac{\sum_{j=1}^{\infty} j^2 P_j \, (t+kq)^{j-2}}{\sum_{j=0}^{\infty} P_j \, (t+kq)^j} \right\} \tag{2.1.14}$$

$$\left(\text{Because} \quad \lim_{k\to\infty} \left\{ \frac{\sum_{j=1}^{\infty} j \, P_j \, (t+kq)^{j-2}}{\sum_{j=0}^{\infty} P_j \, (t+kq)^j} \right\} = 0, \text{ from (2.1.13)} \right)$$

Let us now define a discrete random variable w such that

$$P(w=j) = \frac{P_j \, (t+kq)^j}{\sum_{r=0}^{\infty} P_r \, (t+kq)^r} \tag{2.1.15}$$

Then

$$\text{Var}\left(\frac{w}{t+kq}\right) = \frac{\text{Var } w}{(t+kq)^2} = \frac{1}{(t+kq)^2}\left\{E(w^2) - [E(w)]^2\right\}$$

$$= \frac{1}{(t+kq)^2}\left\{\frac{\sum_{j=1}^{\infty} j^2 P_j \; (t+kq)^j}{\sum_{r=0}^{\infty} P_r \; (t+kq)^r} - \left[\frac{\sum_{j=1}^{\infty} j \; P_j \; (t+kq)^j}{\sum_{r=0}^{\infty} P_r \; (t+kq)^r}\right]^2\right\}$$

$$= \frac{\sum_{j=1}^{\infty} j^2 P_j \; (t+kq)^{j-2}}{\sum_{r=0}^{\infty} P_r \; (t+kq)^r} - \left[\frac{\sum_{j=1}^{\infty} j \; P_j \; (t+kq)^{j-2}}{\sum_{r=0}^{\infty} P_r \; (t+kq)^r}\right]^2 .$$

However, from (2.1.13) and (2.1.14)

$$\text{Var}\left(\frac{w}{t+kq}\right) = \frac{G''(t)}{G(t)} - \left\{\frac{G'(t)}{G(t)}\right\}^2 = \frac{d}{dt}\left\{\frac{G'(t)}{G(t)}\right\} .$$

Since the L.H.S. is always non-negative we have

$$\frac{d}{dt}\left\{\frac{G'(t)}{G(t)}\right\} \geqslant 0.$$

Consequently the function $A(t) = \frac{G'(t)}{G(t)}$ is a monotonic non-decreasing function.

On the other hand

$$A(t) = \frac{G'(t)}{G(t)} = \frac{G'(t+kq)}{G(t+kq)} = A(t+kq), \text{ k integer}$$

i.e. $A(t)$ is periodic as well.

So, we come to the conclusion that $\frac{G'(t)}{G(t)}$ must be a constant independent of t for $-\infty < t < \infty$.

Denoting this constant by λ we have

$$\frac{G'(t)}{G(t)} = \lambda,$$

i.e.

$$G(t) = e^{\lambda(t-1)} . \qquad (2.1.16)$$

This is the p.g.f. of the Poisson distribution.

2 The R-R theorem. The Truncated Case.

Rao and Rubin (1964) extend their result to the case where the original distribution is truncated at a point k-1 (k>0).

The same result appears later in Kagan, Linnik and Rao (1973). It seems, however, that the two versions are different although both are referring to the same kind of situation. Rao and Rubin state their case as follows.

Let X be a discrete r.v. taking the values k, k+1, and let the damage distribution be binomial, i.e. $s(r,n) = \binom{n}{r} p^r q^{n-r}$.

Denote the resulting random variable truncated at k-1 by Y. Then

$$P(Y=r) = P(Y=r|X=Y) = P(Y=r|X>Y) \qquad (2.2.1)$$

if and only if X is a Poisson truncated at k-1. In Kagan, Linnik and Rao (p.423, remark) there is nothing to indicate that the r.v. Y is truncated at k-1. They have erroneously claimed that, under the assumption that s(r,n) is binomial, (2.2.1) is true iff X is Poisson truncated at k-1.

The problem seems to start from the notation that R. and R. use. Since Y usually denotes the resulting r.v. (taking the values 0,1,...), it would probably be better, if we write the R-R condition (2.2.1) in the following form

$$P(Y=r|Y\geq k) = P(Y=r|X=Y) \quad r = k, k+1, \ldots . \qquad (2.2.2)$$

In this case we have

$$P(Y=r|Y\geqslant k) = \frac{\sum_{n=r}^{\infty} P_n\ s(r,n)}{\sum_{r=k}^{\infty}\sum_{n=r}^{\infty} P_n\ s(r,n)} . \tag{2.2.3}$$

It is obvious that the L.H.S. (2.2.2) can also be written as $\frac{P(Y=r)}{P(Y\geqslant k)}$ where

$$P(Y=r) = \sum_{n=r}^{\infty} P_n\ s(r,n) \text{ and } P(Y\geqslant k) = \sum_{r=k}^{\infty} P(Y=r).$$

We shall now establish the correct version of the characterization of the truncated Poisson using partially the proof of theorem 2.1.1.

