Chapter 6

Measurement Error Effect in Control Charts

6.1 Introduction

A problem faced in the context of control charts generally is the measurement error variability. This problem is the result of the inability to measure accurately the variable of interest X. The use of imprecise measurement devices affects the ability of control charts to detect an out-of-control situation. Moreover, the variable of interest may be related through a covariate with the measurement system used.

Section 6.2 presents the research in Shewhart control charts with measurement error in the univariate and multivariate case. A simple linear model together with a model with covariates are given. In Section 6.3 the research work up to now on the effect of the measurement error in the EWMA chart is described along with some new results. Specifically, a covariate model is assumed and investigated together with a detailed examination of some factors that may affect the performance of the EWMA chart.

6.2 Shewhart Control Charts and Measurement Error

The effect of measurement error on Shewhart control charts was studied by several authors. Bennet (1954) examined the effect on the Shewhart chart for the mean using the model $Y = X + \varepsilon$, where X is the actual value of the variable and Y is the measurement we have because of the random error ε. It is additionally assumed that variables Y and X are normally distributed but with different values for the variance. Specifically, variable Y has a larger variance than X because its variance comprises both the variances of X and ε. Bennet (1954) proposed that if the variance due to the measurement error is smaller than the variance due to the process it can be overlooked. Moreover, he investigated the measurement error effect on the Shewhart chart for the mean. Abraham (1977) used the same model as Bennet (1954) and considered the effect of not accurate measurements on the process variation.

6.2.1 Measurement error effect on the joined $\overline{X}$-R and $\overline{X}$-S control charts

Kanazuka (1986) examined the effect of the measurement error on the process variance of the joined $\overline{X}$ and R chart assuming the model of Bennet (1954). He showed that the power of the $\overline{X}$-R chart is given by

$$P_{\overline{X}-R} = 1 - (1 - P_{\overline{X}})(1 - P_R),$$

where

$$P_{\overline{X}} = \Phi\left\{\sqrt{\frac{1+r^2}{k^2+r^2}}\left(-3 + \frac{d\sqrt{n}}{\sqrt{1+r^2}}\right)\right\} + \Phi\left\{\sqrt{\frac{1+r^2}{k^2+r^2}}\left(-3 - \frac{d\sqrt{n}}{\sqrt{1+r^2}}\right)\right\}$$

and

$$P_R = \Pr\left(R \geq D_2\sqrt{\sigma_p^2+\sigma_M^2}\right) + \Pr\left(R \leq D_1\sqrt{\sigma_p^2+\sigma_M^2}\right),$$

$d = (\mu^{'} - \mu)/\sigma_p$, $k^2 = \sigma_p^{'2}/\sigma_p^2$, $r^2 = \sigma_M^2/\sigma_p^2$, Φ stands for the standard normal distribution and D_1, D_2 are control chart factors that depend on the sample size n. Moreover, $\mu, \mu^{'}$ denote the in and out-of-control mean respectively, $\sigma_p^2, \sigma_p^{'2}$ are the in and out-of-control process variance and σ_M^2 is the measurement error variance. In order to compute the probabilities in the formula for the power of the R chart Kanazuka proposed the use of the table given in Pearson (1941). Kanazuka (1986) noted that the power of the chart to detect a shift in the vector (μ, σ^2) is diminished and proposed the use of larger sample sizes to increase this power. Mittag (1995) and Mittag and Stemann (1998) examined the effect of measurement error on the joined $\overline{X}$-S control chart assuming the model of Bennet (1954). Mittag (1995) considered the effect at the onset of a process whereas Mittag and Stemann (1998) considered also the effect in the case of subsequent error occurrence.

Mittag and Stemann (1998) proved that the power function of the $\overline{X}$-S control chart in the case of immediate error occurrence (at the beginning of the implementation of $\overline{X}$-S control chart) is given by

$$G^e(\delta;\varepsilon) = 1 - H^e_{\overline{X}}(\delta;\varepsilon)H^e_S(\varepsilon), \tag{6.1}$$

where

$$H^e_{\overline{X}}(\delta;\varepsilon) = 1 - \left[\Phi\left(\frac{-z_{1-\alpha_1/2}(1+\tau^2)^{1/2}+\delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right) + \Phi\left(\frac{-z_{1-\alpha_1/2}(1+\tau^2)^{1/2}-\delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right)\right],$$

and

$$H^e_S(\varepsilon) = Ch\left(\frac{1+\tau^2}{\varepsilon^2+\tau^2}\chi^2_{n-1;1-\alpha_2}|n-1\right),$$

