APPENDIX

- **Derivation of the Correlated Gamma Ratio Distribution (Panaretos et al. (1997))**

Let (X,Y) be a random vector of dependent components following Kibble's bivariate Gamma distribution. Let Z=X/Y. The distribution function of Z is given by:

$$F_Z(z) = P\left(\frac{X}{Y} \le z\right) = \int_{-\infty}^{+\infty} P\left(X \le zY \mid Y = y\right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{+\infty} F_{X|Y=y}(zy)\, dy$$

and the probability - density function is given by:

$$f_Z(z) = \int_0^{\infty} f_{X|Y=y}(zy) \cdot y \cdot f_Y(y)\, dy = \int_0^{\infty} \frac{f_{X|Y}(zy, y)}{f_Y(y)} \cdot y \cdot f_Y(y)\, dy \quad \Rightarrow$$

$$\boxed{f_Z(z) = \int_0^{\infty} y\; f_{X,Y}(zy,y)\, dy}$$

$$f_{X/Y}(z) = \int_0^{\infty} y\; f_{X,Y}(zy,y)\, dy$$

$$= \frac{\left(1-\rho^2\right)^{-\kappa} z^{\kappa-1}}{\Gamma(\kappa)} \sum_{i=0}^{\infty} \frac{\left(\rho/(1-\rho^2)\right)^{2i}}{\Gamma(i+\kappa)} \frac{z^i}{i!} \int_0^{\infty} e^{-\frac{z+1}{1-\rho^2}\cdot y}\, y^{2(\kappa+i)-1}\, dy$$

But $\int_0^{\infty} e^{-\frac{z+1}{1-\rho^2}\cdot y}\, y^{2(\kappa+i)-1}\, dy = \left(\frac{z+1}{1-\rho^2}\right)^{-2(\kappa+i)} \Gamma(2\kappa+2i)$.

So

$$f_{X/Y}(z) = \frac{\left(1-\rho^2\right)^{-\kappa}}{\Gamma(\kappa)} z^{\kappa-1} \sum_{i=0}^{\infty} \frac{\rho^{2i}(1-\rho^2)^{-2i}}{i!\,\Gamma(i+\kappa)} \frac{\Gamma(2\kappa+2i)\,(z+1)^{-2\kappa-2i}}{(1-\rho^2)^{-2\kappa-2i}}$$

$$= \frac{\left(1-\rho^2\right)^{\kappa}}{\Gamma(\kappa)} z^{\kappa-1}\,(z+1)^{-2\kappa} \sum_{i=0}^{\infty} \frac{\Gamma(2\kappa+2i)}{\Gamma(i+\kappa)} \frac{\left[\rho^2(z+1)^{-2}\right]^i}{i!} z^i$$

But

$$\frac{\Gamma(2\kappa+2i)}{\Gamma(\kappa)\,\Gamma(i+\kappa)} = \frac{\Gamma(2\kappa+2i)\,\Gamma(2\kappa)\,\Gamma(\kappa)}{\Gamma(\kappa)\,\Gamma(i+\kappa)\,\Gamma(2\kappa)\,\Gamma(\kappa)}$$

and

$$B(\alpha,\beta)= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad , \quad \alpha_{(mn)} = n^{nm}\left(\frac{\alpha}{n}\right)_{(m)}\left(\frac{\alpha+1}{n}\right)_{(m)}\cdots\left(\frac{\alpha+n-1}{n}\right)_{(m)} .$$

Using the fact that

$B(\kappa,\kappa)$ and $(2\kappa)_{(2i)}$, we have that:

$$\frac{\Gamma(2\kappa+2i)}{\Gamma(\kappa)\,\Gamma(i+\kappa)} = \frac{(2\kappa)_{(2i)}}{\kappa_{(i)}}\cdot\left(B(\kappa,\kappa)\right)^{-1} = \qquad (*)$$

$$= \left(B(\kappa,\kappa)\right)^{-1}\frac{2^{2i}\left(\frac{2\kappa}{2}\right)_{(i)}\left(\frac{2\kappa+1}{2}\right)_{(i)}}{\kappa_{(i)}} = \frac{2^{2i}\left(\frac{2\kappa+1}{2}\right)_{(i)}}{B(\kappa,\kappa)}$$