<u>Theorem 2.2.1</u> Let X be a random variable taking values, k, k+1,... with distribution $\{P_n\}$ and Y be another r.v. such that for every n with $P_n > 0$

$$P(Y=r|X=n) = \binom{n}{r} p^r q^{n-r} \quad r = 0,1,\dots,n. \tag{2.2.4}$$

Then

$$P(Y=r|Y\geqslant k) = P(Y=r|X=Y) \quad r = k,k+1,\dots \tag{2.2.5}$$

iff X is Poisson truncated at k-1.

<u>Proof</u> The p.g.f. of $Y|Y\geqslant k$ is

$$\frac{\sum_{r=k}^{\infty}\sum_{n=r}^{\infty} P_n \binom{n}{r} p^r q^{n-r} t^r}{\sum_{r=k}^{\infty}\sum_{n=r}^{\infty} P_n \binom{n}{r} p^r q^{n-r}} = \frac{G(q+pt) - Q(t)}{G(1) - Q(1)} \tag{2.2.6}$$

where

$$Q(t) = \sum_{n=k}^{\infty} P_n \sum_{r=0}^{k-1} \binom{n}{r} (pt)^r q^{n-r} \tag{2.2.7}$$

i.e. a polynomial in t of degree k-1.

Hence, (2.2.5) is equivalent to

$$G(q+pt) - Q(t) = a\, G(pt) \tag{2.2.8}$$

where $a = \dfrac{G(1)-Q(1)}{G(p)}$ constant.

Setting z for pt and taking the k-th derivative

$$G^{(k)}(q+z) = a\, G^{(k)}(z). \tag{2.2.9}$$

Observing that after a suitable normalization $G^{(k)}(z)$ is a p.g.f. and taking into consideration theorem 2.1.1, we get

$$G^{(k)}(z) = b\, e^{\lambda z}$$

and therefore

$$G(z) = a\, e^{\lambda z} + f(z) \tag{2.2.10}$$

where f(z) is a polynomial of degree k-1 in z and a, b are constants.

It can now be checked (by equating the first k-1 derivatives of G(z) at z=0 to zero) that

$$G(z) = e^{-\lambda} \left\{e^{\lambda z} - \sum_{j=0}^{k-1} \frac{(\lambda z)^j}{j!}\right\} \div \left\{1 - e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!}\right\} \tag{2.2.11}$$

which is the p.g.f. of the Poisson distribution, truncated at k-1. This establishes the "only if" part. The "if" part is trivial.

3 <u>Characterization of the Survival Distribution</u>

<u>Theorem 2.3.1</u> (Srivastava and Srivastava (1970)).

If the r.v. X follows a Poisson distribution with parameter λ and if s(r,n) is the survival distribution then, the R-R condition (2.2.1) holds iff

$$s(r,n) = \binom{n}{r} p^r q^{n-r}.$$

As it can be checked, their proof is based on the assumption that λ is a variable and not a fixed number as in the R-R theorem. Later (Ch.3) we give a variant of that result, which is based on the assumption that λ is fixed.

It is natural now to ask the following question. What happens if the original distribution is truncated Poisson? Can we make any inference about the survival distribution using the R-R condition in the form (2.2.2)?

It seems that there has not been yet a satisfactory answer to this problem. Srivastava and Singh (1975) using the same argument as Srivastava and Srivastava (1970) come to the conclusion that, under the assumption that the original is Poisson (λ) truncated at k-1, the R-R condition (2.2.2) is equivalent to the functional equation

$$\sum_{i=k}^{m-k} \frac{s(k,m-i)s(i,i)}{(m-i)!\ i!} = \frac{s(k,k)}{k!} \frac{1}{(m-k)!} \sum_{i=k}^{m-k} s(i,m-k) \qquad m \geqslant 2k. \qquad (2.3.1)$$

They, then conjecture that the only solution to that functional equation is a "modified" Binomial probability model defined as follows

$$s(r,n) = \begin{cases} \binom{n}{r} p^r (1-p)^{n-r} & k \leqslant r \leqslant n \quad 0 < p < 1 \\ \text{arbitrary} & 0 \leqslant r < k \end{cases} \qquad (2.3.2)$$

with the restriction of course that

$$\sum_{r=0}^{n} s(r,n) = 1.$$

The fact that (2.3.2) is a solution to equation (2.3.1) can be verified easily. But their conjecture that (2.3.2) is the only solution is false. Because, if their statement were correct, it would imply that with P_n Poisson (λ) truncated at k-1, the R-R condition (2.2.2) holds only if s(r,n) is given by (2.3.2). The following counter-example shows that this is not the case.

