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Original Article

Ruin probability at a given time for a model with liabilities of the fractional Brownian motion type: A partial differential equation approach

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In this paper we study the ruin probability at a given time for liabilities of diffusion type, driven by fractional Brownian motion with Hurst exponent in the range (0.5, 1). Using fractional Itô calculus we derive a partial differential equation the solution of which provides the ruin probability. An analytical solution is found for this equation and the results obtained by this approach are compared with the results obtained by Monte-Carlo simulation.

Keywords: Ruin probability; Fractional Brownian motion; Fractional Itô calculus; Partial differential equations

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1. Introduction

Fractional Brownian motion is used to model a wide variety of stochastic data arising in engineering and physics (network traffic data, solar activity, levels of a river, turbulence in an incompressible fluid flow see e.g. [1]) as well as in financial mathematics (log returns of the stock prices, see e.g. [1–4], the electricity price in a liberated electricity market, see e.g. [5], foreign exhange rates, see e.g. [6] and weather derivatives [7] and references therein). Furthermore, fractional Brownian motion (as a special case of self similar process) has been used recently to model the claims an insurance business may face (see eg. [8–10] etc).

The aim of this paper is to model the liabilities of an insurance business as a fractional Brownian motion and study the ruin probability of the firm under the influence of interest

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force. This problem is interesting from the point of view of applications but presents also considerable theoretical interest. There is recent work on this problem by several authors (see e.g. [8–10], etc.). Most of these works, to the best of our knowledge, deal with the asymptotic properties of ruin probability, using probabilistic techniques and provide upper and lower bounds for the ruin probability in certain limiting situations. For instance, Michna [8, 9] investigates ruin probabilities and first passage times for self-similar processes. He proposes self-similar processes as a risk model with claims appearing in good and bad periods. Then, he gets the fractional Brownian motion with drift as a limit risk process. Some bounds and asymptotics for ruin probability on a finite interval for fractional Brownian motion are derived. Husler and Piterbarg [11] considered the extreme values of fractional Brownian motions, self-similar Gaussian processes and more general Gaussian processes which have a trend $-ct^{\beta}$ for some constants $c, \beta > 0$ and a variance t^{2H}. They derive the tail behaviour of these extremes and show that they occur in the neighborhood of the unique point t_0 where the related boundary function $(u+ct^{\beta})/t^{H}$ is minimal. They consider the case of $H < \beta$. Debicki [10] considered the important role that Pickand constants play in the exact asymptotic of extreme values for Gaussian stochastic processes. The generalized Pickands constant H_n, defined as

$$H_n = \lim_{T\to\infty} \frac{H_n(T)}{T},$$

where

$$H_n = E[\exp(\max_{t\in[0,T]} \sqrt{2}\eta(t) - \sigma^2(t))]$$

and $\eta(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_\eta^2(t)$. Under some mild conditions on $\sigma_\eta^2(t)$ Debicki proves that H_n is well defined and he gives a comparison criterion for the generalised Pickand constants. Moreover he proves a theorem that extends the result of Pickands for certain stationary Gaussian processes. As an application he obtain the exact asymptotic behavior of $\psi(u) = P(\sup_{t\geq 0} \varsigma(t) - ct > u)$ as $u \to \infty$, for a class of integrated Gaussian processes that are important in the fluid model theory. For some bounds and estimators of H_a one can see [12].

The approach we adopt in this work for the treatment of ruin probabilities in models where the claims may present long range dependence is very different from the approach adopted in the above works. In this paper we propose a model for an insurance business facing liabilities presenting long term correlations. The long term correlations are modelled with the use of a fractional Brownian motion with Hurst exponent H. The insurance firm invests in an interest account which is assumed to be deterministic. It is shown that the cash balance process of the firm satisfies an Ornstein-Uhlenbeck stochastic differential equation driven by fractional Brownian motion. Using the recently developed tools of fractional stochastic calculus we show that the probability of ruin at a given date of the firm can be expressed as the solution of a linear parabolic partial differential equation. We have solved this partial differential equation analytically and we provide an exact expression for this quantity in terms of error functions, valid for all times. Using this exact expression one may derive asymptotic results using standard techniques. Finally, the partial differential equation allows an efficient numerical treatment of the problem which may be used as an alternative to Monte-Carlo type simulations. This quantity provides the probability that the cash balance of the firm goes negative at time t provided a given initial cash balance of the firm, i.e. $P[X_t < 0 \mid X_0 = x]$. The ruin probability at a given date is a very interesting quantity for the regulating authority of the company as it provides a lower bound to the quantity traditionally called ruin probability, given by $P[\inf_{s \in [0,T]} X_s < 0 \mid X_0 = x]$. Our model and treatment is inspired by a very interesting model proposed by Norberg [13] for the study of ruin probability in a model with diffusive type liabilities (Brownian motion type), with the use of partial differential equations. In some sense our treatment is an extension of Norberg's model to the case of fractional Brownian motion type liabilities. This extension is by no means trivial since the inclusion of fractional Brownian motion in the model presents difficulties which need different mathematical techniques in order to be overcome.

2. A short introduction to fractional stochastic calculus

In this section we review some fundamental results in fractional stochastic calculus for the convenience of the reader. The approach is based on the approach in [14].

The fractional Brownian motion (FBM) is a self affine stochastic process displaying long term correlation. FBM is characterized by the Hurst exponent H. Let us denote by W_t^H the FBM with Hurst exponent $H \in (0.5, 1)$. The process W_t^H is defined with respect to some probability space $(\Omega, \mathscr{F}, P^H)$, has continuous sample paths, is a zero-mean Gaussian random variable for all $t \geq 0$ and has autocorrelation function

$$E[W_t^H W_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for all $t, s \geq 0$, where by E we denote the expectation with respect to the probability measure P^H. For $H = 1/2$ we recover the usual Brownian motion. For $H > 1/2$ the FBM has a long range dependence.

Fractional Brownian motion does not have the 'nice' properties of Brownian motion. In particular, it is not a Markov process and it is not a semi-martingale. This presents problems in the definition of a stochastic integral and a stochastic calculus with respect to fractional Brownian motion, as we may not apply the standard theory of stochastic integration over a semimartingale to define a stochastic integral over fractional Brownian motion. Several approaches to this subject has been proposed (see e.g. [15,16] etc). We will adopt here the theory of Duncan, Hu and Pasik-Duncan [14] in which a stochastic integral over fractional Brownian motion of Hurst exponent $1/2 < H < 1$ has been defined, having some properties that have similarities with the corresponding properties of the stochastic integral over the usual Brownian motion.