So

$$f_{X/Y}(z)= \left(1-\rho^2\right)^{\kappa}\frac{z^{\kappa-1}(1+z)^{-2\kappa}}{B(\kappa,\kappa)}\sum_{i=0}^{\infty}\left(\frac{2\kappa+1}{2}\right)_{(i)}\frac{\left[4\rho^2(z+1)^{-2}\right]^{i} z^{i}}{i!}$$

$$= \left(1-\rho^2\right)^{\kappa}\frac{z^{\kappa-1}(1+z)^{-2\kappa}}{B(\kappa,\kappa)}\;{}_1F_0\left(\frac{2\kappa+1}{2}\,;\,\frac{4\rho^2 z}{(1+z)^2}\right)$$

where ${}_1F_0$ is the hypergeometric function

i.e.,

$$f_{X/Y}(z) = \left(1-\rho^2\right)^{\kappa}\frac{z^{\kappa-1}(1+z)^{-2\kappa}}{B(\kappa,\kappa)}\left(1-\frac{4\rho^2 z}{(z+1)^2}\right)^{-\frac{2\kappa+1}{2}} \Rightarrow$$

$$\boxed{f_{X/Y}(z) = \frac{\left(1-\rho^2\right)^{\kappa}}{B(\kappa,\kappa)}\, z^{\kappa-1}(1+z)^{-2\kappa}\left(1-\left(\frac{2\rho}{z+1}\right)^2 z\right)^{-\frac{2\kappa+1}{2}}}$$

The distribution with p.d.f $f_{X|Y}(Z)$ given by the above formula is called Correlated Gamma-Ratio distribution (Panaretos et al.(1997)).

- **Proof that the Correlated Gamma Ratio Distribution is a well-defined distribution (Panaretos et al.(1997))**

It should be shown that:

$$\int_0^\infty f_{X/Y}(z)\ dz = 1$$

It is

$$\int_0^\infty f_{X/Y}(z)\ dz = \frac{\left(1-\rho^2\right)^\kappa}{B(\kappa,\kappa)} \int_0^\infty z^{\kappa-1}(1+z)^{-2\kappa}\left(1-\frac{4\rho^2 z}{(z+1)^2}\right)^{-\frac{2\kappa+1}{2}} dz$$

$$= \frac{\left(1-\rho^2\right)^\kappa}{B(\kappa,\kappa)}\ I$$

where

$$I = \sum_{r=0}^\infty \left(\frac{2\kappa+1}{2}\right)_{(r)} \frac{\left(4\rho^2\right)^r}{r!} \int_0^\infty z^{r+\kappa-1}(1+z)^{-2r-2\kappa}\ dz$$

and

$$I_r = \int_0^\infty z^{r+\kappa-1}(1+z)^{-2r-2\kappa}\ dz = \frac{\Gamma(\kappa+r)\Gamma(\kappa+r)}{\Gamma(2\kappa+2r)}$$

So,

$$I = \sum_{r=0}^\infty \left(\frac{2\kappa+1}{2}\right)_{(r)} \frac{\left(4\rho^2\right)^r}{r!} \frac{\Gamma(\kappa+r)\Gamma(\kappa+r)}{\Gamma(2\kappa+2r)}.$$

Because of the relationship (*) of the previous proof we have :

$$I = \sum_{r=0}^\infty \frac{B(\kappa,\kappa)\ \Gamma(2\kappa+2r)}{\Gamma(\kappa)\Gamma(\kappa+r)} \frac{2^{-2r}\quad 4^r \rho^{2r}}{r!} \frac{\Gamma(\kappa+r)\Gamma(\kappa+r)}{\Gamma(2\kappa+2r)}.$$

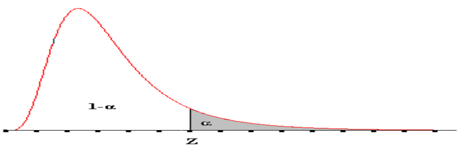
So

$$I = B(\kappa,\kappa) \sum_{r=0}^\infty \kappa_{(r)} \frac{\rho^{2r}}{r!} = B(\kappa,\kappa)\left(1-\rho^2\right)^{-\kappa}$$

and, finally, $\int_0^\infty f_{X/Y}(z)\ dz = 1$.

Table 24 :Percentiles of the Correlated Gamma Ratio distribution for α=0.1 (Panaretos et al. (1997))

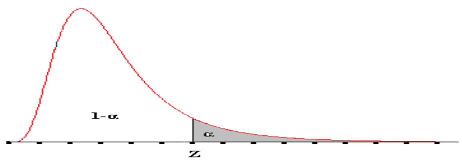
$$\int_0^{+\infty} \frac{\left(1-\rho^2\right)^{\kappa}}{B(\kappa,\kappa)} t^{\kappa-1}(1+t)^{-2\kappa}\left[1-\left[\frac{2\rho}{t+1}\right]^2 t\right]^{-\frac{2\kappa+1}{2}} dt = 1-\alpha = 0.90$$