Let us consider

$$s(r,n) = \begin{cases} \binom{n}{r} p^r q^{n-r} & \text{if } 1 \leqslant r \leqslant n-1 \\ (2-a)\, p^n & \text{if } r=n \\ (1-p)^n + (a-1)\, p^n & \text{if } r=0 \end{cases} \qquad (2.2.3)$$

where $0 < p < 1$ and $1 < a < 2$.

(Note that since $a \neq 1$ the binomial distribution is excluded from the family (2.3.3)).

It is immediate that s(r,n), as given by (2.3.3), is a valid probability distribution. Also it satisfies the R-R condition (2.2.2) or equivalently (2.3.1). This is so, because for s(r,n) given by (2.3.3), (2.3.2) becomes

$$\frac{s(k,k)p^{m-2k}}{k!(m-k)!} \sum_{r=0}^{m-2k-1} \frac{(m-k-r)!\; p^{k+r}\, q^{m-2k-r}}{k!\;(m-2k-r)!\;(k+r)!}$$

$$= \frac{1}{k!(m-k)!}\left\{s(m-k,m-k) + \sum_{r=0}^{m-2k-1} s(k+r,m-k)\right\}$$

i.e.

$$(2-a)p^{m-k} + \sum_{r=0}^{m-2k-1} \binom{m-k}{k+r} p^{k+r} q^{m-2k-r} = (2-a)p^{m-k} + \sum_{r=0}^{m-2k-1} \binom{m-k}{k+r} p^{k+r} q^{m-2k-r}$$

which is an identity.

Since $s(n,n) \neq p^n$ it is clear that (2.3.3) is not the same as (2.3.2) and hence the conjecture in question is false.

The Bivariate Extension

Theorem 2.4.1 (Talwalker 1970)

Let (X_1, X_2) be a non-degenerate, random vector such that X_1, X_2 take non-negative integral values. Let $\binom{n_1}{r_1} p_1^{r_1} q_1^{n_1-r_1}$ and $\binom{n_2}{r_2} p_2^{r_2} q_2^{n_2-r_2}$ be the independent probabilities that the observations n_1, n_2 on X_1, X_2 are reduced to r_1, r_2 respectively during the destructive process. In other words

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=n_1, X_2=n_2) = \binom{n_1}{r_1} p_1^{r_1} q_1^{n_1-r_1} \binom{n_2}{r_2} p_2^{r_2} q_2^{n_2-r_2} \qquad (2.4.1)$$

$$r_1 = 0,\ldots,n_1$$

$$r_2 = 0,\ldots,n_2$$

Then

$$P(Y_1=r_1, Y_2=r_2) = P(Y_1=r_1, Y_2=r_2 \mid X_1 > Y_1, X_2 > Y_2)$$

$$= P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2=Y_2) = P(X_1=Y_1, X_2=Y_2 \mid X_1=Y_1, X_2 > Y_2)$$

$$= P(Y_1=r_1, Y_2=r_2 \mid X_1 > Y_1, X_2=Y_2), \qquad (2.4.2)$$

$$r_1, r_2 = 0,1,2,\ldots$$

iff (X_1, X_2) has a double Poisson distribution.

To prove the extension, Talwalker uses a method similar to the one given by Rao and Rubin in the univariate case. Shanbhag (1974) provides again a simpler way of arriving at an improved version of that result. His method is similar to the one explained in Section 2.1.

Theorem 2.4.2 (Srivastava and Srivastava (1970))

Consider (X_1, X_2, Y_1, Y_2) as in Theorem 2.4.1. Suppose that (X_1, X_2)

follows the double Poisson distribution with parameters λ and μ, i.e.,

$$P(X_1=n_1, X_2=n_2) = e^{-\lambda-\mu} \frac{\lambda^{n_1}\mu^{n_2}}{n_1!n_2!} \qquad (2.4.3)$$

$$\lambda,\mu > 0, \; n_1, n_2 = 0,1,\ldots$$

Then,

$$P(Y_1=r_1, Y_2=r_2) = P(Y_1=r_1, Y_2=r_2 \mid \text{undam.}) = P(Y_1=r_1, Y_2=r_2 \mid \text{dam.}) \qquad (2.4.4)$$

$$r_1, r_2 = 0,1,\ldots$$

iff the conditional probability of Y_1 and Y_2 given X_1 and X_2 is double binomial, i.e. iff it is of the form (2.4.1).

<u>Proof</u> Similar to the one suggested by the authors in the Univariate case.

It must be stressed again that, as it is obvious in the course of their proof, λ and μ are assumed variables.