α_1, α_2 are the probabilities of a false signal on the $\overline{X}$ and S charts respectively, ε equals σ/σ_0, δ equals $(\mu-\mu_0)/\sigma_0$, where μ_0,σ_0 are the process parameters target values. The symbol Φ stands for the standard normal distribution again, z_ω denotes its ω quantile, Ch is the central chi-squared distribution and $\chi^2_{n-1;\omega}$ represents its ω quantile with $n-1$ degrees of freedom. In the case of subsequent error occurrence they proved that the power function of the $\overline{X}$-S control chart is given by relationship (6.1) as in the immediate error occurrence but now with

$$H^e_{\overline{X}}(\delta;\varepsilon) = 1 - \left[\Phi\left(\frac{-z_{1-\alpha_1/2} + \delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right) + \Phi\left(\frac{-z_{1-\alpha_1/2} - \delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right)\right]$$

and

$$H^e_S(\varepsilon) = Ch\left(\frac{\chi^2_{n-1;1-\alpha_2}}{\varepsilon^2+\tau^2}|n-1\right).$$

In both cases we have a reduced ability of the $\overline{X}$-S control chart to identify when a process is out-of-control.

6.2.2 Model with Covariates

Linna and Woodall (2001) extended the model considered by Bennet (1954) assuming one with covariates examining the effect on the $\overline{X}$ and S^2 control charts. Specifically, they considered the model $Y = A + BX + \varepsilon$, where X is a normally distributed variable with mean μ and variance σ_p^2 and ε is normally distributed with mean 0 and variance σ_m^2. If the mean of X shifts to some value μ' then the probability of a signal on the Shewhart chart for the mean is

$$1 - \Phi\left(3 + \frac{(\mu-\mu')\sqrt{n}}{\sqrt{\sigma_p^2 + \sigma_m^2/B^2}}\right) + \Phi\left(-3 + \frac{(\mu-\mu')\sqrt{n}}{\sqrt{\sigma_p^2 + \sigma_m^2/B^2}}\right).$$

The probability of a signal on the Shewhart chart for the variance if the characteristic X shifts from σ_p^2 to $\sigma_p^{2'}$ is

$$1 - \Pr\left(\frac{B^2\sigma_p^2 + \sigma_m^2}{B^2\sigma_p^{2'} + \sigma_m^2}\chi^2_{a/2,n-1} < Q < \frac{B^2\sigma_p^2 + \sigma_m^2}{B^2\sigma_p^{2'} + \sigma_m^2}\chi^2_{1-a/2,n-1}\right),$$

where Q is a chi-square random variable with $n-1$ degrees of freedom. They proved that in both charts the effect of the measurement error variance and the value B on their power is significant. Linna and Woodall among others, proposed the use of multiple measurements per item as a solution to this problem. If we assume that we take k successive independent measurements on each of n items then the variance of the sample mean of the subgroup is $\frac{B^2\sigma_p^2}{n}+\frac{\sigma_m^2}{nk}$, indicating that as k increases this variance decreases. In the case of linearly increasing variance we assume that ε is normally distributed with mean 0 and variance $C+D\mu^{'}$ where C and D are assumed known constants. The probability of a signal in this case is

$$1-\Phi\left(\frac{\left(\mu-\mu^{'}\right)\sqrt{n}+3\sqrt{\sigma_p^2+C/B^2+D\mu/B^2}}{\sqrt{\sigma_p^2+C/B^2+D\mu^{\prime}/B^2}}\right)$$
$$+\Phi\left(\frac{\left(\mu-\mu^{'}\right)\sqrt{n}-3\sqrt{\sigma_p^2+C/B^2+D\mu/B^2}}{\sqrt{\sigma_p^2+C/B^2+D\mu^{\prime}/B^2}}\right).$$

In this case, both charts for the mean and the variance are affected.

Linna, Woodall and Busby (2001) examined the same model with covariates in the multivariate case, in the case of the X^2 chart. In particular, let $\mathbf{Y}_i=\mathbf{A}+\mathbf{B}\mathbf{X}_i+\boldsymbol{\varepsilon}_i$, $i=1,\ 2,...$ where $\mathbf{A}$ is a $p\times 1$ vector of constants, $\mathbf{B}$ is an invertible $p\times p$ matrix of constants and $\boldsymbol{\varepsilon}_i$ is a $p\times 1$ normal random vector independent of $\mathbf{X}$ with a mean vector of zeroes and variance covariance matrix $\boldsymbol{\Sigma}_m$. Linna, Woodall and Busby proved that multivariate control charts under measurement error effect can detect in a more powerful way shift in one direction than in other. Moreover, they have shown that multivariate control charts are affected and this effect can be very serious because of the loss of the directional invariance property.