κ \ ρ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	9	8.93	8.72	8.36	7.85	7.2	6.4	5.45	4.33	3.02
2	4.11	4.08	4.01	3.88	3.71	3.48	3.2	2.85	2.44	1.93
3	3.055	3.04	3.00	2.92	2.81	2.67	2.49	2.27	2.00	1.66
4	2.59	2.58	2.55	2.49	2.41	2.3	2.17	2.00	1.8	1.53
5	2.32	2.31	2.29	2.24	2.18	2.09	1.98	1.84	1.67	1.46
6	2.15	2.14	2.12	2.08	2.02	1.95	1.85	1.74	1.59	1.41
7	2.02	2.01	2.00	1.96	1.91	1.85	1.76	1.66	1.54	1.37
8	1.93	1.92	1.90	1.87	1.83	1.77	1.70	1.61	1.49	1.34
9	1.85	1.846	1.83	1.80	1.76	1.71	1.64	1.56	1.455	1.315
10	1.79	1.785	1.775	1.75	1.71	1.665	1.6	1.525	1.425	1.295
11	1.745	1.74	1.725	1.705	1.67	1.62	1.565	1.49	1.4	1.277
12	1.705	1.70	1.685	1.665	1.63	1.59	1.535	1.465	1.38	1.265
13	1.665	1.664	1.65	1.63	1.60	1.56	1.51	1.44	1.36	1.253
14	1.635	1.63	1.62	1.6	1.57	1.53	1.485	1.423	1.345	1.24
15	1.605	1.604	1.59	1.575	1.546	1.51	1.465	1.405	1.33	1.31
16	1.585	1.58	1.57	1.55	1.525	1.49	1.445	1.39	1.32	1.225
17	1.56	1.553	1.546	1.53	1.505	1.471	1.43	1.376	1.307	1.216
18	1.54	1.535	1.525	1.510	1.486	1.455	1.415	1.364	1.297	1.207
19	1.52	1.519	1.51	1.495	1.471	1.44	1.402	1.351	1.287	1.203
20	1.505	1.504	1.495	1.48	1.456	1.426	1.39	1.341	1.28	1.197
21	1.49	1.489	1.48	1.465	1.44	1.415	1.377	1.331	1.274	1.193
22	1.475	1.474	1.466	1.451	1.43	1.404	1.379	1.323	1.353	1.187
23	1.465	1.460	1.455	1.440	1.567	1.391	1.358	1.315	1.259	1.183
24	1.454	1.450	1.442	1.428	1.408	1.382	1.35	1.306	1.252	1.178
25	1.442	1.44	1.432	1.418	1.4	1.374	1.34	1.3	1.246	1.174
26	1.432	1.43	1.422	1.408	1.39	1.366	1.344	1.292	1.240	1.17
27	1.422	1.42	1.412	1.4	1.382	1.356	1.326	1.286	1.238	1.166
28	1.412	1.410	1.402	1.39	1.372	1.35	1.32	1.28	1.23	1.163
29	1.404	1.402	1.394	1.382	1.366	1.342	1.312	1.274	1.226	1.16
30	1.396	1.394	1.386	1.375	1.358	1.336	1.306	1.27	1.222	1.157
40	1.333	1.332	1.326	1.316	1.302	1.284	1.259	1.228	1.189	1.134
50	1.293	1.291	1.287	1.279	1.267	1.249	1.229	1.203	1.168	1.119
60	1.265	1.264	1.259	1.252	1.24	1.226	1.207	1.183	1.152	

Table 25: Percentiles of the Correlated Gamma Ratio distribution for α=0.05 (Panaretos et al. (1997))

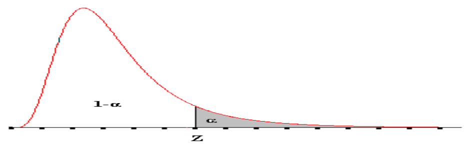
$$\int_0^{+\infty} \frac{\left(1-\rho^2\right)^{\kappa}}{B(\kappa,\kappa)} t^{\kappa-1}(1+t)^{-2\kappa}\left[1-\left[\frac{2\rho}{t+1}\right]^2 t\right]^{-\frac{2\kappa+1}{2}} dt = 1-\alpha = 0.95$$