We summarize here the basic results of [14] that we will use in this paper. In the following we assume that $1/2 < H < 1$. The stochastic integral $\int_0^t f_s \, dW_s^H$ over deterministic functions f is defined easily to provide a zero mean, Gaussian random variable with variance $\int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt$ where $\phi(s,t) = H(2H-1)\,|\,s-t\,|^{2H-2}$. The stochastic integral $\int_0^t F_s \, dW_s^H$ can be defined over stochastic processes F as the limit

$$\int_0^t F_s \, dW_s^H = \lim_{\Delta \to 0} \sum_{i=0}^{n-1} F_{t_i} \diamond (W_{t_{i+1}}^H - W_{t_i}^H)$$

where $\{t_i\}$ is some partition of the interval $(0, t)$, $\Delta = \sup_i |\, t_{i+1} - t_i \,|$. By $\diamond$ we denote the Wick product which is defined by

$$\varepsilon(f) \diamond \varepsilon(g) = \varepsilon(f+g)$$

where

$$\varepsilon(f) := \exp\left\{ \int_0^\infty f_t dW_t^H - \frac{1}{2} \int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt \right\}$$

is the stochastic exponential of the deterministic function f which is such that $|\int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt\,| < \infty$.

Duncan *et al.* [14] provide the following generalization of Itô's lemma in the case of fractional Brownian motion. For a proof of this result and generalizations to more complicated integrands, we refer to [14].

Proposition 2.1. *Let $\eta_t = \int_0^t a_s dW_s^H$ where a_t is some deterministic function such that $|\int_0^\infty \int_0^\infty a_s a_t \phi(s,t) ds dt\,| < \infty$. Let $f \in C^{1,2}$ and assume that $\frac{\partial f}{\partial x}(s, \eta_s) a_s \in \mathcal{L}(0,T)$. Then,*

$$f(t,\eta_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,\eta_s) ds + \int_0^t \frac{\partial f}{\partial x}(s,\eta_s) a_s dW_s^H + \int_0^t \frac{\partial^2 f}{\partial x^2}(s,\eta_s) a_s \int_0^s \phi(s,v) a_v dv ds, \quad a.s.$$

3. The model

Following the spirit of the original model proposed for liabilities of the Brownian motion type by Norberg [13] let us consider the following model for an insurance firm: The firm is characterized by its value at time t, which is assumed to be a stochastic process X_t. The firm invests its value X_t to an interest account with logarithmic interest force δ_t. The interest is assumed to be a deterministic function of time. This assumption is not unreasonable for models valid for short times. The firm has to face liabilities B_t. Assuming an insurance portfolio that is made up of a large number of individual risks, none of which is large enough to affect the total result significantly we approximate the liabilities or the payment function B_t by a fractional Brownian motion with drift

$$dB_t = -b_t dt - \sigma_t dW_t^H$$

where b_t represents the expected gain per time unit due to a safety loading in the premium, and σ_t is the standard deviation of the payments per time unit and is thus a measure of the size of the liability risk. In the above W_t^H is a fractional Brownian motion with Hurst exponent H and b_t, σ_t are deterministic functions of time. Allowing b_t, σ_t to be given functions of time we allow for seasonality in the claims. This seasonality is relevant in a number of models, for instance road accidents are more likely to happen during holiday periods, fires which may lead to property damage are more likely to happen during the hot months of the summer etc. The introduction of the fractional Brownian motion allows for the modeling of correlations in the claims. We consider only the case where $H > 1/2$ which corresponds to positive correlation between the claims. Such models may be relevant in models of claims related to health, disability insurance, accident or whole life insurance. The case $H < 1/2$ will correspond to negative correlation between claims. An example of a risk process with long range dependence was developed by Michna [8], who constructed a risk model in which claims appear in good and bad periods (e.g. good weather and bad weather), and under the assumption that the claims in bad periods are bigger than the claims of the good periods.

Following [13] the cash balance equation for the firm at time t has the following form

$$X_t = e^{\Delta_t}\left(X_0 - \int_0^t e^{-\Delta_s} dB_s\right) \tag{1}$$

where

$$\Delta_t = \int_0^t \delta_s \, ds$$

The cash-balance process is the solution of a stochastic differential equation, driven by fractional Brownian motion. We have the following proposition:

PROPOSITION 3.1. *The cash-balance process X_t given by the book-keeping equation* (*1*) *is the solution of the fractional Ornstein-Uhlenbeck equation*

$$dX_t = (\delta_t X_t + b_t)dt + \sigma \, dW_t^H, \tag{2}$$

$$X_0 = x \tag{3}$$

Proof. Define the process $K_t = \exp(\int_0^t \delta_s ds)$. Then we may rewrite (1) as

$$X_t = xK_t + K_t \int_0^t K_s^{-1} b_s ds + K_t \int_0^t K_s^{-1} \sigma_s dW_s^H$$

Let us further define the stochastic process $\eta_s = \int_0^t \sigma_s K_s^{-1} dW_s^H$ and the function

$$f(t, \eta) = xK_t + K_t \int_0^t K_s^{-1} b_s ds + K_t \eta$$

We see that $f(t, \eta_t) = X_t$.

We now apply the fractional Itô lemma on the function $f(t, \eta)$. We have that

$$\frac{\partial f}{\partial t}(t, \eta) = \delta_t f(t, \eta) + b_t$$

$$\frac{\partial f}{\partial \eta}(t,\eta) = K_t$$

$$\frac{\partial^2 f}{\partial \eta^2}(t,\eta) = 0$$

A straightforward application of Proposition 2.1 yields

$$f(t,\eta_t) = f(0,0) + \int_0^t (\delta_s f(s,\eta_s) + b_s)ds + \int_0^t \sigma_s dW_s^H$$

or equivalently

$$X_t = x + \int_0^t (\delta_s X_s + b_s)ds + \int_0^t \sigma_s dW_s^H$$

This concludes the proof. □

We may also state the following:

PROPOSITION 3.2. *The cash-balance process is a Gaussian process with mean*

$$m_t = xK_t + K_t \int_0^t b_s K_s^{-1} ds$$

and variance

$$V_t = K_t^2 \int_0^t \int_0^s \sigma_u \sigma_s K_u^{-1} K_s^{-1} \phi(u,s) du ds$$

where

$$K_t = \exp\left(\int_0^t \delta_s ds\right)$$

and

$$\phi(u,s) = H(2H-1)\,|\,u-s\,|^{2H-2}$$

Proof. The proof follows using the properties of the stochastic integral over fractional Brownian motion. □

The mean and the variance can be computed using special functions for the particular case of constant parameters, $\delta_s = \delta$, $b_s = b$.