6.3 EWMA Charts and Measurement Error

Stemann and Weihs (2001) were the first to investigate the effect of measurement error on the EWMA chart. They considered the EWMA-X-S chart, as they name it, which is a combination of the EWMA charts for the mean and the standard deviation. Specifically, they showed through simulation that the ARL behavior of this chart is affected by the measurement error. The model assumed is the one proposed by Bennet (1954). They checked both the cases of the presence of measurement error in the beginning of the process and subsequently during production. However, the model with covariates proposed by Linna and Woodall (2001) was not investigated. This model is investigated in the case of measurement error for the EWMA chart for the mean in Maravelakis et al. (2004) and is presented in the following subsections.

6.3.1 The EWMA Chart Using Covariates

Assume again that we have a process where the true value of the characteristic X under investigation is normally distributed with mean μ and variance σ^2 when the process is in-control. However, we are not able to observe this true value but rather a value Y, which is related to X with the formula $Y = A + BX + \varepsilon$, where A and B are constants and ε is the random error distributed independently of X as a normal random variable with mean zero and variance σ_m^2. We assume here that all model parameters are known.

From the formula relating Y and X it is straightforward that Y is normally distributed with mean $A + B\mu$ and variance $B^2\sigma^2 + \sigma_m^2$. We need to construct an EWMA chart for the measured quantity Y since in this way we can keep under control the variable X. Assume that at each sampling point we collect n values of Y, we compute the mean of these observations $\overline{Y_i}$ and we compute the EWMA statistic z_i using the formula

$$\begin{aligned} z_i &= \lambda\overline{Y_i} + (1-\lambda)z_{i-1}, \\ z_0 &= A + B\mu \end{aligned}$$

where $\overline{Y_i}$ is the mean of the observations collected at time $i = 1, 2, ...$ and λ is the smoothing parameter.

The control limits are

$$\begin{aligned} UCL &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\frac{(B^2\sigma^2+\sigma_m^2)}{n}} \\ LCL &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\frac{(B^2\sigma^2+\sigma_m^2)}{n}}, \end{aligned} \tag{6.2}$$

where L is a constant used to specify the width of the control limits and $A + B\mu$ and $\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\frac{(B^2\sigma^2+\sigma_m^2)}{n}}$ are the mean and standard deviation of Z_i respectively, when the process is in-control. In case the EWMA chart is used for some time, instead of the control limits (6.2), we may use their limiting values

$$\begin{aligned} UCL &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{(B^2\sigma^2+\sigma_m^2)}{n}} \\ LCL &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{(B^2\sigma^2+\sigma_m^2)}{n}}, \end{aligned} \tag{6.3}$$

(see e.g., Lucas and Saccucci (1990)). In this case $\sqrt{(\lambda/(2-\lambda))\left((B^2\sigma^2+\sigma_m^2)/n\right)}$ is the asymptotic standard deviation of Z_i.

6.3.2 Multiple measurements

In order to decrease the measurement error effect, a technique that is suggested by Linna and Woodall (2001) is to take more than one measurements in each sampled unit. Taking more than one measurements and averaging them leads to a more precise measurement. Moreover, the variance of the measurement error component in the average of the multiple observations becomes smaller as the number of multiple measurements increases. Therefore, ideally if the number of multiple measurements becomes infinite the variance of the measurement error component will become zero. Consequently, the larger the number of multiple measurements the better, keeping in mind always the additional cost and time needed for these observations. We must understand also that in the absence of measurement error multiple measurements will not contribute to the control charting methodology anything (in fact they will add the cost of measuring the extra observations).

In the case of sufficient number of multiple measurements we can assume that our process actually operates without measurement error. However, the cost of extra measurements and the time are factors that can not be overlooked. Therefore, a careful examination of these factors in the specific application we are working on is essential. We have to stress that the measurement error variance has to be large enough and the two factors small enough for the extra observations to have a practical value.