κ \ ρ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	19	18.80	18.3	17.4	16.27	14.73	12.84	10.60	8.02	5.04
2	6.39	6.34	6.20	5.97	5.64	5.22	4.7	4.07	3.34	2.46
3	4.284	4.26	4.18	4.04	3.85	3.61	3.31	2.945	2.51	1.97
4	3.44	3.42	3.36	3.27	3.145	2.96	2.74	2.48	2.16	1.76
5	2.98	2.96	2.92	2.84	2.74	2.6	2.43	2.22	1.965	1.64
6	2.687	2.675	2.65	2.57	2.485	2.37	2.23	2.06	1.835	1.56
7	2.49	2.47	2.44	2.39	2.31	2.21	2.09	1.935	1.75	1.51
8	2.335	2.325	2.29	2.25	2.18	2.1	1.985	1.85	1.675	1.46
9	2.22	2.21	2.19	2.14	2.18	2	1.95	1.775	1.63	1.427
10	2.125	2.115	2.095	2.055	2	1.93	1.837	1.725	1.585	1.4
11	2.05	2.04	2.02	1.983	1.935	1.87	1.783	1.677	1.55	1.375
12	1.983	1.977	1.955	1.925	1.876	1.815	1.735	1.635	1.515	1.355
13	1.93	1.922	1.905	1.875	1.83	1.775	1.697	1.605	1.49	1.338
14	1.884	1.876	1.86	1.83	1.787	1.733	1.663	1.577	1.47	1.324
15	1.843	1.835	1.82	1.794	1.752	1.7	1.63	1.552	1.453	1.31
16	1.805	1.798	1.783	1.757	1.72	1.675	1.61	1.527	1.427	1.297
17	1.775	1.767	1.753	1.727	1.697	1.644	1.582	1.508	1.414	1.287
18	1.745	1.74	1.723	1.697	1.667	1.620	1.563	1.493	1.397	1.277
19	1.717	1.711	1.697	1.678	1.644	1.59	1.543	1.472	1.387	1.27
20	1.695	1.69	1.676	1.653	1.624	1.576	1.527	1.46	1.375	1.262
21	1.672	1.667	1.654	1.633	1.604	1.564	1.511	1.447	1.362	1.254
22	1.654	1.647	1.635	1.613	1.584	1.549	1.498	1.434	1.353	1.247
23	1.633	1.629	1.617	1.597	1.567	1.531	1.484	1.424	1.344	1.242
24	1.615	1.612	1.6	1.581	1.553	1.516	1.469	1.412	1.336	1.236
25	1.6	1.596	1.585	1.566	1.54	1.504	1.458	1.401	1.328	1.229
26	1.585	1.581	1.57	1.552	1.526	1.491	1.447	1.390	1.320	1.224
27	1.57	1.566	1.558	1.54	1.514	1.48	1.437	1.383	1.314	1.22
28	1.558	1.556	1.544	1.528	1.502	1.47	1.426	1.374	1.307	1.215
29	1.546	1.543	1.532	1.516	1.492	1.459	1.418	1.367	1.302	1.211
30	1.534	1.531	1.522	1.505	1.482	1.45	1.41	1.359	1.296	1.207
40	1.447	1.445	1.437	1.423	1.404	1.378	1.346	1.303	1.249	1.175
50	1.391	1.390	1.382	1.37	1.355	1.332	1.304	1.267	1.22	1.156
60	1.353	1.35	1.345	1.334	1.319	1.299	1.274	1.241	1.199	

Table 26: Percentiles of the Correlated Gamma Ratio distribution for α=0.01 (Panaretos et al. (1997))

$$\int_0^{+\infty} \frac{\left(1-\rho^2\right)^{\kappa}}{B(\kappa,\kappa)} t^{\kappa-1}(1+t)^{-2\kappa}\left[1-\left[\frac{2\rho}{t+1}\right]^2 t\right]^{-\frac{2\kappa+1}{2}} dt = 1-\alpha = 0.99$$



κ \ ρ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	99	98.10	95.2	90.3	83.5	74.8	64.1	51.7	36.7	20.4
2	15.98	15.84	15.42	14.71	13.72	12.45	10.90	9.05	6.91	4.45
3	8.47	8.40	8.20	7.87	7.40	6.8	6.05	5.17	4.13	2.91
4	6.03	5.99	5.86	5.64	5.34	4.95	4.47	3.89	3.2	2.38
5	4.85	4.82	4.73	4.57	4.34	4.05	3.69	3.25	2.73	2.11
6	4.155	4.13	4.06	3.93	3.75	3.52	3.23	2.88	2.46	1.94
7	3.7	3.68	3.62	3.51	3.36	3.16	2.92	2.62	2.27	1.83
8	3.37	3.36	3.30	3.21	3.08	2.91	2.7	2.45	2.14	1.75
9	3.13	3.12	3.07	2.99	2.87	2.72	2.53	2.31	2.03	1.68
10	2.94	2.93	2.88	2.81	2.705	2.565	2.405	2.2	1.95	1.63
11	2.785	2.775	2.735	2.67	2.575	2.45	2.3	2.11	1.88	1.59
12	2.66	2.65	2.61	2.55	2.465	2.35	2.21	2.04	1.825	1.555
13	2.555	2.545	2.51	2.455	2.375	2.27	2.135	1.975	1.78	1.525
14	2.465	2.455	2.425	2.37	2.295	2.195	2.075	1.925	1.74	1.497
15	2.39	2.38	2.35	2.3	2.23	2.135	2.025	1.88	1.705	1.475
16	2.32	2.31	2.285	2.235	2.17	2.08	1.975	1.84	1.675	1.46
17	2.26	2.25	2.225	2.18	2.117	2.035	1.935	1.805	1.645	1.437
18	2.208	2.195	2.172	2.13	2.07	1.99	1.895	1.773	1.62	1.418
19	2.16	2.15	2.127	2.086	2.03	1.955	1.86	1.744	1.599	1.41
20	2.115	2.105	2.085	2.046	1.994	1.92	1.83	1.72	1.58	1.395
21	2.075	2.07	2.049	2.01	1.956	1.89	1.801	1.695	1.56	1.384
22	2.04	2.034	2.01	1.976	1.925	1.86	1.775	1.675	1.544	1.374
23	2.005	2	1.98	1.946	1.897	1.835	1.754	1.654	1.53	1.364
24	1.978	1.972	1.952	1.918	1.872	1.810	1.732	1.634	1.512	1.352
25	1.95	1.944	1.924	1.892	1.848	1.788	1.712	1.618	1.5	1.344
26	1.924	1.918	1.90	1.868	1.824	1.766	1.694	1.602	1.488	1.336
27	1.9	1.894	1.876	1.846	1.804	1.748	1.676	1.588	1.476	1.328
28	1.878	1.872	1.854	1.826	1.784	1.73	1.66	1.574	1.464	1.32
29	1.856	1.852	1.834	1.806	1.766	1.712	1.645	1.561	1.455	1.314
30	1.838	1.832	1.816	1.788	1.748	1.696	1.632	1.55	1.446	1.308
40	1.69	1.685	1.672	1.65	1.619	1.578	1.525	1.458	1.374	1.259
50	1.597	1.594	1.583	1.565	1.538	1.502	1.456	1.4	1.327	1.229
60	1.536	1.532	1.522	1.506	1.48	1.449	1.409	1.359	1.294	-