4. A partial differential equation for the ruin probability at a given date

We are interested in the derivation of the quantity $P(X_t \leq 0 \mid X_0 = x)$. This quantity provides the probability that the cash balance of the firm goes negative at time t provided the initial cash balance of the firm is x. Thus it corresponds to the ruin probability of the firm at the given date t and thus using the terminology of [13] we call it the ruin probability at a given date.

In a number of studies, the quantity $P[\inf_{s\in[0,T]}X_s < 0 \mid X_0 = x]$ is proposed as a measure for the ruin probability. The ruin probability we calculate in our model may serve as a lower bound for this quantity and may thus serve as an alert to the regulating authority of the company. We also think that the quantity we use is more interesting as it allows us to monitor the positivity of the cash balance of the insurance business at any given time rather than checking the fact of ruin at any time instant within a given time interval. In [13] a partial differential equation was obtained for this quantity as well using the Markovian property of the Brownian motion driving the liabilities. However, in the model with fractional Brownian motion, the Markovian property is no longer valid and we do not expect similar results to hold. We content here to perform a numerical evaluation of $P[\inf_{s\in[0,T]}X_s < 0 \mid X_0 = x]$ using Monte-Carlo simulation (see section 6.3).

We will show in this section that the ruin probability at a given date can be determined by the (classical) solution of a linear parabolic partial differential equation. The analysis follows the lines of Brody, Syroka and Zervos [7] where the value of a weather derivative whose underlying (the temperature) is modelled by a fractional Brownian motion is expressed through the use of a partial differential equation. We restrict ourselves to the case where $H > 0.5$.

We have the following proposition

PROPOSITION 4.1. *Assume that $H > 0.5$ and σ_s has no singularities. Then the ruin probability $u(t,x) := P(X_t \le 0 \mid X_0 = x)$ satisfies the following parabolic partial differential equation (Cauchy problem)*

$$-\frac{\partial u}{\partial \tau} + (\delta_{t-\tau}x + b_{t-\tau})\frac{\partial u}{\partial x} + K_{t-\tau}\sigma_{t-\tau}\left(\int_0^{t-\tau}\phi(s,t-\tau)\sigma_s K_s^{-1}ds\right)\frac{\partial^2 u}{\partial x^2} = 0$$

$$u(0,x) = \mathbf{1}_{\{x\le 0\}}$$

REMARK 4.1. It is useful to make a comment on the meaning and use of the above equation. Since the equation depends on the parameter t, the solution of the equation is a function $u(\tau,x) = u(\tau,x;t)$. The ruin probability at time t, given that the initial capital is x is the solution of this equation calculated at $\tau = t$, i.e. $u(t,x) = u(t,x;t)$. That means that fixing t we have to solve the equation for $u(\tau,x) = u(\tau,x;t)$ and then take the limit as $\tau \to t$.

Proof of Proposition 4.1. Consider the function $g(\tau_0,\eta;t) = w(t-\tau_0, f(\tau_0,\eta))$ where

$$f(\tau_0,\eta) = xK_{\tau_0} + K_{\tau_0}\int_0^{\tau_0} K_s^{-1}\beta_s\,ds + K_{\tau_0}\eta.$$

We now apply the fractional Itô formula to $g(\tau_0,\eta_{\tau_0};t)$ for τ_0 taking values between $\tau_0 = 0$ and $\tau = t$. Note that t is considered as a fixed parameter while τ_0, η_{τ_0} are considered as variables.

Since

$$\frac{\partial g}{\partial \tau_0} = -\frac{\partial w}{\partial \tau} + (\delta_{\tau_0} f(\tau_0, \eta) + b_{\tau_0}) \frac{\partial w}{\partial x}$$

$$\frac{\partial g}{\partial \eta} = \frac{\partial w}{\partial x} K_{\tau_0}$$

$$\frac{\partial^2 g}{\partial \eta^2} = \frac{\partial^2 w}{\partial x^2} K_{\tau_0}^2,$$

where we consider $w = w(\tau, x)$, with $\tau = t - \tau_0$, we see that

$$g(t, \eta_t) = g(0,0) + \int_0^t \frac{\partial g}{\partial \tau_0}(\tau_0, \eta_{\tau_0}) d\tau_0 + \int_0^t \frac{\partial g}{\partial \eta}(\tau_0, \eta_{\tau_0}) \sigma_{\tau_0} K_{\tau_0}^{-1} dW_{\tau_0}^H$$
$$+ \int_0^t \frac{\partial^2 g}{\partial \eta^2}(\tau_0, \eta_{\tau_0}) \sigma_{\tau_0} K_{\tau_0}^{-1} \left(\int_0^{\tau_0} \phi(s, \tau_0) \sigma_s K_s^{-1} ds \right) d\tau_0 \tag{4}$$

where

$$\phi(s,t) = H(2H-1) \,|\, s-t \,|^{2H-2}.$$

We observe that

$$g(t, \eta_t) = w(0, X_t), \quad g(0, x) = w(t, x)$$

Thus, equation (4) assumes the form

$$w(0, X_t) = w(t, x) + \int_0^t \left(-\frac{\partial w}{\partial \tau} + (\delta_{\tau_0} f + b_{\tau_0}) \frac{\partial w}{\partial x} + K_{\tau_0} \sigma_{\tau_0} \right.$$
$$\left. \times \left(\int_0^{\tau_0} \phi(s, \tau_0) \sigma_s K_s^{-1} ds \right) \frac{\partial^2 w}{\partial x^2} \right) d\tau_0 + \int_0^t \sigma_{\tau_0} \frac{\partial w}{\partial x} dW_{\tau_0}^H$$