In order to compute the EWMA statistic we assume that at each sampling point we collect k measurements for each of n observations of Y, we compute the overall mean of these observations $\overline{\overline{Y_i}}$ and we compute the EWMA statistic q_i using the formula

$$\begin{aligned} q_i &= \lambda\overline{\overline{Y_i}} + (1-\lambda)q_{i-1}, \\ q_0 &= A + B\mu \end{aligned}$$

where $\overline{\overline{Y_i}}$ is the mean of the observations collected at time $i = 1, 2, ...$, λ is a smoothing parameter that takes values between 0 and 1 and q_0 is the initial value. Moreover, we assume that the k observations collected at the same sampling unit are independent. If $k = 1$ we face the measurement error case discussed in Section 6.3.1.

It is straightforward to prove (Linna and Woodall (2001)) that the variance of the overall mean is $\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}$. Therefore, the control limits are

$$\begin{aligned} UCL_q &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}\right)} \qquad (6.4) \\ LCL_q &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}\right)}, \end{aligned}$$

where L is a constant used to specify the width of the control limits and $A + B\mu$ and $\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)}$ are the mean and standard deviation of q_i respectively, when the process is in-control. In case the EWMA chart is used for some time, instead of the control limits (6.4), we may use their limiting values

$$
\begin{aligned}
UCL_q &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)} \\
LCL_q &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)}.
\end{aligned}
\tag{6.5}
$$

6.3.3 Linearly increasing variance

Although the model with covariates considered assumes constant variance it is not unlikely to have a model with variance that depends on the mean level of the process. Montgomery and Runger (1994) and Linna and Woodall (2001) refer to practical problems indicating situations where this phenomenon occurs in industry.

We assume that the variance changes linearly with variable X. The model we use is again $Y = A + BX + \varepsilon$ with the same assumptions as in Section 6.3.1, except that this time ε is distributed as a normal variable with mean 0 and variance $C + D\mu$. As in Section 6.3.1 all model parameters are assumed known. From the relation between Y and X we deduce that Y is normally distributed with mean $A + B\mu$ and variance $B^2\sigma^2 + C + D\mu$. The EWMA statistic will be exactly the same as in Section 6.3.1.

It can be shown that the control limits of the EWMA statistic are

$$
\begin{aligned}
UCL_l &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)} \\
LCL_l &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)},
\end{aligned}
\tag{6.6}
$$

where L is again a constant used to specify the width of the control limits and $A + B\mu$ and $\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)}$ are the mean and standard deviation of the

EWMA statistic respectively, when the process is in-control. When the EWMA chart is used for a suitable number of points in time, instead of the control limits (6.6), we can use their limiting values

$$\begin{aligned} UCL_l &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2 + C + D\mu}{n}\right)} \\ LCL_l &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2 + C + D\mu}{n}\right)}. \end{aligned} \tag{6.7}$$

6.3.4 ARL computations

In order to compute the probability density function, the cumulative distribution function and the first moment of the run length distribution of the EWMA chart for the mean we may approximate it as a discrete Markov Chain by dividing the distance between the control limits in 2m+1 states each of which has width 2δ (see, e.g. Brook and Evans (1972)). We say that the statistic Z_i remains in state j as long as $S_j - \delta < Z_i \leq S_j + \delta$ where $-m \leq j \leq m$ and S_j is the midpoint in the jth interval. When Z_i crosses the control limits we say that it is in the absorbing state. On the other hand when the process is in-control we say that it is in a transient state.

The transition probability matrix for the EWMA chart for the mean is computed as

$$\mathbf{P} = \begin{bmatrix} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{1} \\ \mathbf{0}^T & 1 \end{bmatrix},$$

where $\mathbf{R}$ is a submatrix containing the transient states, $\mathbf{I}$ is a $(t \times t)$ identity matrix and $\mathbf{1}$ is a $(t \times 1)$ vector of unities. The $jkth$ element of the submatrix $\mathbf{R}$ is given by $p_{jk} = P\left[S_j - \delta < \lambda y_i + (1-\lambda)S_j \leq S_j + \delta\right]$. In the case of the normal distribution with the model with covariates of our case the probabilities are given by

$$p_{jk} = \Phi\left[\frac{(S_k+\delta) - (1-\lambda)S_j - \lambda\left(A + B\mu\right)}{\lambda\sqrt{\left(B^2\sigma^2 + \sigma_m^2\right)/n}}\right] - \Phi\left[\frac{(S_k-\delta) - (1-\lambda)S_j - \lambda\left(A + B\mu\right)}{\lambda\sqrt{\left(B^2\sigma^2 + \sigma_m^2\right)/n}}\right].$$