«Table 27» : Iowa Crop Yield Data

YEA	Yield.	1-Serial Number	2-Precip.	3-May Temp	4-June Rain.	5-June Temp	6-Aug. Rain	7-July Temp	8-Aug. Rain	9-Aug. Temp
1930	34	1	17.75	60.2	5.83	69	1.49	77.9	2.42	74.4
1931	32.9	2	14.76	57.5	3.83	75	2.72	77.2	3.3	72.6
1932	43	3	27.99	62.3	5.17	72	3.12	75.8	7.1	72.2
1933	40	4	16.76	60.5	1.64	77.8	3.45	76.1	3.01	70.5
1934	23	5	11.36	69.5	3.49	77.2	3.85	79.7	2.84	73.4
1935	38.4	6	22.71	55	7	65.9	3.35	79.4	2.42	73.6
1936	20	7	17.91	66.2	2.85	70.1	0.51	83.4	3.48	79.2
1937	44.6	8	23.31	61.8	3.8	69	2.63	75.9	3.99	77.8
1938	46.3	9	18.53	59.5	4.67	69.2	4.24	76.5	3.82	75.7
1939	52.2	10	18.56	66.4	5.32	71.4	3.15	76.2	4.72	70.7
1940	52.3	11	12.45	58.4	3.56	71.3	4.57	76.7	6.44	70.7
1941	51	12	16.05	66	6.2	70	2.24	75.1	1.94	75.1
1942	59.9	13	27.1	59.3	5.93	69.7	4.89	74.3	3.17	72.2
1943	54.7	14	19.05	57.5	6.16	71.6	4.56	75.4	5.07	74
1944	52	15	20.79	64.6	5.88	71.7	3.73	72.6	5.88	71.8
1945	43.5	16	21.88	55.1	4.7	64.1	2.96	72.1	3.43	72.5
1946	56.7	17	20.02	56.5	6.41	69.8	2.45	73.8	3.56	68.9
1947	30.5	18	23.17	55.6	10.39	66.3	1.72	72.8	1.49	80.6
1948	60.5	19	19.15	59.2	3.42	68.6	4.14	75	2.54	73.9
1949	46.1	20	18.28	63.5	5.51	72.4	3.47	76.2	2.34	73
1950	48.2	21	18.45	59.8	5.7	68.4	4.65	69.7	2.39	67.7
1951	43.1	22	22	62.2	6.11	65.2	4.45	72.1	6.21	70.5
1952	62.2	23	19.05	59.6	5.4	74.2	3.84	74.7	4.78	70
1953	52.9	24	15.67	60	5.31	73.2	3.28	74.6	2.33	73.2
1954	53.9	25	15.92	55.6	6.36	72.9	1.79	77.4	7.1	72.1
1955	48.4	26	16.75	63.6	3.07	67.2	3.29	79.8	1.79	77.2
1956	52.8	27	12.34	62.4	2.56	74.7	4.51	72.7	4.42	73
1957	62.1	28	15.82	59	4.84	68.9	3.54	77.9	3.76	72.9
1958	66	29	15.24	62.5	3.8	66.4	7.55	70.5	2.55	73
1959	64.2	30	21.72	62.8	4.11	71.5	2.29	72.3	4.92	76.3
1960	63.2	31	25.08	59.7	4.43	67.4	2.76	72.6	5.36	73.2
1961	75.4	32	17.79	57.4	3.36	69.4	5.51	72.6	3.04	72.4
1962	76	33	26.61	66.6	3.12	69.1	6.27	71.6	4.31	72.5

• NOTATION AND TERMINOLOGY FOR TABLES 28-30

⇒ Response (YIELD) : Y°_{t+1}

⇒ Regression Coefficients (BETA): $\hat{\mathbf{b}}_{\mathbf{t}} = \left(\mathbf{X}'_{\mathbf{t}}\mathbf{X}_{\mathbf{t}}\right)^{-1}\mathbf{X}'_{\mathbf{t}}\mathbf{Y}_{\mathbf{t}}$