<u>The Truncated Bivariate Extension</u>

<u>Theorem 2.5.1.</u>

Consider the random vectors (X_1, X_2) as previously. Suppose that k_1 and k_2 are two non-negative integers for which

$$P(X_1 \geqslant k_1, X_2 \geqslant k_2) = 1, \; P(X > k_1) > 0, \; P(X > k_2) > 0.$$

Also assume that

$$P(Y_1=r_1, Y_2=r_2 \mid X_1=n_1, X_2=n_2) = \binom{n_1}{r_1} p_1^{r_1} q_1^{n_1-r_1} \binom{n_2}{r_2} p_2^{r_2} q_2^{n_2-r_2} \qquad (2.5.1)$$

$$r_i = 0,1,\ldots,n_i$$

$$n_i = k_i, k_i+1, \ldots, \; i = 1,2.$$

i.e. double Binomial.

Then

$$P(Y_1=r_1, Y_2=r_2 \mid Y_1 \geq k_1, Y_2 \geq k_2) = P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2=Y_2) \qquad (2.5.2\alpha)$$

$$= P(Y_1=r_1, Y_2=r_2 \mid X_1=Y_1, X_2>Y_2, Y_2 \geq k_2) \qquad (2.5.2\beta)$$

$$r_i = k_i, k_i+1, \ldots; \; i=1,2$$

iff P_{n_1,n_2} is truncated double Poisson.

It can be checked that the same result is true if the R.H.S. of (2.5.2α) is replaced by

$$P(Y_1=r_1, Y_2=r_2 \mid X_1>Y_1, X_2>Y_2, Y_1 \geq k_1, Y_2 \geq k_2)$$

of if the R.H.S. of (2.5.2β) is replaced by

$$P(Y_1=r_1, Y_2=r_2 \mid X_1>Y_1, X_2=Y_2, Y_1 \geq k_1).$$

Proof Proof of an improved version of this theorem will be given later (Chapter 4).

.6 The Multivariate Extension

Theorem 2.6.1 (Talwalker (1970)).

Let $\underset{\sim}{X} = (X_1, \ldots, X_s)$ be a non-degenerate random vector such that X_i $i=1,\ldots,s$ takes non-negative integer-values. Let $\binom{n_i}{r_i} p_i^{r_i} q_i^{n_i-r_i}$ where $q_i = 1-p_i$ $i=1,\ldots,s$ be the independent probabilities that the observation n_i on X_i is reduced to r_i for $i=1,2,\ldots,s$, during the destructive process. Denote by $\underset{\sim}{Y} = (Y_1, \ldots, Y_s)$ the resulting random vector where $Y_1, Y_2, \ldots, Y_s$ take the values $0,1,\ldots$ Then,

$$P_r(\underset{\sim}{Y}=\underset{\sim}{r}) = P_r(\underset{\sim}{Y}=\underset{\sim}{r} \mid \text{damaged}) = P_r(\underset{\sim}{Y}=\underset{\sim}{r} \mid \text{partially damaged})$$

$$= P_r(\underset{\sim}{Y}=\underset{\sim}{r} \mid \text{undamaged}), \quad r_i=0,1,\ldots; \; i=1,2,\ldots,s$$

iff $\underset{\sim}{X}$ follows multiple Poisson distribution.

Proof The proof is straightforward.

Theorem 2.6.2 (Characterization of the Truncated Multiple Poisson)

Consider $\underset{\sim}{X} = (X_1, \dots, X_s)$, $\underset{\sim}{Y} = (Y_1, \dots, Y_s)$ as in Theorem 2.6.1. Suppose that k_i $i=1,\dots,s$ are non-negative integers for which

$$P(X_1 \geq k_1, X_2 \geq k_2, \dots, X_s \geq k_s) = 1, \quad P(X_i > k_i) > 0 \quad i=1,\dots,s.$$

Suppose that the destructive process has the same form as in Theorem 2.6.1, i.e. multiple binomial.

Then

$$P(\underset{\sim}{Y}=\underset{\sim}{r} \mid Y_1 \geq k_1, Y_2 \geq k_2, \dots, Y_s \geq k_s) = P(\underset{\sim}{Y}=\underset{\sim}{r} \mid \text{undamaged})$$

$$= P(\underset{\sim}{Y}=\underset{\sim}{r} \mid \text{damaged}) = P(\underset{\sim}{Y}=\underset{\sim}{r} \mid \text{partially damaged})$$

$$r_i \geq k_i \quad i=1,2,\dots,s$$

iff $P_{\underset{\sim}{n}} = P_{n_1, \dots, n_s}$ is multiple Poisson truncated at $k_1, k_2, \dots, k_s$.
As with theorem 2.5.1 an improved version of this theorem will be stated and proved in Chapter 4.