We now take expectations and use the properties of the stochastic integral to obtain

$$E[w(0, X_t)] = w(t, x) + E\left[\int_0^t \left(-\frac{\partial w}{\partial \tau} + (\delta_{\tau_0} f + b_{\tau_0}) \frac{\partial w}{\partial x} \right. \right.$$
$$\left. \left. + K_{\tau_0} \sigma_{\tau_0} \left(\int_0^{\tau_0} \phi(s, \tau_0) \sigma_s K_s^{-1} ds \right) \frac{\partial^2 w}{\partial x^2} \right) d\tau_0 \right] \tag{5}$$

We rephrase the ruin probability at a given date as

$$P(X_t \leq 0 \,|\, x) = u(t, x) = E[\mathbf{1}_{\{X_t \leq 0\}}]$$

and add this to equation (5) to obtain

$$E[w(0, X_t)] + u(t, x) = w(t, x) + E[\mathbf{1}_{\{X_t \leq 0\}}] + E\left[\int_0^t \left(-\frac{\partial w}{\partial \tau} + (\delta_{\tau_0} f + b_{\tau_0}) \frac{\partial w}{\partial x} \right. \right.$$
$$\left. \left. + K_{\tau_0} \sigma_{\tau_0} \left(\int_0^{\tau_0} \phi(s, \tau_0) \sigma_s K_s^{-1} ds \right) \frac{\partial^2 w}{\partial x^2} \right) d\tau_0 \right] \tag{6}$$

where inside the integrals $w := w(t - \tau_0, X_{\tau_0}) = w(\tau, X_{\tau_0})$. If we choose w to be the solution of the PDE

$$-\frac{\partial w}{\partial \tau}+(\delta_{\tau_0} f+b_{\tau_0})\frac{\partial w}{\partial x}+K_{\tau_0}\sigma_{\tau_0}\left(\int_0^{\tau_0}\phi(s,\tau_0)\sigma_s K_s^{-1}ds\right)\frac{\partial^2 w}{\partial x^2}=0$$

with $w(0,x)=\mathbf{1}_{\{x\leq 0\}}$ we see that $u(t,x)=w(t,x)$. Observing that the coefficients of the equation are calculated in τ_0 whereas w is calculated in $\tau := t-\tau_0$ we may redefine time so as to express this equation in the equivalent form

$$-\frac{\partial w}{\partial \tau}+(\delta_{t-\tau} x+b_{t-\tau})\frac{\partial w}{\partial x}+K_{t-\tau}\sigma_{t-\tau}\left(\int_0^{t-\tau}\phi(s,t-\tau)\sigma_s K_s^{-1}ds\right)\frac{\partial^2 w}{\partial x^2}=0$$

$$w(0,x)=\mathbf{1}_{\{x\leq 0\}}$$

This concludes the proof of the proposition. □

We may further obtain a PDE for the computation of the ruin probability at a given date $u(t,y;s,x) := P(X_t \leq y \mid X_s = x)$. We have the following proposition

Proposition 4.2. *Under the same assumptions as in proposition 4.1 the ruin probability* $u(t,x;s,x) := P(X_t \leq y | X_s = x)$ *satisfies the following parabolic partial differential equation* (*Cauchy problem*)

$$-\frac{\partial w}{\partial \tau}+(\delta_{t+s-\tau} x+b_{t+s-\tau})\frac{\partial w}{\partial x}+K_{t+s-\tau}\sigma_{t+s-\tau}\left(\int_s^{t+s-\tau}\phi(s',t+s-\tau)\sigma_{s'} K_{s'}^{-1}ds'\right)\frac{\partial^2 w}{\partial x^2}=0$$

$$w(s,x)=\mathbf{1}_{\{x\leq y\}}$$

in the sense that $u(t,x;s,y)=w(t,x)$.

Proof. We may show that

$$X_t=\bar{K}_{s,t}x+K_t\left[\int_s^t K_{s'}^{-1}b_{s'}ds'+\int_s^t K_{s'}^{-1}\sigma_{s'}dW_{s'}^H\right]$$

where $\bar{K}_{s,t}=\exp(\int_s^t \delta_{s'}ds')$. We now apply the fractional Itô formula to the function $g(\tau,\eta_\tau)=w(t+s-\tau, f(\tau,\eta_\tau))$ where

$$f(\tau,\eta)=x\bar{K}_{s,\tau}+K_\tau\int_s^t K_{s'}^{-1}b_{s'}ds'+K_\tau\eta_\tau$$

$$\eta_\tau=\int_s^t K_{s'}^{-1}\sigma_{s'}dW_{s'}^H$$

The rest follows as in the proof of proposition 4.1. □

5. Solution of the PDE

We now deal with the solution of the PDE for the ruin probability at a given date.

5.1. An analytical solution

The PDE for the ruin probability at a given date can be solved analytically in its most general form. This facilitates immensely the calculation of the ruin probability.

We start our presentation of the analytical solution of the PDE for the ruin probability at a given date in the case where the coefficients δ_t, σ_t and b_t are constants. This facilitates the arguments. Then we provide the solution for the general case of time dependent coefficients.

In the case of constant coefficients the time dependent factor multiplying the second derivative term becomes

$$\begin{aligned} f(t) &= e^{\delta t}\sigma^2 H(2H-1)\int_0^t |t-s|^{2H-2}e^{-\delta s}ds \\ &= \sigma^2 H(2H-1)\int_0^t s^{2H-2}e^{\delta s}ds \\ &= \sigma^2 H(2H-1)\sum_{n=0}^{\infty}\frac{\delta^n}{n!}\frac{t^{2H-1+n}}{2H-1+n} \\ &= \sigma^2 H(2H-1)(-1)^{2H-1}\delta^{1-2H}\gamma(2H-1,-\delta t) \end{aligned} \tag{7}$$

where $\gamma(z, a)$ is the incomplete gamma function (see for instance [17]). In this case we may obtain an expression for the ruin probability at a given date in terms of the complementary error function. We have the following proposition.