⇒ Prediction (PRED) : $\hat{Y}^{\circ}_{t+1} = \mathbf{X}^{\circ}_{\mathbf{t+1}}\hat{\mathbf{b}}_{\mathbf{t}}$

⇒ Difference (DIFF) : $Y^{\circ}_{t+1} - \hat{Y}^{\circ}_{t+1}$

⇒ Mean Square Error (MSE) : $S^2_t = \dfrac{\left[\mathbf{Y}_{\mathbf{t}} - \mathbf{X}_{\mathbf{t}}\,\hat{\mathbf{b}}_{\mathbf{t}}\right]'\left[\mathbf{Y}_{\mathbf{t}} - \mathbf{X}_{\mathbf{t}}\,\hat{\mathbf{b}}_{\mathbf{t}}\right]}{\left[\ell_t - m\right]}$

⇒ Standard Error of Prediction (SEOFP): SEOFP: $S_t\sqrt{\left(\mathbf{X}^{\circ}_{\mathbf{t+1}}\left(\mathbf{X}'_{\mathbf{t}}\mathbf{X}_{\mathbf{t}}\right)^{-1}\mathbf{X}^{\circ\,\prime}_{\mathbf{t+1}} + 1\right)}$

⇒ Residual (RESID): $r_{t+1} = \dfrac{\hat{Y}^{\circ}_{t+1} - Y^{\circ}_{t+1}}{S_t\sqrt{\left(\mathbf{X}^{\circ}_{\mathbf{t+1}}\left(\mathbf{X}'_{\mathbf{t}}\mathbf{X}_{\mathbf{t}}\right)^{-1}\mathbf{X}^{\circ\,\prime}_{\mathbf{t+1}} + 1\right)}}$

«Table 28» : Iowa Data - Model with variables {1,2,3,4,5,6,7,8,9}

YEAR	YIELD	PRED	DIFF	MSE	SEOFP	RESID	BETA1	BETA2	BETA3	BETA4	BETA5	BETA6	BETA7	BETA8	BETA9	BETA10
1941	51	52,453	-1,453	2,643	4,536	0,320	482,650	1,270	0,076	-0,052	-0,995	-1,328	-0,888	-3,073	0,177	-1,487
1942	59,9	59,546	0,354	1,438	2,218	-0,159	497,026	1,143	0,076	0,001	-1,323	-1,523	-0,623	-3,022	0,193	-1,572
1943	54,7	51,462	3,238	0,972	2,332	-1,389	494,716	1,161	0,090	-0,006	-1,307	-1,501	-0,615	-3,012	0,173	-1,572
1944	52	66,735	-14,735	1,184	1,475	9,988	450,799	1,309	0,085	-0,161	-0,741	-1,119	-0,828	-2,930	0,407	-1,352
1945	43,5	75,557	-32,057	24,564	9,309	3,444	432,376	0,955	0,095	-0,200	-1,246	-1,415	0,590	-2,489	-0,173	-1,218
1946	56,7	53,018	3,682	69,017	14,723	-0,250	125,388	0,563	0,173	-0,056	1,817	0,162	2,469	-1,539	0,799	-0,007
1947	30,5	60,432	-29,932	59,770	17,231	1,737	121,048	0,698	0,207	-0,145	1,964	0,316	1,954	-1,427	0,760	-0,148
1948	60,5	51,688	8,812	74,842	11,701	-0,753	234,087	0,470	0,292	0,365	-0,019	-0,778	3,142	-0,937	0,688	-1,483
1949	46,1	51,201	-5,101	71,257	11,019	0,463	226,982	0,676	0,366	0,358	-0,581	-0,813	3,479	-0,819	0,270	-1,466
1950	48,2	70,308	-22,108	65,672	11,021	2,006	249,718	0,532	0,302	0,349	-0,633	-0,958	3,538	-1,019	0,425	-1,393
1951	43,1	64,980	-21,880	83,735	11,847	1,847	133,439	0,605	0,455	0,027	-0,422	-0,436	3,047	-0,343	1,340	-0,845
1952	62,2	54,989	7,211	100,57	13,779	-0,523	100,172	0,487	0,568	-0,284	-0,487	0,200	2,763	-0,380	0,360	-0,662
1953	52,9	51,355	1,545	94,949	11,711	-0,132	74,418	0,665	0,631	-0,396	-0,427	0,460	2,433	-0,161	0,524	-0,735
1954	53,9	53,128	0,772	88,286	15,328	-0,050	71,449	0,690	0,627	-0,409	-0,436	0,493	2,394	-0,151	0,509	-0,723
1955	48,4	46,551	1,849	82,415	13,065	-0,142	70,474	0,700	0,622	-0,422	-0,425	0,502	2,344	-0,135	0,550	-0,726
1956	52,8	57,605	-4,805	77,368	11,419	0,421	66,703	0,726	0,630	-0,413	-0,488	0,476	2,388	-0,078	0,512	-0,718
1957	62,1	54,776	7,324	73,618	9,929	-0,738	66,655	0,702	0,681	-0,405	-0,319	0,425	2,340	0,069	0,433	-0,843
1958	66	61,902	4,098	71,745	12,020	-0,341	60,403	0,770	0,670	-0,432	-0,294	0,371	2,430	0,287	0,419	-0,920
1959	64,2	54,240	9,960	68,406	10,883	-0,915	61,502	0,766	0,626	-0,416	-0,304	0,318	2,765	0,249	0,408	-0,859
1960	63,2	64,461	-1,261	67,853	9,823	0,128	55,904	0,842	0,749	-0,386	-0,640	0,433	2,271	0,022	0,537	-0,688
1961	75,4	67,683	7,717	64,676	9,135	-0,845	54,962	0,834	0,730	-0,386	-0,600	0,437	2,315	0,025	0,529	-0,681
1962	76	75,131	0,869	63,829	10,372	-0,084	52,957	0,872	0,768	-0,470	-0,768	0,477	2,533	0,052	0,414	-0,658

