Penalty shoot-outs

The quest for fairness

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Extreme shoot-outing

20th of June, 2007. Heerenveen, Abe Lenstra Stadion. After a last-minute equalizer in the semi-finals of the U21-European Championship between England and The Netherlands, the extra time provides no goals and a penalty shoot-out needs to decide who wins...

Extreme shoot-outing

Figure 1: As it happened

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At the end, the Dutch prevailed, which was a great show of character by
the team, as they had the added pressure of having to level after most of

Extreme shoot-outing

These were the results of the penalties round by round.

Round	1	2	3	4	5	6	7	8
England	O	O	O	X	O	O	O	O
Netherlands	O	X	O	O	O	O	O	O
Score	1-1	2-1	3-2	3-3	4-4	5-5	6-6	7-7
Round	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
England	X	O	O	O	O	O	X	X
Netherlands	X	O	O	O	O	O	X	O
Score	7-7	8-8	9-9	10-10	11-11	12-12	12-12	12-13

Figure 2: Penalty shoot-out result.

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Score	7-7	8-8	9-9	10-10	11-11	12-12	12-12	12-13

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The Dutch team had to level a total of 9 times during this shoot-out, whereas the English team never faced the pressure of immediate loss.

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Outline

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- Two teams, Team A and Team B, played a match.
- To determine a winner, a shoot-out takes place (usually over 5 rounds).
- In every round, every team takes one penalty.
- If, after 5 rounds, the scores are equal, the sudden death phase starts.
- The sudden death ends if and only if a team scores in a round, while the other team does not.

- The best known and abundantly present penalty shoot-out has one team starting all the rounds, and one team shooting second - A(B)A(B)A(B). . . .
- Palacios-Huerta and Apesteguia (2010) showed in an empirical study that there is an advantage for the team that shoots first.

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- Palacios-Huerta and Apesteguia (2010) showed in an empirical study that there is an advantage for the team that shoots first.
- This is usually explained by the extra pressure of having to catch up every round; known as **First Mover Advantage.** (Brams and Ismail 2017; Vandebroek et al., 2018)

Problem case
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We say there is a probability p of scoring if a team is at least equal in score, and a probability q if a team is trailing in score, with $p > q$.
If the score is equal at the beginning of a round started by A, this results in probabilities:

$$\begin{aligned}\mathbb{P}(\text{Team A wins round}) &= p \cdot (1-q) =: P_+ \\ \mathbb{P}(\text{Team B wins round}) &= (1-p) \cdot p =: P_- \\ \mathbb{P}(\text{The round is drawn}) &= (1-p)^2 + pq =: P_\pm\end{aligned}$$

Difference $P_+ - P_- = \lambda$ is referred to as the FMA.

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- Alternatives have been suggested and experimented with, such as the one called ABBA with football, that has A(B)B(A) as sequence.
- Also, field hockey has shoot-outs in which the eventual sudden death is started by the team that did not start the shoot-out.

Problem case
Fairness

It is undesirable to have an unfair shoot-out to decide the outcome of a game. What do we consider a fair shoot-out?

We want the order of the penalties to be fixed before the first penalty. Then:

Definition

We say a penalty shoot-out S between A and B is fair, if:

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We assume both teams are as good as the other.

What are the requirements for a shoot-out to be fair?

Sudden-death stage

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The probability that team A wins the sudden death is equal to:

$$\mathbb{P}(\text{Team A wins}) = \sum_{r=1}^{\infty} \mathbb{P}(\text{r-1 rounds drawn})\mathbb{P}(\text{Team A wins round r})$$

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An analogue expression can be derived for Team B.