PROPOSITION 5.1. *In the case where δ_t, σ_t, b_t are constants the ruin probability at a given date may be expressed as*

$$P(X_t \leq 0 \mid X_0 = x) = \frac{1}{\sqrt{\pi}}\operatorname{erfc}(k(t,x))$$

where

$$k(t,x) = \frac{e^{\delta t}(\delta x + b) - b}{\sqrt{2T(t)}}$$

with

$$T(t) = 2\delta^2\int_0^t f(t-t_1')e^{2\delta t_1'}dt_1'.$$

Proof. We proceed to the solution of the PDE

$$-\frac{\partial w}{\partial t_1} + (\delta x + b)\frac{\partial w}{\partial x} + f(t-t_1)\frac{\partial^2 w}{\partial x^2} = 0$$

with initial condition $w(0, x) = \mathbf{1}_{\{x \leq 0\}}$. We will use the change of variables $(t, x) \to (T, X)$ where

$$\begin{cases} X = e^{\delta t_1}(\delta x + b) \\ T = t_1 \end{cases}$$

Since

$$\frac{\partial}{\partial x} = \delta e^{\delta t}\frac{\partial}{\partial X} = \delta e^{\delta T}\frac{\partial}{\partial X},$$

$$\frac{\partial^2}{\partial x^2} = \delta^2 e^{2\delta t}\frac{\partial^2}{\partial X^2} = \delta^2 e^{2\delta T}\frac{\partial^2}{\partial X^2},$$

$$\frac{\partial}{\partial t_1} = \delta e^{\delta t}(\delta x + b)\frac{\partial}{\partial X} + \frac{\partial}{\partial T} = \delta X\frac{\partial}{\partial X} + \frac{\partial}{\partial T}$$

the PDE becomes in the new variables

$$-\frac{\partial w}{\partial T} + \delta^2 f(T)e^{2\delta T}\frac{\partial^2 w}{\partial X^2} = 0$$

By further defining the new set of variables

$$\begin{cases} X' = X \\ T' = 2\delta^2\int_0^T e^{2\delta t} f(t - t_1)dt_1 \end{cases}$$

we see that the PDE assumes the form of the heat equation

$$-\frac{\partial w}{\partial T'} + \frac{1}{2}\frac{\partial^2 w}{\partial X'^2} = 0$$

with initial condition $w(0, X') = \mathbf{1}_{\{X' \leq b\}}$. This can be solved using the Green's function (heat kernel) for the diffusion equation $G(X' - Y, T')$. The solution is given by the integral formula

$$w(T', X') = \int_{-\infty}^{\infty} G(X' - Y, T')w(0, Y)dY$$

where

$$G(X' - Y, T') = \frac{1}{\sqrt{2\pi T'}}\exp\left(-\frac{(X' - Y)^2}{2T'}\right)$$

and $w(0, Y) = \mathbf{1}_{\{Y \leq b\}}$. Using the integral formula

$$w(T', X') = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T'}}\exp\left(-\frac{(X' - Y)^2}{2T'}\right)dY$$

The last integral may be expressed in terms of the complementary error function as follows

$$w(T', X') = \frac{1}{\sqrt{\pi}}\operatorname{erfc}\left(\frac{X' - b}{\sqrt{2T'}}\right)$$

and returning to the original variables we may find that

$$w(t_1, x) = \frac{1}{\sqrt{\pi}}\operatorname{erfc}(k(t_1, x)),$$

$$k(t_1, x) = \frac{e^{\delta t_1}(\delta x + b) - b}{\sqrt{2T'}}$$

$$T' = 2\delta^2 \int_0^{t_1} f(t - t_1') e^{2\delta t_1'} dt_1'$$

The ruin probability at a given date is obtained setting $t_1 = t$ in the above expression. This completes the proof. □

Remark 5.1. The new variable T' may be expressed as a function of t_1 in the form of series using the expression

$$T' = 2\delta^2\sigma^2 H(2H-1)\sum_{n,m=0}^{\infty}\frac{2^m\delta^{n+m}}{m!n!(n+2H-1)}t^{n+m+2H}B_{t_1/t}(m+1, n+2H)$$

where by $B_x(\alpha, \beta)$ we denote the incomplete Beta function

$$B_x(\alpha,\beta) = \int_0^x s^{\alpha-1}(1-s)^{\beta-1}ds$$

Setting $t = t_1$ in the above series we may obtain a series expression for $T(t)$ of the form

$$T(t) = 2\delta^2\sigma^2 H(2H-1)\sum_{n,m=0}^{\infty}\frac{2^m\delta^{n+m}}{m!n!(n+2H-1)}t^{n+m+2H}B(m+1, n+2H)$$

where by $B(\alpha, \beta)$ we denote the complete Beta function

$$B(\alpha,\beta) = \int_0^x s^{\alpha-1}(1-s)^{\beta-1}ds$$

We now give the solution of the PDE of the ruin probability at a given date in the general case of time dependent coefficients:

Proposition 5.2. *The solution of the PDE of the ruin probability at a given date in the general case is given in the form*

$$u(t,x) = \int_{-\infty}^{0}\frac{1}{\sqrt{2T'}}\exp\left(-\frac{(X'-Y)^2}{2T'}\right)dY = \frac{1}{\sqrt{\pi}}\text{erfc}\left(\frac{X'}{\sqrt{2T'}}\right)$$

where

$$X' = \exp\left(\int_0^t \delta_{t-s}ds\right)x + \int_0^t b_{t-s}\exp\left(\int_0^s \delta_{t-s'}ds'\right)ds$$

$$T' = \int_0^t \exp\left(\int_0^{t_1} 2\delta_{t-s}ds\right)f(t-t_1)dt_1$$

$$f(t-t_1) = K_{t-t_1}\sigma_{t-t_1}\left(\int_0^{t-t_1}\phi(s, t-t_1)\sigma_s K_s^{-1}ds\right) \tag{8}$$

Proof. As before we seek for a new set of variables in which the PDE assumes the form of the heat equation. To this end we try the new set of variables

$$\begin{cases} X = f_1(t_1)x + f_2(t_1) \\ T = t_1 \end{cases}$$

where f_1 and f_2 are functions to be specified. Choosing f_1 and f_2 to be the solutions of the differential equations

$$-\frac{df_1}{dt_1}(t_1) + \delta_{t-t_1} f_1(t_1) = 0$$

$$-\frac{df_2}{dt_1} + b_{t-t_1} f_1(t_1) = 0$$

we see that in the new coordinates the equation becomes

$$-\frac{\partial u}{\partial T} + f_1(t_1)^2 f(t-t_1) \frac{\partial^2 u}{\partial X^2} = 0 \tag{9}$$

where f_1 and f_2 can be readily found from the solution of the above ODEs as

$$f_1(t_1) = \exp\left(\int_0^{t_1} \delta_{t-s} ds\right)$$

$$f_2(t_1) = \int_0^{t_1} b_{t-s} \exp\left(\int_0^{s} \delta_{t-s'} ds'\right) ds$$