«Table 29» : Iowa Data - Model with variables {1,2,3,4,5,6,8,9}

YEAR	YIELD	PRED	DIFF	MSE	SEOFP	RESID	BETA1	BETA2	BETA3	BETA4	BETA5	BETA6	BETA7	BETA8	BETA9
1940	52,3	83,476	-31,176	156,583	54,579	0,571	778,669	-0,445	-0,955	0,090	-8,080	-5,760	4,710	4,457	-4,165
1941	51	35,908	15,092	103,672	25,836	-0,584	294,442	1,177	0,712	-0,299	-1,758	-1,233	1,623	0,574	-2,289
1942	59,9	68,170	-8,270	80,852	15,856	0,522	84,311	2,677	0,836	-0,985	2,021	1,134	-1,050	0,474	-1,421
1943	54,7	63,678	-8,978	66,185	18,239	0,492	109,902	2,340	0,544	-0,867	1,847	0,775	-1,249	0,978	-1,427
1944	52	61,556	-9,556	56,177	9,888	0,966	214,443	1,980	0,598	-0,457	0,351	-0,225	-0,640	0,317	-2,101
1945	43,5	66,732	-23,232	55,549	13,242	1,754	226,212	1,669	0,552	-0,454	-0,107	-0,520	0,311	-0,070	-1,932
1946	56,7	58,687	-1,987	72,006	14,164	0,140	41,366	1,131	0,474	-0,259	2,020	0,480	1,919	0,685	-0,731
1947	30,5	55,457	-24,957	63,180	17,204	1,451	40,374	1,074	0,467	-0,213	1,939	0,400	2,208	0,703	-0,679
1948	60,5	52,719	7,781	70,947	11,308	-0,688	164,876	0,758	0,463	0,259	0,193	-0,594	3,181	0,657	-1,698
1949	46,1	53,740	-7,640	67,217	10,039	0,761	166,323	0,910	0,510	0,265	-0,334	-0,646	3,479	0,286	-1,659
1950	48,2	67,295	-19,095	64,644	10,427	1,831	179,692	0,762	0,461	0,211	-0,320	-0,822	3,579	0,555	-1,616
1951	43,1	65,045	-21,945	77,321	11,383	1,928	113,408	0,687	0,506	-0,008	-0,317	-0,412	3,087	1,342	-0,955
1952	62,2	56,254	5,946	93,473	12,597	-0,472	77,850	0,577	0,625	-0,324	-0,371	0,228	2,806	0,359	-0,784
1953	52,9	51,449	1,451	88,285	11,271	-0,129	66,023	0,693	0,652	-0,406	-0,377	0,452	2,480	0,510	-0,786
1954	53,9	53,639	0,261	82,497	14,275	-0,018	63,712	0,715	0,647	-0,418	-0,389	0,483	2,441	0,497	-0,772
1955	48,4	47,147	1,253	77,343	11,754	-0,107	63,659	0,717	0,645	-0,422	-0,387	0,487	2,422	0,512	-0,771
1956	52,8	57,265	-4,465	72,845	10,352	0,431	62,949	0,732	0,644	-0,415	-0,449	0,472	2,431	0,495	-0,750
1957	62,1	54,534	7,566	69,553	9,112	-0,830	70,485	0,694	0,671	-0,402	-0,348	0,426	2,294	0,445	-0,819
1958	66	62,467	3,533	68,415	11,632	-0,304	77,137	0,744	0,620	-0,425	-0,423	0,368	2,229	0,472	-0,821
1959	64,2	55,263	8,937	65,310	10,122	-0,883	76,149	0,744	0,588	-0,412	-0,417	0,321	2,547	0,457	-0,779
1960	63,2	64,473	-1,273	64,625	9,583	0,133	57,399	0,839	0,744	-0,386	-0,647	0,432	2,254	0,540	-0,682
1961	75,4	67,684	7,716	61,739	8,917	-0,865	56,645	0,830	0,724	-0,386	-0,608	0,436	2,297	0,532	-0,674
1962	76	75,034	0,966	61,068	10,059	-0,096	56,421	0,865	0,756	-0,470	-0,786	0,475	2,496	0,422	-0,643