When we have multiple measurements the probabilities are

$$p_{jk} = \Phi \left[\frac{(S_k + \delta) - (1-\lambda)S_j - \lambda\,(A + B\mu)}{\lambda\sqrt{\left(\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}\right)}} \right] - \Phi \left[\frac{(S_k - \delta) - (1-\lambda)S_j - \lambda\,(A + B\mu)}{\lambda\sqrt{\left(\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}\right)}} \right]$$

and in the case of linearly increasing variance the probabilities are

$$p_{jk} = \Phi \left[\frac{(S_k + \delta) - (1-\lambda)S_j - \lambda\,(A + B\mu)}{\lambda\sqrt{\left(\frac{B^2\sigma^2 + C + D\mu}{n}\right)}} \right] - \Phi \left[\frac{(S_k - \delta) - (1-\lambda)S_j - \lambda\,(A + B\mu)}{\lambda\sqrt{\left(\frac{B^2\sigma^2 + C + D\mu}{n}\right)}} \right].$$

Let τ denote the run length of the EWMA, then $P(\tau \leq t) = (\mathbf{I} - \mathbf{R}^t)\,\mathbf{1}$ and therefore $P(\tau = t) = (\mathbf{R}^{t-1} - \mathbf{R}^t)\,\mathbf{1}$ for $t \geq 1$. The ARL can be computed using the formula $E(\tau) = \sum_{i=1}^{\infty} tP(\tau = t) = (\mathbf{I} - \mathbf{R}^{-1})\mathbf{1}$.

6.3.5 Effect of the measurement error

In the context of EWMA charts as we have already said there are three ways of computing the ARL. The integral equation method, the Markov chain method and simulation. Here, we use the Markov Chain method in all the computations.

In Table 6.1, we can see the ARL results of the covariate model for different values of the ratio σ_m^2/σ^2 when B=1. The in-control ARL value is the same for all combinations in order to achieve a fair comparison. From the table we see that there is an increasing effect on the out-of-control ARL as the ratio of σ_m^2/σ^2 increases. This result is similar to the one in Linna and Woodall (2001). In Table 6.2, we can see the ARL results of the covariate model for different values of B. The results are displayed for the same parameters as in Table 6.1 when σ_m^2/σ^2=1. We observe that as the value of B increases the effect on the ARL diminishes. This result is again in accordance with Linna and Woodall (2001). Furthermore, in both Tables 6.1 and 6.2 the effect of the measurement error on the ARL values lessens as the shift increases. We have to state also that A does not affect the ARL performance in this study.

Table 6.1. ARL for the covariate model for different values of σ_m^2/σ^2

Shift	No Error	0.1	0.2	0.3	0.5	1
0	370.22	370.27	370.27	370.27	370.27	370.26
0.5	41.13	45.22	49.26	53.23	60.96	79.06
1	10.25	11.21	12.18	13.16	15.15	20.26
1.5	5.18	5.57	5.96	6.36	7.16	9.20
2	3.46	3.69	3.91	4.13	4.57	5.67
2.5	2.65	2.80	2.94	3.09	3.37	4.08
3	2.19	2.29	2.40	2.50	2.71	3.22

Table 6.2. ARL for the covariate model for different values of B

Shift	No Error	1	2	3	5
0	370.22	370.26	370.27	370.26	370.27
0.5	41.13	79.06	51.25	45.67	42.78
1	10.25	20.26	12.67	11.31	10.63
1.5	5.18	9.20	6.16	5.61	5.33
2	3.46	5.67	4.02	3.71	3.55
2.5	2.65	4.08	3.01	2.81	2.71
3	2.19	3.22	2.45	2.31	2.23

In Table 6.3, we can see the ARL results for the covariate model with multiple measurements for different values of σ_m^2/σ^2 when $k = 5$ and $B = 1$. It is obvious that if the practitioner has the ability to take five measurements in each unit then for values of σ_m^2/σ^2 less than 0.3 we may say that the process operates actually without measurement error. For values larger than 0.3 the effect is seriously reduced in comparison to the $k = 1$ case, which corresponds to the results in Table 6.1, even for $\sigma_m^2/\sigma^2 = 1$.