Equation (9) can be reduced to a diffusion equation of the form

$$-\frac{\partial u}{\partial T'} + \frac{1}{2} \frac{\partial^2 u}{\partial X'^2} = 0$$

through a further change of variables

$$\begin{cases} X' = X \\ T' = 2\int_0^T f_1(t_1)^2 f(t-t_1)\, dt_1 \end{cases}$$

The solution of this equation can be given in terms of the Green's function (heat kernel) for the diffusion equation in a way analogous to the constant coefficient case. □

REMARK 5.2. Note that we use Lebedev's [17] convention for the complementary error function

$$\mathrm{erfc}(x) = \int_x^\infty e^{-z^2} dz$$

To avoid confusion note that software packages such as e.g. Mathematica or Matlab use a slightly different definition. These two are related by a simple scaling factor of $\frac{2}{\sqrt{\pi}}$.

5.2. Asymptotics

Using the well known asymptotic expansions for the error function (see e.g. [17]) we may obtain asymptotics for the probability of ruin at a given date for various limiting cases of interest.

One particularly interesting case is the limit of large initial capital $x \to \infty$. In the constant coefficients case for example we have

$$P(X_t \leq 0 \mid X_0 = x) \simeq \frac{1}{\sqrt{\pi}} \exp(-k(t,x)^2) \left[\frac{1}{2k(t,x)} - \frac{1}{2^2 k(t,x)^3} + \cdots \right]$$

From that we see that the ruin probability at a given date decreases as $\exp(-\lambda x^2)$ for large x, for some properly chosen constant λ. This is in accordance with the results obtained in [13] for the Brownian motion case.

Of interest are also the large time asymptotics. The case of general H is complicated to handle (due to the complicated form of the integral defining T'), but some insight can be obtained by studying the special cases $H = 1/2$ and $H = 1$ (see next section).

Finally of interest is the asymptotic formulae for the ruin probability at a given date as the interest force tends to 0 ($\delta \to 0$). When $\frac{1}{2} < H < 1$ and $\delta \to 0$ we have from the asymptotics of the general solution that

$$\lim_{\delta \to 0} u(t,x) = \frac{1}{2} \left[1 - \Phi\left(\frac{tb + x}{t^H \sigma \sqrt{2}} \right) \right]$$

5.3. Two special cases

We now provide results for two special values for the Hurst parameter H. We will only consider the constant coefficient case.

5.3.1. The case of Brownian motion ($H = 1/2$). In the case $H = 1/2$ the only term in the series 7 that survives is the term corresponding to $n = 0$. This gives $f(t) = \frac{\sigma^2}{2}$ which is a constant.

Then the PDE for the ruin probability at a given date becomes

$$-\frac{\partial w}{\partial t_1} + (\delta x + b)\frac{\partial w}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} = 0$$

$$w(0,x) = \mathbf{1}_{\{x \leq 0\}}$$

The ruin probability at a given date $u(t,x) = w(t,x)$. This is the same equation as the one derived by Norberg [13] for the case of Brownian motion driven liabilities.

Using the consecutive transformations

$$\begin{cases} X = e^{\delta t_1}(\delta x + b) \\ T = t_1 \end{cases}$$

and

$$\begin{cases} X' = X \\ T' = \int_0^T \delta^2 \sigma^2 e^{2\delta t} dt = \frac{\delta \sigma^2}{2}(e^{2\delta T} - 1) \end{cases}$$

we see that the equation transforms to

$$-\frac{\partial w}{\partial T'} + \frac{1}{2}\frac{\partial^2 w}{\partial X'^2} = 0$$

The initial condition is

$$w(t_1 = 0, x) = \mathbf{1}_{\{x \leq 0\}}$$

which translates to

$$w\left(T' = 0, \frac{X - b}{\delta}\right) = \mathbf{1}_{\left\{\frac{X-b}{\delta} \leq 0\right\}} = \mathbf{1}_{\{X \leq b\}}$$

The general solution to this equation is

$$w(X', T') = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi T'}} \exp\left(-\frac{(X' - Y)^2}{2T'}\right) \mathbf{1}_{\{Y \leq b\}} dY$$
$$= \frac{1}{\sqrt{\pi}} \int_k^{\infty} \exp(-z^2) dz = \frac{1}{\sqrt{\pi}} \operatorname{erfc}(k)$$

where

$$k = k(T', X') = \frac{X - b}{\sqrt{2T'}}$$

or in terms of the original coordinates

$$k = k(x, t) = \frac{1}{\sqrt{\delta}\sigma} \frac{e^{\delta t}(\delta x + b) - b}{\sqrt{e^{2\delta t} - 1}}$$

It is interesting to look at the limiting behaviour of the above formula.

- $t \to 0$. In this case

$$k(x, t) \simeq \frac{x}{\sigma\sqrt{t}}$$

  and

$$u(x, t) \simeq \frac{1}{\sqrt{\pi}} \operatorname{erfc}\left(\frac{x}{\sigma\sqrt{t}}\right)$$

  The limiting behaviour is different depending on whether x is positive or negative.
- $t \to \infty$. In this case

$$k(x, t) \simeq \frac{\sqrt{2}}{\sigma\sqrt{\delta}}(\delta x + b - be^{-2\delta t})$$

  so that

$$u(x, t) \simeq \frac{1}{\sqrt{\pi}} \operatorname{erfc}\left(\frac{\sqrt{2}}{\sigma\sqrt{\delta}}(\delta x + b - be^{-2\delta t})\right)$$

5.3.2. The case $H = 1$. In the case $H = 1$ we have

$$f(t) = e^{\delta t}\sigma^2 \int_0^t e^{-\delta s} ds = \frac{\sigma^2}{\delta}(e^{\delta t} - 1)$$

The equation for the ruin probability becomes

$$-\frac{\partial w}{\partial t_1} + (\delta x + b)\frac{\partial w}{\partial x} + \frac{\sigma^2}{\delta}(e^{\delta(t-t_1)} - 1)\frac{\partial^2 w}{\partial x^2}$$

and the ruin probability is $u(t, x) = w(t_1 = t, x)$.