«Table 30» : Iowa Data - Model with variables {1,2,3,5,6,8}

YEAR	YIELD	PRED	DIFF	MSE	SEOFP	RESID	BETA1	BETA2	BETA3	BETA4	BETA5	BETA6	BETA7
1938	46,3	28,77629	17,52371	83,7461	20,5241	-0,85381	-87,7813	-0,41157	2,63272	-0,38234	1,50953	1,25564	-4,07246
1939	52,2	34,40498	17,79502	72,3294	11,5528	-1,54032	43,36649	0,52342	0,89691	-1,0047	0,34016	2,95125	0,48431
1940	52,3	59,58465	-7,28465	105,4203	29,9616	0,243133	66,07274	1,39317	0,21544	-1,04138	0,08391	2,64521	2,83389
1941	51	35,5467	15,4533	80,6226	13,1144	-1,17834	40,12198	1,32436	0,5916	-0,85651	0,24376	2,72456	1,75309
1942	59,9	65,90416	-6,00416	86,8878	16,342	0,367406	13,13821	2,25308	0,81209	-0,89704	0,64269	1,69349	0,50037
1943	54,7	62,63452	-7,93452	74,3604	11,6269	0,68243	27,84427	2,0283	0,59625	-0,82959	0,43973	1,46783	0,89916
1944	52	56,61182	-4,61182	68,6844	10,3429	0,445893	34,70778	1,65084	0,58162	-0,55492	0,12556	2,10372	0,68849
1945	43,5	59,07674	-15,5767	61,8059	10,7759	1,445514	43,21256	1,50258	0,54816	-0,55839	0,02795	2,38713	0,50687
1946	56,7	50,18439	6,51561	69,2826	12,2034	-0,53392	13,49046	1,0642	0,58101	-0,1657	0,0729	3,65277	0,64458
1947	30,5	52,71826	-22,2183	64,334	10,8356	2,050494	4,40385	1,2995	0,62974	-0,35918	0,37143	2,79403	0,58464
1948	60,5	53,83595	6,66405	83,0835	10,6548	-0,62545	17,24415	0,8327	0,39094	-0,05572	-0,08128	4,41913	1,52096
1949	46,1	52,31962	-6,21962	78,8639	10,8471	0,573391	15,99304	0,92242	0,3758	-0,0223	-0,09719	4,75669	1,26295
1950	48,2	58,11941	-9,91941	74,7907	10,0654	0,985492	30,16879	0,80148	0,33865	-0,06093	-0,25363	4,81079	1,46166
1951	43,1	63,5784	-20,4784	74,6336	11,1135	1,842659	25,79727	0,7355	0,38954	-0,12607	-0,13187	4,18829	1,74428
1952	62,2	56,16521	6,03479	86,5596	12,1214	-0,49786	5,78942	0,61486	0,50845	-0,39941	0,43473	3,76252	0,76315
1953	52,9	52,52844	0,37156	82,4942	10,7979	-0,03441	-6,48714	0,73208	0,53551	-0,48224	0,66365	3,43543	0,91877
1954	53,9	54,70222	-0,80222	77,6473	13,7837	0,058201	-6,81797	0,73725	0,53406	-0,48476	0,67108	3,42222	0,91438
1955	48,4	48,47548	-0,07548	73,3482	10,0917	0,007479	-6,95797	0,72915	0,54064	-0,4722	0,66103	3,48736	0,86994
1956	52,8	58,8745	-6,0745	69,488	9,6706	0,62814	-6,99238	0,72858	0,54121	-0,47313	0,66213	3,48726	0,87141
1957	62,1	54,35952	7,74048	67,384	8,9431	-0,86552	-2,93261	0,68429	0,59837	-0,48339	0,61121	3,36579	0,84001
1958	66	65,14793	0,85207	66,5798	11,2503	-0,07574	2,73197	0,73213	0,53351	-0,49949	0,55606	3,3274	0,87342
1959	64,2	56,50502	7,69498	63,5708	9,5587	-0,80503	3,38748	0,73209	0,52585	-0,49473	0,54232	3,39437	0,8645
1960	63,2	62,3989	0,8011	62,599	9,0825	-0,0882	-9,06925	0,81188	0,63593	-0,43328	0,6345	3,10787	0,90948
1961	75,4	67,32594	8,07406	60,0112	8,6009	-0,93875	-9,46808	0,81758	0,64571	-0,43104	0,63485	3,09131	0,9192
1962	76	74,33085	1,66915	59,7241	9,8645	-0,16921	-10,2275	0,85545	0,66037	-0,49296	0,68528	3,31544	0,79969