Sports betting strategies: an experimental review

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July 2, 2019

Problem Statement

- Optimal wealth allocation across presented betting opportunities.
- **Goal:** Finding portfolio vector

$$\boldsymbol{b} = \left[b_1, ..., b_n\right]^T \tag{1}$$

b_i is fraction of our wealth allocated on i-th opportunity.

Coin Toss

- Heads - We enlarge our wealth by 50%. Tails - We loose 40% of our wealth.

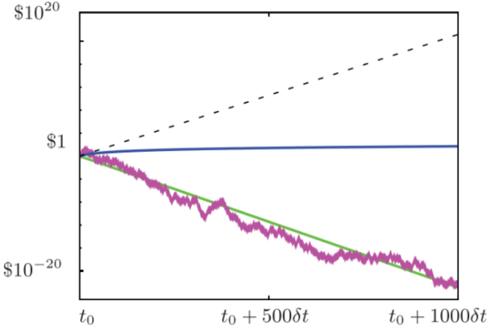
$$r = \begin{cases} 1.5 & \text{with probability } 1/2 \\ 0.6 & \text{with probability } 1/2 \end{cases} \tag{2}$$

- If we assume repeated play with reinvestment of our **whole** bank.
- $ev = 1.5 \cdot 0.5 + 0.6 \cdot 0.5 = 1.05$

Q: Should we take the deal?

Problem Statement

Coin Toss



The magenta line represents median wealth trajectory in 1000 time steps of a coin toss game. The dashed line is the expected value.

Observations:

- Overbetting = ruin (overvaluation of presented opportunity)
- Long term performance for single individual does not follow EV

Outline

1. Strategies
   - MPT
   - Kelly
2. Uncertainty
   - Fractional approach
   - Drawdown constraint
   - DRO
3. Experiments
   - Basketball
   - Football

Strategies

MPT

Likely the most famous approach to portfolio optimization.
In simple terms we maximize the following:

$$\mathbb{E}[gain] - \gamma \cdot risk \tag{3}$$

E.g. risk of a portfolio can be measured by it's variance defined through a covariance matrix $\boldsymbol{\Sigma}$.

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \boldsymbol{\mu}^T \boldsymbol{b} - \gamma \boldsymbol{b}^T \Sigma \boldsymbol{b} \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0,\ b_i \geq 0 \end{aligned}$$

- $\boldsymbol{b}$ is portfolio, $\boldsymbol{\mu}$ is the expected gains vector.
- γ is a risk aversion parameter $\longrightarrow$ set of "efficient portfolios"
- Criterion to choose one portfolio $\longrightarrow$ maximum sharpe ratio $\frac{r_p - r_f}{\sigma_p}$

Growth Optimal Strategy a.k.a. Kelly Criterion

$$\underset{\boldsymbol{b}}{\text{maximize}} \quad \mathbb{E}[\log(\boldsymbol{R} \cdot \boldsymbol{b})]$$

$$\text{subject to} \quad \sum_{i=1}^{K} b_i = 1.0, \; b_i \geq 0$$

- K probabilistic outcomes $p_1, p_2, ..., p_K$.
- n opportunities, (assets), $n-1$ risky, 1 risk-less. $\boldsymbol{a_1}, \boldsymbol{a_2}, ..., \boldsymbol{c}$.

$$\boldsymbol{p} = \begin{bmatrix} p_1 & p_2 & ... & p_K \end{bmatrix} \qquad \boldsymbol{a_i} = \begin{bmatrix} r_{1,i} \\ r_{2,i} \\ ... \\ r_{K,i} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} \tag{4}$$

Cash asset can have a different payoff if money can be **risk-free invested** elsewhere. (e.g. bank acc interest rate). return matrix $\boldsymbol{R}$ and "portfolio" vector $\boldsymbol{b}$

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & ... & \boldsymbol{a_{n-1}} & \boldsymbol{c} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_{n-1} \\ b_c \end{bmatrix} \tag{5}$$