We perform the consecutive change of variables

$$X = e^{\delta t_1}(\delta x + b)$$
$$T = t_1$$

and

$$X' = X$$
$$T' = 2\delta\sigma^2 \int_0^T (e^{\delta(t-T)} - 1)e^{2\delta T} dT = 2\sigma^2 \left\{ (e^{\delta T} - 1)e^{\delta t} - \frac{e^{2\delta T} - 1}{2} \right\}$$

In the new variables the equation becomes

$$-\frac{\partial w}{\partial T'} + \frac{1}{2}\frac{\partial^2 w}{\partial X'^2} = 0$$

with initial condition $w(t_1 = 0, x) = \mathbf{1}_{\{x \leq 0\}}$, or in the new variables

$$w(T' = 0, X') = \mathbf{1}_{\left\{\frac{X'-b}{\delta} \leq 0\right\}} = \mathbf{1}_{\{X' \leq b\}}.$$

Using the integral formula for the solution of the diffusion equation we find that

$$w(T', X') = \frac{1}{\sqrt{2T'}} \int_{-\infty}^{b} \exp\left(-\frac{(X - Y)^2}{2T'}\right) dY = \frac{1}{\sqrt{\pi}} \int_{\frac{X'-b}{\sqrt{2T'}}}^{\infty} \exp(-z^2) dz$$

or in terms of the original variables

$$w(t_1, x) = \frac{1}{\sqrt{\pi}} \int_{k(t_1,x)}^{\infty} \exp(-z^2) dz = \frac{1}{\sqrt{\pi}} \mathrm{erfc}(k(t_1, x))$$
$$k(t_1, x) = \frac{e^{\delta t_1}(\delta x + b) - b}{2\sigma\sqrt{(e^{\delta t_1} - 1)e^{\delta t} - \frac{(e^{2\delta t_1} - 1)}{2}}}$$

The ruin probability $u(t, x) = w(t, x)$ is obtained by setting $t_1 = t$ in the above formula. We thus find

$$u(t, x) = \frac{1}{\sqrt{\pi}} \mathrm{erfc}\left(\frac{e^{\delta t}(\delta x + b) - b}{2\sigma\sqrt{\frac{1}{2}e^{2\delta t} - e^{\delta t} + \frac{1}{2}}}\right) = \frac{1}{\sqrt{\pi}} \mathrm{erfc}\left(\frac{e^{\delta t}(\delta x + b) - b}{\sigma\sqrt{2}(e^{\delta t} - 1)}\right)$$

Two limiting cases are interesting.

- $t \to 0$. Then

$$k(t,x) \simeq \frac{x}{\sqrt{2}\sigma t}$$

and

$$u(t,x) \simeq \frac{1}{\sqrt{\pi}} \operatorname{erfc}\left(\frac{x}{\sqrt{2}\sigma t}\right)$$

- $t \to \infty$. Then

$$k(t,x) = \frac{1}{\sqrt{2}\sigma}(\delta x + b - be^{-\delta t})$$

and

$$u(t,x) \simeq \frac{1}{\sqrt{\pi}} \operatorname{erfc}\left(\frac{1}{\sqrt{2}\sigma}(\delta x + b - be^{-\delta t})\right)$$

Finally we present the asymptotics for zero interest force in the $H=1$ case. When $H=1$ and $\delta \to 0$ we have from the asymptotics of the general solution that

$$\lim_{\delta \to 0} u(t,x) = \frac{1}{2}\left[1 - \Phi\left(\frac{tb + x}{t\sigma\sqrt{2}}\right)\right]$$

6. Numerical treatment of the problem

In this section we propose some possible approaches to the numerical study of ruin probabilities for our insurance business model.

6.1. Monte Carlo method

As an alternative to the pde approach one can use the Monte Carlo method in order to find the probability of ruin $w(x,t)$. Furthermore since the Monte Carlo method is attacking the problem from a quite different point of view it can be used as an independent test of the validity of the pde approach proposed here.

As we have seen the cash-balance process X_t is given by the solution of equations (2) and (3). In order to implement the Monte Carlo method we simulate a large number M of paths of X_t in the time interval [0, T]. We use for each path $N=2^L$, number of points. Then the probability of ruin at time T can be found as:

$$w(x_0, T) = \frac{\text{number of } X_T \leq 0}{M}.$$

In order to simulate the paths of fractional Brownian motion, we have used the method of Wood and Chan [18]. Some other methods that could be used are the methods described in Mandelbrot *et al*. [19], Abry and Sellan [20], Norros *et al*. [21] (and the references therein).

6.2. Finite difference methods

As we have seen we have derived an analytical solution of the pde that governs the ruin probability at a given date. Instead of using the analytical solution we could alternatively use the finite difference method for the numerical resolution of the pde. We have implemented the full implicit finite difference method and the Crank–Nicolson finite difference method.

6.3. Numerical results

In the following tables we present some numerical results using the analytic expressions, the Monte Carlo method and the finite difference method. We consider the following values for the parameters of the model. For the interest force we have assumed that $\delta = 0.05$, for the volatility in claims liabilities we assume that $\sigma = 0.20$, for the expected gain per time unit due to a safety loading in the premium we assume that $b = 0.10$. The parameters used for the implementation of the various numerical schemes are included in the Appendix.

As an indication of the results obtained for the ruin probability using the various methods proposed we present tables comparing the estimates for the ruin probability at time T for different values of the initial capital $X_0 = 0, 0.5, -0.5$.

Table 1. Probability of ruin for initial capital $X_0 = 0$, at $T = 100$.

H	Exact	Monte Carlo	Implicit	Crank–Nicolson
0.5	0.00084174	0.000867	0.00084170	0.00084169
0.6	0.0132523	0.0131667	0.01310549	0.01319379
0.7	0.060585	0.0605333	0.06097709	0.06110057
0.8	0.141854	0.145800	0.14438584	0.14459275
0.9	0.231166	0.234100	0.23129410	0.23160310
1	0.308538	0.309100	0.30000028	0.30017402

Table 2. Probability of ruin for initial capital $X_0 = 0.5$, at $T = 100$.