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We have P_+, P_- as probabilities of the first and second shooting team winning a round, and $P_\pm$ for a drawn round. Then:

$$\mathbb{P}(\text{Team A wins}) = \frac{P_-}{1 - P_\pm} + \lambda \sum_{r \in I} \mathbb{P}_\pm^{r-1}$$

Similarly, we derive for Team B:

$$\mathbb{P}(\text{Team B wins}) = \frac{P_-}{1 - P_\pm} + \lambda \sum_{r \in \bar{I}} P_\pm^{r-1}$$

A sequence S where Team A shoots first in rounds I is fair with respect to $P_+, P_-, P_\pm$ if:

$$\sum_{r \in I} P_\pm^{r-1} = \sum_{r \in \mathbb{N} \setminus I} P_\pm^{r-1} \qquad (1)$$

A sequence S where Team A shoots first in rounds I is fair with respect to $P_+, P_-, P_\pm$ if:

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Notice that the fairness of I only depends on I and the value of $P_\pm$ - not on P_+, P_- or the FMA λ.

Theorem

If we assume that $p, q \in \mathbb{Q}$, every finite repeated sequence S_n is unfair.

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This means that if we assume $p, q \in \mathbb{Q}$, and we try to find a fair shoot-out policy with a finite repeated sequence, we will fail.

Proof.

Let $f_I(x)$ be the polynomial $f_I(x) = \sum_{i \in I} x^{i-1}$, then the corresponding sequence S is fair if $P_{\pm}$ is a solutions of the polynomial:

$$f_S(x) = f_I(x) - f_{\bar{I}}(x) = \sum_{i \in I} x^{i-1} - \sum_{j \in \bar{I}} x^{j-1} = 0$$

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If we have any finite sequence S_n of n rounds, that is repeated, the characteristic polynomial of the resulting sequence S will be:

$$f_S = f_{S_n} \cdot (1 + x^n + \dots)$$

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Thus, all zeroes of $f_S(x)$ need to be zeroes of $f_{S_n}(x)$. However, since $f_{S_n}(x)$ is a polynomial of finite degree with coefficients ± 1, there will be no rational zero $x \in (0, 1)$ (Gauss' Lemma). $\square$

Sudden-death stage
Unfairness

Even worse:

- The common sequence $A(B)A(B)\dots$ with polynomial $f_A = 1 + x + \cdots = 0$ has **no** solution with $x \in (0,1)$.

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A well known infinite penalty shoot-out is based on the Prouhet-Thue-Morse (PTM) sequence. Team A starts round 1, Team B round 2. From then, the following rounds will be an inverse copy of the first rounds.

$$AB\ BA\ BAAB\ BAABABBA\dots \tag{2}$$

This can be proven to be unfair for all values of $\mathbb{P}_\pm \in (0,1)$.

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Fair algorithm

The following algorithm creates a fair shoot-out, for any given value of $P_{\pm} \geq \frac{1}{2}$.

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Algorithm

Let Team A shoot first in round 1 *and set* $I_1 = \{1\}$*. Then, starting from* $n = 1$*, construct* I_{n+1} *from* I_n *in the following way:*

- *If* $f_{I_n}(P_{\pm}) < f_{\bar{I}_n}(P_{\pm})$ *:* $I_{n+1} = I_n \cup \{n+1\}$*.*
- *Otherwise:* $I_{n+1} = I_n$

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- *Otherwise: $I_{n+1} = I_n$*

By this construction, the resulting set $I = \lim_{n\to\infty} I_n$ will have:

$$f_I(P_{\pm}) = f_{\overline{I}}(P_{\pm})$$

And therefore be fair.

Sudden-death stage

Fair algorithm

If we assume $p = \frac{3}{4}, q = \frac{2}{3}$, this would be the sequence generated by the algorithm:

Round	1	2	3	4	5	6	7	8
First shooter	?	?	?	?	?	A	B	B
Second shooter	?	?	?	?	?	B	A	A
Score								
Round	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
First shooter	B	A	B	A	B	B	A	A
Second shooter	A	B	A	B	A	A	B	B
Score								

Figure 3: Fair sudden death sequence

The sudden death in the match of the example should have had, as sequence: $ABBBABABBAA\ldots$.

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For any p, q, a fair sequence can be calculated using the algorithm.

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- It is possible that, at the start of a round, one team has scored more goals than the other - adding pressure to the team lagging behind, as they have to make up a goal.
- This also influences the advantage a team gets from starting a round.