Example

Assume horse race with 16 running horses. Bet type quinella denoted $QNL(i,j)$ pays off if pair of horses (i,j) win the race. Order does not matter. There are hence 120 different pairs, 121 different assets including cash asset and 120 probabilities in the vector $\boldsymbol{p}$. $o_{i,j}$ denotes posted odds for given $QNL(i,j)$.

$$\boldsymbol{R} = \begin{bmatrix} o_{1,2} & 0 & 0 & ... & 1 \\ 0 & o_{1,3} & 0 & ... & 1 \\ 0 & 0 & o_{1,4} & ... & 1 \\ ... & ... & ... & ... & 1 \end{bmatrix} \tag{6}$$

This is a bet on an exclusive outcome, hence R matrix is almost completely made up of zeros and odds diagonally.

$$\boldsymbol{p} = \left[p_{1,2}, p_{1,3}, ..., p_{15,16}\right] \tag{7}$$

$$\boldsymbol{b} = \left[b_{1,2}, b_{1,3}, ..., b_{15,16}, b_c\right] \tag{8}$$

Review

MPT

- Maximize return, minimize risk
- Modular approach $\rightarrow$ Utility functions and risk definitions.
- Different set up $\rightarrow$ different optimal allocation.

Kelly

- Long term growth optimal. (geom. mean)
- Ruin avoidance. (no risk definitions or utility functions)
- *"Make sure you show up tomorrow"* approach to risk.
- Unique optimal allocation.

Both assume we know the true probability distributions of the outcomes
$\rightarrow$ Uncertainty of estimates breaks the guarantees.

Uncertainty

- Bad news $\rightarrow$ we do not know the true probability of the outcomes.
- Good news $\rightarrow$ neither does bookmaker.

Goal: Avoid overbetting and maintain growth.

1. Calculate optimal fractions by MPT or Kelly on "Train" dataset.
2. Adjust fractions (Fractional, Drawdown, DR) to reach desired performance.
3. Validate on unseen data.

Fractional Approach

Idea: Adjust portfolio by a fixed fraction, (e.g. "half kelly" or "fractional MPT")

Fractional MPT

We define a trade-off index ω for a portfolio as:

$$\boldsymbol{b_\omega} = \omega \boldsymbol{b_s} + (1 - \omega)\boldsymbol{b_c} \tag{9}$$

- $\boldsymbol{b_s}$ stands for portfolio suggested by MPT strategy
- $\boldsymbol{b_c}$ is a portfolio with only cash asset.

Drawdown constraint

Drawdown

- Fractional approach uses fixed fraction. (Too static).
- A better approach is to add a drawdown constraint.

$$P(W^{MIN} < 0.7) \leq 0.1 \tag{10}$$

Probability of our wealth falling below 0.7 is at most 0.1, in general:

$$P(W^{MIN} < \alpha) \leq \beta \tag{11}$$

The drawdown constraint is approximately satisfied if the following is satisfied, (Boyd et al., 2016).

$$\mathbb{E}[(\boldsymbol{R} \cdot \boldsymbol{b})^{-\lambda}] \leq 1 \;\; \textit{where} \;\; \lambda = \log(\beta)/\log(\alpha) \tag{12}$$

Drawdown constraint

Risk Constrained Kelly

$$\begin{aligned} \underset{\boldsymbol{b}}{\text{maximize}} \quad & \sum_{i=1}^{K} p_i \cdot log(\boldsymbol{R_i} \cdot \boldsymbol{b}) \\ \text{subject to} \quad & \sum_{i=1}^{K} b_i = 1.0 \\ & b_i \geq 0 \\ & \log(\sum_{i=1}^{K} \exp[\log(p_i) - \lambda \log(\boldsymbol{R_i} \cdot \boldsymbol{b})]) \leq 0 \end{aligned}$$

