APPENDIX 2

Theorems – Corollaries

Identification Theorem for the Joint Distribution of Outcomes

Let outcomes Y_1 and Y_0 be written as functions of observed variables X and unobserved variables u_1 and u_0 respectively:

$$Y_1 = g_1(X_1, X_c) + u_1$$
$$Y_0 = g_0(X_0, X_c) + u_0$$

where X_1 (a k_1-dimensional vector) and X_0 (a k_0-dimensional vector) are variables unique to g_1 and g_0 respectively and X_c (a k_c-dimensional vector) includes variables common to the two functions. The variables u_1 and u_0 are unobserved from the point of view of the econometrician. The Roy model assumes that selection into the program depends only on the gain from the program, that is an individual participates iff:

$$\Delta = Y_1 - Y_0 = g_1(X_1, X_c) - g_0(X_0, X_c) + u_1 - u_0 \succ 0$$

Assuming that $F(u_0, u_1)$ denotes the joint distribution of (u_0, u_1), Heckman and Smith (1998) state the following theorem:

Theorem

Let $Y_1 = g_1(X_1,X_c) + u_1$ and $Y_0 = g_0(X_0,X_c) + u_0$. Assume

(i) $(u_0, u_1) \perp (X_0, X_1, X_c)$

(ii) $D = 1(Y_1 \geq Y_0)$

(iii) (u_0, u_1) absolutely continuous with $Support(u_0,u_1) = R_1 \times R_1$

(iv) For each fixed X_c

$$g_0(X_0, X_c) : R_{k_0} \to R_1 \text{ for all } X_1$$

$$g_1(X_1, X_c) : R_{k_1} \to R_1 \text{ for all } X_0$$

$$Support\big(g_0(X_0, X_c) \big| X_c, X_1\big) = R_1 \text{ for all } X_c, X_1$$

$$Support\big(g_1(X_1, X_c) \big| X_c, X_0\big) = R_1 \text{ for all } X_c, X_0$$

$$Support\big(X_0 \big| X_c, X_1\big) = Support(X_0) = R_1 \text{ for all } X_c, X_1$$

$$Support\big(X_1 \big| X_c, X_0\big) = Support(X_1) = R_1 \text{ for all } X_c, X_0$$

(v) The marginal distributions of u_0, u_1 have zero medians.

Then g_0, g_1 and $F(u_0, u_1)$ are non-parametrically identified from data on participation choices and outcomes.

Proof:

By assumption, we know for all (X_0, X_1, X_c) in the support of (X_0, X_1, X_c) and for all y:

1. $\Pr(Y_1 \le Y_0 | X_0, X_1, X_c) = \Pr(g_1(X_1, X_c) + u_1 \le g_0(X_0, X_c) + u_0)$

2. $\Pr(Y_1 \le y, Y_1 \succ Y_0 | X_0, X_1, X_c) =$

$$\Pr\big(g_1(X_1, X_c) + u_1 \le y, g_1(X_1, X_c) + u_1 \succ g_0(X_0, X_c) + u_0\big)$$

3. $\Pr(Y_0 \le y, Y_0 \ge Y_1) = \Pr\big(g_0(X_0, X_c) + u_0 \le y, g_0(X_0, X_c) + u_0 \succ g_1(X_1, X_c) + u_1\big)$

Fix X_c. Let $\overline{X}_1$ and $\overline{X}_0$ *be the support of X_1 and X_0, respectively. Using the information in (1), we can define sets of values (X_0, X_1) corresponding to contours of constant probability:*

A. $S\big(X_0, X_1 \big| X_c, \overline{p}\big)$

$$= \big\{(x_0, x_1) : \Pr\big(g_1(X_1, X_c) + u_1 \succ g_0(X_0, X_c) + u_0\big) = \Pr\big(g_1(\overline{X}_1, X_c) + u_1 \succ g_0(\overline{X}_0, X_c) + u_0\big) = \overline{p}\big\}$$
$$= \big\{(X_0, X_1) : g_1(X_1, X_c) + l = g_0(X_0, X_c)\big\}$$

for some unknown constant l.

For any point in S we can use the information in (2) to write:

B. $\Pr\left(g_1(X_1, X_c) + u_1 \leq y_1, u_1 \succ u_0 + l\right)$

for all y_1. Varying X_1 over its full support from assumption (iv), we can find a compensating value X_0 within the set defined by (A) so that $\Pr(D = 1 | X_0, X_1, X_c) = \overline{p}$ *is constant. This keeps fixed the second argument in (B). The variation in X_1 produces a set of (y, X_1) values for each value of X_c which identifies the function $g_1(X_1, X_c)$ over the support of X, up to an unknown constant. By similar reasoning, we can identify $g_0(X_0, X_c)$ up to an unknown constant using (3).*

Tracing out (B) for all values g_1 and y identifies $F(u_1, u_0\text{-}u_1)$ except for a location parameter. Using (3) we identify $F(u_0, u_0\text{-}u_1)$. The location of u_0 and u_1 is determined by using the assumption that the medians of u_0, u_1 are zero using the marginals obtained by letting $g_0 \rightarrow -\infty$ (in (2)) and $g_1 \rightarrow -\infty$ (in (3)) respectively. (The information in (2) is actually all we need). With knowledge of the locations of u_0, u_1, we can determine the unknown additive constant absorbed in g_1 and g_0. By a standard transformation of variables, we obtain $F(u_0, u_1)$ from either (2) or (3). Since X_c is arbitrary, this completes the proof because we can recover everything for all X_c.