Best of 5

Parameter results

The shoot-out with respect to p, q that is the least unfair:

p/q	0.6	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.7	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.8
0.6																					
0.61	ABABB																				
0.62	ABABB	ABABB																			
0.63	ABBAB	ABABB	ABABB																		
0.64	ABBAB	ABABB	ABABB	ABABB																	
0.65	ABBAB	ABBAB	ABABB	ABABB	ABABB																
0.66	ABBBA	ABBAB	ABBAB	ABABB	ABABB	ABABB															
0.67	ABBBA	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB														
0.68	ABBBA	ABBBA	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB													
0.69	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB												
0.7	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB											
0.71	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB										
0.72	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	AABBB									
0.73	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB								
0.74	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB							
0.75	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB						
0.76	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB					
0.77	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB				
0.78	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB			
0.79	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB		
0.8	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB	

Figure 4: Fairest Bo5 penalty sequence for given p, q

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0.61	ABABB																				
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0.74	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB							
0.75	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB						
0.76	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB					
0.77	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB				
0.78	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB			
0.79	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB		
0.8	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB	

Figure 4: Fairest Bo5 penalty sequence for given p, q

Note that both *AAAAA* or *ABABA* (common and ABBA series) are never suggested.

We can compare the unfairness of the preferred penalty sequence, to the one currently used.

Best of 5

p/q	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80
0.60																					
0.61	0.0094																				
0.62	0.0192	0.0096																			
0.63	0.0294	0.0196	0.0098																		
0.64	0.0400	0.0301	0.0201	0.0101																	
0.65	0.0511	0.0410	0.0309	0.0206	0.0104																
0.66	0.0627	0.0524	0.0421	0.0317	0.0212	0.0106															
0.67	0.0747	0.0643	0.0538	0.0432	0.0325	0.0218	0.0109														
0.68	0.0873	0.0767	0.0660	0.0552	0.0444	0.0334	0.0224	0.0112													
0.69	0.1003	0.0896	0.0788	0.0678	0.0567	0.0456	0.0343	0.0230	0.0115												
0.70	0.1139	0.1030	0.0920	0.0809	0.0697	0.0583	0.0469	0.0353	0.0237	0.0119											
0.71	0.1281	0.1170	0.1059	0.0946	0.0832	0.0717	0.0600	0.0483	0.0364	0.0244	0.0122										
0.72	0.1428	0.1316	0.1203	0.1089	0.0973	0.0856	0.0738	0.0618	0.0497	0.0375	0.0251	0.0126									
0.73	0.1581	0.1468	0.1354	0.1238	0.1121	0.1002	0.0882	0.0760	0.0637	0.0512	0.0386	0.0259	0.0130								
0.74	0.1740	0.1626	0.1511	0.1394	0.1275	0.1154	0.1032	0.0909	0.0784	0.0657	0.0529	0.0399	0.0267	0.0135							
0.75	0.1906	0.1791	0.1674	0.1556	0.1435	0.1314	0.1190	0.1065	0.0938	0.0809	0.0678	0.0546	0.0412	0.0276	0.0139						
0.76	0.2078	0.1962	0.1844	0.1725	0.1603	0.1480	0.1355	0.1228	0.1099	0.0968	0.0835	0.0701	0.0564	0.0426	0.0286	0.0144					
0.77	0.2257	0.2140	0.2022	0.1901	0.1778	0.1654	0.1527	0.1398	0.1268	0.1135	0.1000	0.0864	0.0725	0.0584	0.0441	0.0296	0.0149				
0.78	0.2443	0.2326	0.2206	0.2085	0.1961	0.1835	0.1707	0.1577	0.1445	0.1310	0.1174	0.1035	0.0894	0.0750	0.0605	0.0457	0.0307	0.0155			
0.79	0.2637	0.2519	0.2399	0.2277	0.2152	0.2025	0.1896	0.1764	0.1631	0.1494	0.1356	0.1215	0.1072	0.0926	0.0778	0.0627	0.0474	0.0318	0.0160		
0.80	0.2839	0.2720	0.2600	0.2477	0.2351	0.2223	0.2093	0.1960	0.1825	0.1687	0.1547	0.1404	0.1259	0.1111	0.0960	0.0807	0.0651	0.0492	0.0331	0.0167	