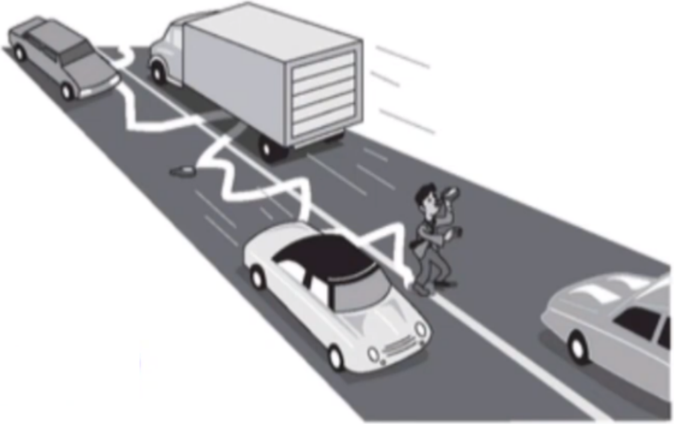
where $\lambda = \frac{\log(\beta)}{\log(\alpha)}$ for some $\alpha, \beta \in (0, 1)$

Result

Portfolio satisfies $P(W^{MIN} < \alpha) \leq \beta$ and is as "growth optimal" as possible.

DRO

Idea: We replace the single probability distributions of outcomes with ambiguity set of probability distributions: $\mathbf{\Pi}$.



The state of the drunk at his average position is alive. But the average state of the drunk is dead.

DRO

DR Kelly

Goal: Find portfolio that performs best in the worst case scenario.

$$\begin{aligned} \max_{\boldsymbol{b}} \quad & \min_{\boldsymbol{p}} \sum_{i=1}^{K} p_i \cdot log(\boldsymbol{R_i} \cdot \boldsymbol{b}) \\ \text{subject to} \quad & \dots \end{aligned}$$

- DRO - game between player and adversary
- Player maximizes the growth rate and the adversary, (nature) picks the distribution $\boldsymbol{p}$ from the ambiguity set $\boldsymbol{\Pi}$ to inflict maximum damage to the player.

Ambiguity set $\boldsymbol{\Pi}$ can be defined in a number of ways.

- Divergence based(**KL divergence**)
- Wasserstein distance

  ...

Experiments

Basketball

The basketball market has the following properties.

m-acc	*b-acc*	*n*	*margin*	*odds*
≈ 0.68	≈ 0.7	2	≈ 0.038	$\in [1.01, 41.]$

- > 14000 games
- We assume 10 game "rounds" i.e. always 10 games happening in parallel. 2^{10} possible outcomes.
- Train and Test dataset always randomly shuffled
- Fixed number of games always randomly removed from the dataset.

The risk constraint and fractional parameters are selected according to.

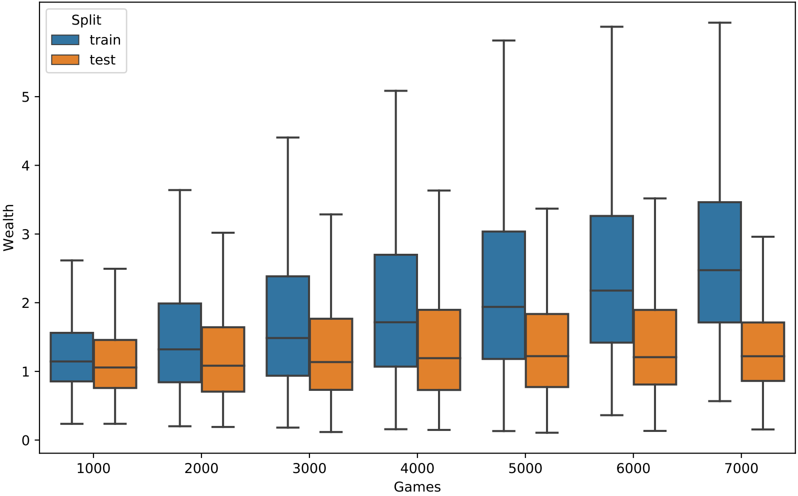
$$\begin{aligned} &\text{maximize} && \mathit{median}(W_F) \\ &\text{subject to} && Q_5 > 0.90 \end{aligned}$$

- Maximal median final wealth W_F
- Only 5% of all the wealth positions can go below 0.90 of initial wealth.