Table 6.3. ARL for multiple measurements k=5, B=1 for different values of σ_m^2/σ^2

Shift	No Error	0.1	0.2	0.3	0.5	1
0	370.22	370.26	370.26	370.27	370.27	370.27
0.5	41.13	41.96	42.78	43.59	45.22	49.26
1	10.25	10.44	10.63	10.82	11.21	12.18
1.5	5.18	5.25	5.33	5.41	5.57	5.96
2	3.46	3.51	3.55	3.60	3.69	3.91
2.5	2.65	2.68	2.71	2.74	2.80	2.94
3	2.19	2.21	2.23	2.25	2.29	2.40

Table 6.4. ARL for multiple measurements k=5, σ_m^2/σ^2=1 for different values of B

Shift	No Error	1	2	3	5
0	370.22	370.27	370.26	370.28	370.27
0.5	41.13	49.26	43.18	42.05	41.46
1	10.25	12.18	10.73	10.46	10.33
1.5	5.18	5.96	5.37	5.26	5.21
2	3.46	3.91	3.57	3.51	3.48
2.5	2.65	2.94	2.72	2.68	2.66
3	2.19	2.40	2.24	2.21	2.20

Table 6.4 presents the results in the case of multiple measurements for different values of B. We see that as the value of B increases the effect on the ARL diminishes. This result is in accordance with the results in Table 6.2. Moreover, in Table 6.5 we have results in the case of multiple measurements for different k values. As the value of k increases the measurement error effect lessens. However, since cost and time needed for the extra measurements are important factors, the practitioner will have to do a trade-off

between these two concerns and the measurement error he can put up with. We have to stress here that the results displayed refer to the worst case, since we choose $B = 1$ and $\sigma_m^2/\sigma^2 = 1$, that correspond to the most affected combination. Therefore, one may conclude that the results in the other cases will be even better.

Table 6.5. ARL for multiple measurements for different values of k

Shift	No Error	5	10	20	50
0	370.22	370.27	370.27	370.26	370.26
0.5	41.13	49.26	45.22	43.18	41.96
1	10.25	12.18	11.21	10.73	10.44
1.5	5.18	5.96	5.57	5.37	5.25
2	3.46	3.91	3.69	3.57	3.51
2.5	2.65	2.94	2.80	2.72	2.68
3	2.19	2.40	2.29	2.24	2.21

Table 6.6. ARL for linearly increasing variance for different values of D

Shift	No Error	1	2	3	5
0	370.22	370.27	370.28	370.27	370.28
0.5	41.13	231.40	282.70	306.34	328.76
1	10.25	102.95	161.16	198.80	244.30
1.5	5.18	50.14	90.14	122.03	168.52
2	3.46	28.10	53.72	76.89	115.28
2.5	2.65	17.79	34.49	50.85	80.43
3	2.19	12.39	23.73	35.38	57.77

The results in the case of linearly increasing variance are displayed on Tables 6.6 and 6.7. In Table 6.6 we have the ARL values when $B = 1$, $C = 0$, $\sigma_m^2/\sigma^2 = 1$ for different values of D. We see that even for small values of D there is a more serious effect than in the no error case. Additionally, as the value of D increases, this effect is getting larger. This result is expected because as D increases so does the variance of the error component in the model. In this special case of measurement error, extra precaution is needed because the ability of the EWMA chart to detect fast small shifts is canceled out. Consequently, serious distortion factors may go undetected for a long time costing a lot in money, time and credibility.

Table 6.7 presents the ARL results when $B = 1$, $D = 1$ and $\sigma_m^2/\sigma^2 = 1$ for different values of C. In analogy to Table 6.6, increasing values of C cause an increasing measurement error effect on the ARL. However, this effect is not of the same magnitude as the effect of D. This result is also expected since D is multiplied by the mean μ, thus increasing faster the error variance as D increases whereas C is just added to this variance.

Table 6.7. ARL for linearly increasing variance for different values of C

Shift	No Error	0	1	2	3
0	370.22	370.27	370.29	370.27	370.27
0.5	41.13	231.40	239.08	245.96	252.14
1	10.25	102.95	110.13	116.95	123.44
1.5	5.18	50.14	54.53	58.84	63.06
2	3.46	28.10	30.72	33.34	35.95
2.5	2.65	17.79	19.43	21.09	22.75
3	2.19	12.39	13.49	14.59	15.71

In all the computations we used 211 states for the Markov Chain method. The values of the constants are $\lambda = 0.25$ and $L = 2.898$. In order to detect small shifts fast the λ value usually used is 0.1 or less. However, such small values are not able to detect small to moderate shifts and this is the reason for choosing this particular value of λ. Note also that in all the cases the control limits used are the ones with the limiting values.

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