Figure 5: Compared fairness

Best of 5

p/q	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80
0.60																					
0.61	0.0094																				
0.62	0.0192	0.0096																			
0.63	0.0294	0.0196	0.0098																		
0.64	0.0400	0.0301	0.0201	0.0101																	
0.65	0.0511	0.0410	0.0309	0.0206	0.0104																
0.66	0.0627	0.0524	0.0421	0.0317	0.0212	0.0106															
0.67	0.0747	0.0643	0.0538	0.0432	0.0325	0.0218	0.0109														
0.68	0.0873	0.0767	0.0660	0.0552	0.0444	0.0334	0.0224	0.0112													
0.69	0.1003	0.0896	0.0788	0.0678	0.0567	0.0456	0.0343	0.0230	0.0115												
0.70	0.1139	0.1030	0.0920	0.0809	0.0697	0.0583	0.0469	0.0353	0.0237	0.0119											
0.71	0.1281	0.1170	0.1059	0.0946	0.0832	0.0717	0.0600	0.0483	0.0364	0.0244	0.0122										
0.72	0.1428	0.1316	0.1203	0.1089	0.0973	0.0856	0.0738	0.0618	0.0497	0.0375	0.0251	0.0126									
0.73	0.1581	0.1468	0.1354	0.1238	0.1121	0.1002	0.0882	0.0760	0.0637	0.0512	0.0386	0.0259	0.0130								
0.74	0.1740	0.1626	0.1511	0.1394	0.1275	0.1154	0.1032	0.0909	0.0784	0.0657	0.0529	0.0399	0.0267	0.0135							
0.75	0.1906	0.1791	0.1674	0.1556	0.1435	0.1314	0.1190	0.1065	0.0938	0.0809	0.0678	0.0546	0.0412	0.0276	0.0139						
0.76	0.2078	0.1962	0.1844	0.1725	0.1603	0.1480	0.1355	0.1228	0.1099	0.0968	0.0835	0.0701	0.0564	0.0426	0.0286	0.0144					
0.77	0.2257	0.2140	0.2022	0.1901	0.1778	0.1654	0.1527	0.1398	0.1268	0.1135	0.1000	0.0864	0.0725	0.0584	0.0441	0.0296	0.0149				
0.78	0.2443	0.2326	0.2206	0.2085	0.1961	0.1835	0.1707	0.1577	0.1445	0.1310	0.1174	0.1035	0.0894	0.0750	0.0605	0.0457	0.0307	0.0155			
0.79	0.2637	0.2519	0.2399	0.2277	0.2152	0.2025	0.1896	0.1764	0.1631	0.1494	0.1356	0.1215	0.1072	0.0926	0.0778	0.0627	0.0474	0.0318	0.0160		
0.80	0.2839	0.2720	0.2600	0.2477	0.2351	0.2223	0.2093	0.1960	0.1825	0.1687	0.1547	0.1404	0.1259	0.1111	0.0960	0.0807	0.0651	0.0492	0.0331	0.0167	

Figure 5: Compared fairness

Even with very small FMA, there is still an improvement of 1% in fairness. The model suggest an improvement of 10% in fairness applying *ABBAB* instead of *AAAA* for $p = \frac{3}{4}, q = \frac{2}{3}$.

So, in summary, if it comes to a penalty shoot-out, this is how the first 16 rounds in the example should have been taken:

Round	1	2	3	4	5	6	7	8
First shooter	A	B	B	A	B	A	B	B
Second shooter	B	A	A	B	A	B	A	A

Round	9	10	11	12	13	14	15	16
First shooter	B	A	B	A	B	B	A	A
Second shooter	A	B	A	B	A	A	B	B

Figure 6: First 16 penalties

Conclusions

- The current penalty shoot-out is inherently unfair.
- It is possible to construct fair sudden death sequences.
- An improvement in fairness can also be made within the best of 5 shoot-out.
- ABBAB in the best-of-5 would be fair compared to the current AAAAA.

Questions?