Experiments

Basketball

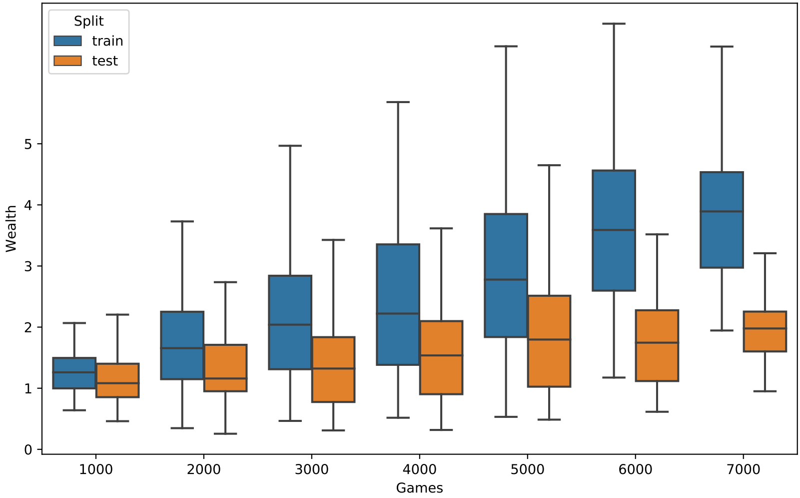
Fractional MPT



Experiments

Basketball

Risk Constrained Kelly



Experiments

Football

The football betting market has the following properties in relation to the model prediction.

m-acc	*b-acc*	*n*	*margin*	*odds*
0.523	0.537	3	0.03	$[1.03, 66]$

- Dataset consists of > 28000 games
- 10 games happening simultaneously.
- Train and Test always randomly shuffled with fixed number of games removed.

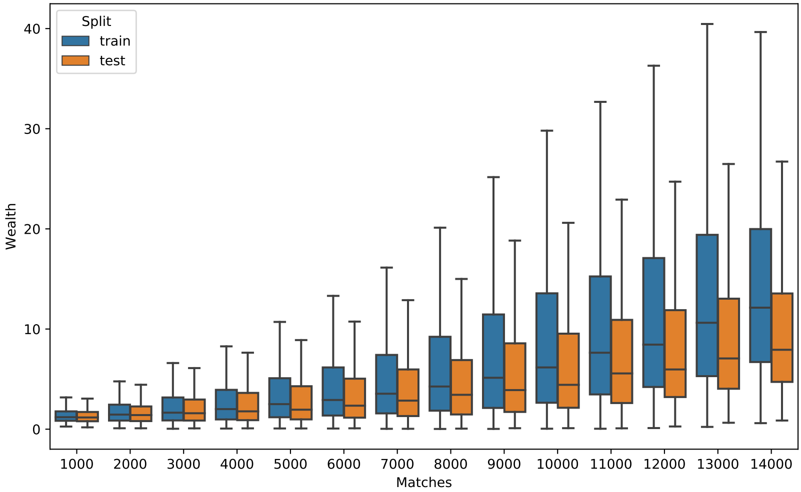
The parameters for all methods are selected according to the following criterion:

$$\begin{aligned} &\text{maximize} && \mathit{median}(W_F) \\ &\text{subject to} && Q_5 > 0.95 \end{aligned}$$