Identification Theorem for the Joint Distribution of Outcomes (generalization)

Let $\left(u_0, u_1, u_I\right)$ *be median-zero, independently and identically distributed random variables with distribution* $F\left(u_0, u_1, u_I\right)$*. Assume structure (i) – (iii) of the previous theorem and knowledge of* $F\left(Y_0 | D = 0, X_0, X_I, X_c\right), F\left(Y_1 | D = 0, X_1, X_I, X_c\right)$ *and* $P\left(D = 1 | X_I, X_c\right)$*. Assume:*

a) $\left(u_0, u_I\right) \perp \left(X_0, X_I, X_c\right)$ *or* $\left(u_1, u_I\right) \perp \left(X_1, X_I, X_c\right)$

b) g_I *is concave and Support*$\left[g_I\left(X_0, X_c\right)\right] \supset$ *Support*$\left[u_I\right]$ *or there exists a subset* $\overline{T}$ *of the support of* $X = \left(X_I, X_c\right)$ *such that (i) for all* $g_{1I}, g_{2I} \in G$*, and all* $x \in \overline{T}$*,*

$g_{1I}(X) = g_{2I}(X)$ and (ii) for all t in the support of u_I, there exists $X \in \overline{T}$ such that $g_I(X) = t$.

c) $Support(u_I, u_0) = R_{1_0} \to R_1$

$Support(u_I, u_1) = R_{1_0} \to R_1$

d)

$$g_0(X_0, X_c) : R_{k_0} \to R_1 \quad \text{for all } X_c$$

$$g_1(X_1, X_c) : R_{k_1} \to R_1 \quad \text{for all } X_c$$

$$g_I(X_I, X_c) : R_{k_I} \to R_1 \quad \text{for all } X_c$$

$$Support(g_0(X_0, X_c) | X_c) = R_1 \quad \text{for all } X_c$$

$$Support(g_1(X_1, X_c) | X_c) = R_1 \quad \text{for all } X_c$$

$$Support(X_0 | X_c) = Support(X_0)$$

$$Support(X_1 | X_c) = Support(X_1)$$

Then,

- *Under (a), (b), (c) and (d),* F_I *and* g_I *are identified. If the first part of (b) is used* g_I *is understood to be the least-concave version of the original* g_I.
- *Under the first part of (a), (b), (c) and (d),* $g(X_0, X_c)$ *and* $F(u_0, u_I)$ *are identified over the supports of* (X_0, X_c) *and* (u_0, u_I)*, respectively.*
- *Under the second part of (a), (b), (c) and (d),* $g(X_1, X_c)$ *and* $F(u_1, u_I)$ *are identified over the supports of* (X_1, X_c) *and* (u_1, u_I)*, respectively.*

The proof is provided in Heckman and Smith (1998).

Matching Corollaries

Corollary 4.1: Pair matching on balancing scores

Suppose treatment assignment is strongly ignorable. Further suppose that a value of a balancing score $b(X_i)$ is randomly sampled from the population of units, and then one treated, $D_i = 1$, unit and one control, $D_i = 0$, unit are sampled with this value ob $b(X_i)$. Then the expected difference in response to the two treatments for the units in the matched pair equals the average treatment effect at $b(X_i)$. Moreover, the mean of matched pair differences obtained by this two-step sampling process is unbiased for the average treatment effect $E(Y_{1i}) - E(Y_{0i})$.

Corollary 4.2: Subclassification on balancing scores

Suppose treatment assignment is strongly ignorable. Suppose further that a group of units is sampled using $b(X_i)$ such that: (i) $b(X_i)$ is constant for all units in the group, and (ii) at least one unit in the group received each treatment. Then, for these units, the expected difference in treatment means equals the average treatment effect at the value of $b(X_i)$. moreover, the weighted average of such differences, that is, the directly adjusted difference, is unbiased for the treatment effect $E(Y_{1i}) - E(Y_{0i})$, *when the weights equal the fraction of the population at $b(X_i)$.*

Corollary 4.3: Covariance adjustment on balancing scores

Suppose treatment assignment is strongly ignorable, so that $E(Y_{ti} | D_i = t, b(X_i)) = E(Y_{ti} | b(X_i))$ *for balancing score $b(X_i)$. Further suppose that the conditional expectation of Y_{ti} given $b(X_i)$ is linear:*

$$E(Y_{ti} | D_i = t, b(X_i)) = \alpha_t + \beta_t b(X) \qquad (t = 0,1)$$

Then the estimator

$$(\hat{\alpha}_1 - \hat{\alpha}_0) + (\hat{\beta}_1 - \hat{\beta}_0) \times b(X_i)$$

is conditionally unbiased given b(X_i), (i = 1, …,N) for the treatment effect at b(X_i), namely $E\left(Y_{1i} - Y_{0i} \middle| b(X_i)\right)$, *if* $\hat{\alpha}_t$ *and* $\hat{\beta}_t$ *are conditionally unbiased estimators of* α_t *and* β_t, *such as least squares estimators. Moreover,*

$$(\hat{\alpha}_1 - \hat{\alpha}_0) + (\hat{\beta}_1 - \hat{\beta}_0) \times \overline{b}$$

where $\overline{b} = \sum b(X_i)/N$, *is unbiased for the average treatment effect* $E(Y_{1i}) - E(Y_{0i})$ *if the units in the study are a simple random sample from the population.*