Experiments

Football

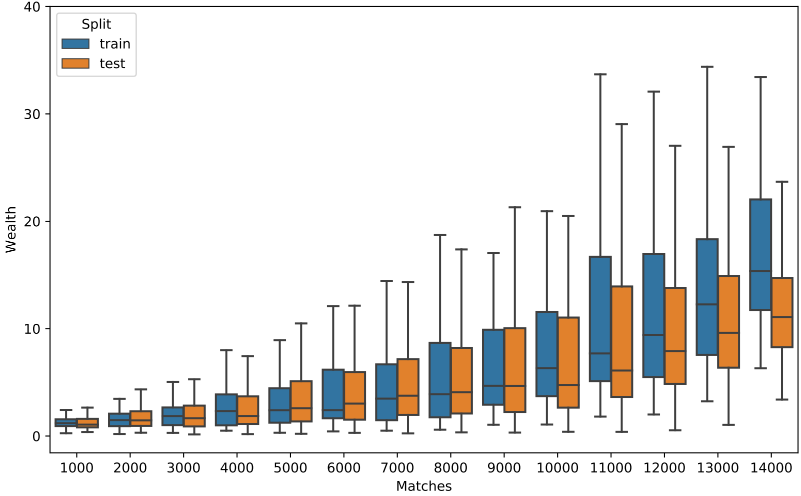
Fractional MPT



Experiments

Football

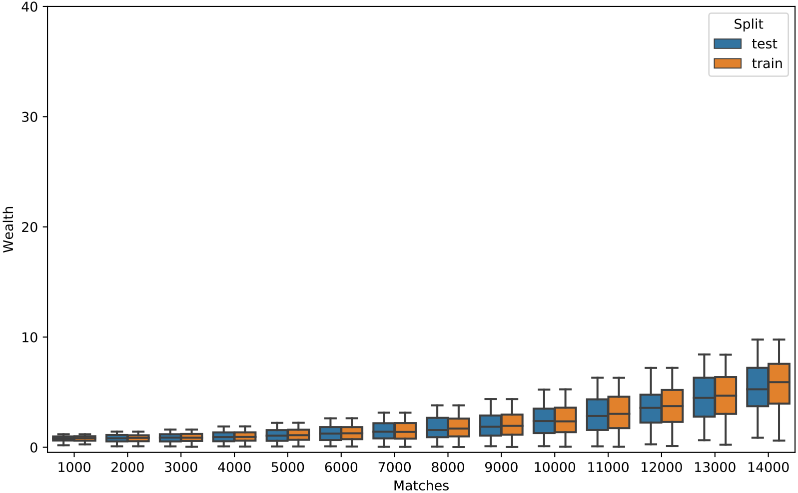
Risk Constrained Kelly



Experiments

Football

DR Kelly



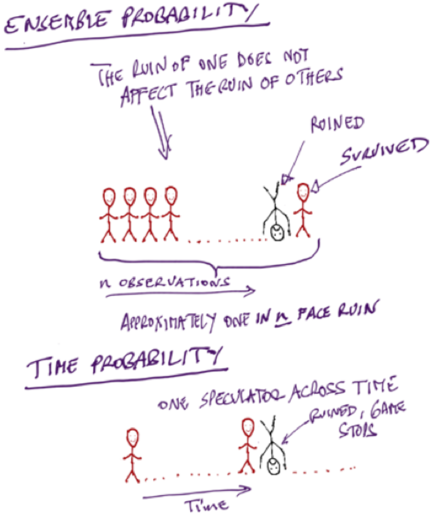
Conclusion

- Two general approaches to betting strategies: MPT and Kelly.
- MPT $\rightarrow$ max return, min risk. Kelly $\rightarrow$ max growth rate.
- Guarantees hold if true probability of outcomes is known.
- If not known $\rightarrow$ overbetting $\rightarrow$ ruin.
- Avoid ruin by adjustment of portfolio using Fixed Fraction, Drawdown constraint, DRO...
- Can a good strategy save a bad model?(No) Can it significantly improve a reasonable one?(Yes)

Future Work

- End to end strategies.
- Focus on profit, not accuracy.

Pictures



6 players playing russian roulette vs one player playing russian roulette 6 times.