H	Exact	Monte Carlo	Implicit	Crank–Nicolson
0.5	0.000042186	0.00003333	0.00004218	0.000042178
0.6	0.00274159	0.00316667	0.00270194	0.00272269
0.7	0.026191	0.02706667	0.02636487	0.02639296
0.8	0.0898221	0.09073300	0.09051296	0.09062079
0.9	0.178783	0.17716600	0.17352670	0.17379322
1	0.265707	0.26480000	0.24598212	0.24616236

Table 3. Probability of ruin for initial capital $X_0 = -0.5$, at $T = 00$.

H	Exact	Monte Carlo	Implicit	Crank–Nicolson
0.5	0.00937525	0.009133	0.00937540	0.00937526
0.6	0.048428	0.04943333	0.04807283	0.04830152
0.7	0.123069	0.12323333	0.12391473	0.12414554
0.8	0.211218	0.20900000	0.21659835	0.21685958
0.9	0.291155	0.29193330	0.29843905	0.29873668
1	0.354146	0.35490000	0.35870696	0.35885493

The following observation is of interest. For $H = 1$ and $x = 0$ the Monte-Carlo approach gives the following results for the ruin probability as a function of time:

Table 4. The probability of ruin for $H = 1$ at $x = 0$ for different times.

T	Exact	Monte Carlo
1	0.308538	0.3094333
10	0.308538	0.3106333
100	0.308538	0.309100

It is interesting to see that this behaviour, i.e. the fact that the ruin probability is independent with respect to variations in time is predicted by the exact analytical solution for $H = 1$. The case $H = 1$ is a limiting situation for which the results of this work are questionable since the theory of stochastic integration with respect to fractional Brownian motion used here is strictly valid for values of the Hurst index in the interval (0, 1). However, the fact that this behaviour is reproduced by the Monte Carlo simulation poses questions on the validity of the theory in the limit $H = 1$. This is a point which probably deserves further attention.

We now present graphically the dependence of probability of ruin with the Hurst index, the initial capital and time.

In figure 1 we present the variation of the ruin probability at $x = 0$ and time $T = 100$ with the Hurst exponent. We see that as H is taking bigger values the probability of ruin is also growing. This result indicates the effect of long time correlations in the probability of ruin for the insurance business.

In figure 2 we present the variation of the probability of ruin at time $T = 100$ as a function of the initial capital The probability of ruin decreases as the initial capital increases as is expected. The Hurst index is taken to be $H = 0.7$.

In figure 3 we present the variation of the probability of ruin with time for initial capital equal to $x = 0$ and $H = 0.7$. As time increases the probability of ruin decreases.

In all the above figures we have taken $\delta = 0.05$, $\sigma = 0.20$, $b = 0.10$.

As a final application of the simulation approach we present the calculation of a slightly different form of the ruin probability $P^* = P[\inf_{s \in [0,\, T]} X_s < 0 \mid X_0 = x]$. We have taken 10000 paths and 2^{14} points in each path, for initial capital $X_0 = 0$, 0.25, 0.5. The results are shown in table 5. We observe that for initial capital zero as H increases the probability of ruin in a finite time decreases. For initial capital 0.25, 0.50 we see that as H increases the

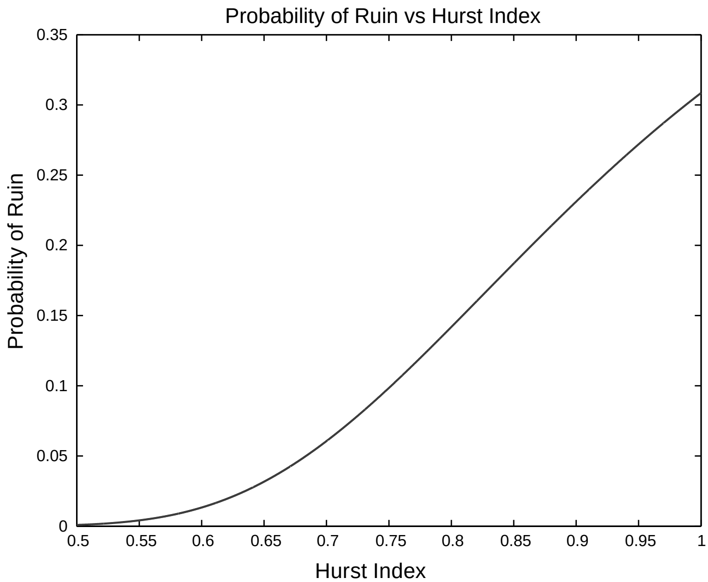

Figure 1. Probability of ruin at $x=0$ as a function of the Hurst exponent.

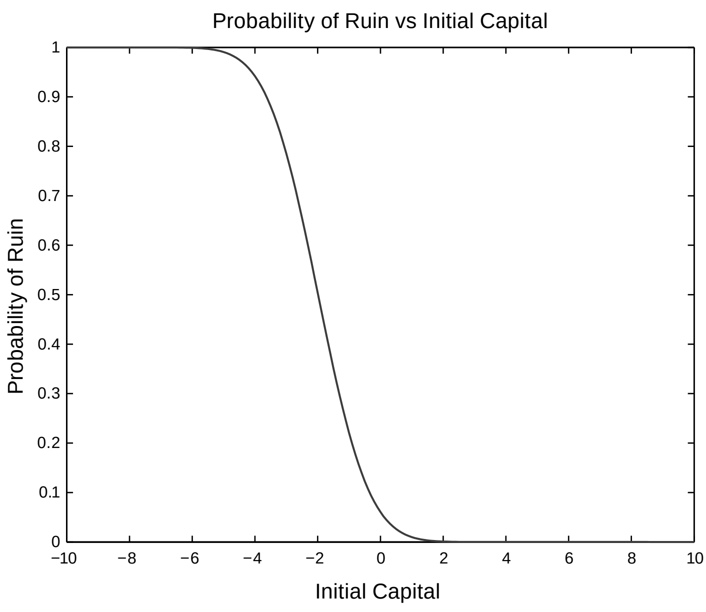

Figure 2. Probability of ruin as a function of the initial capital for $H=0.7$.

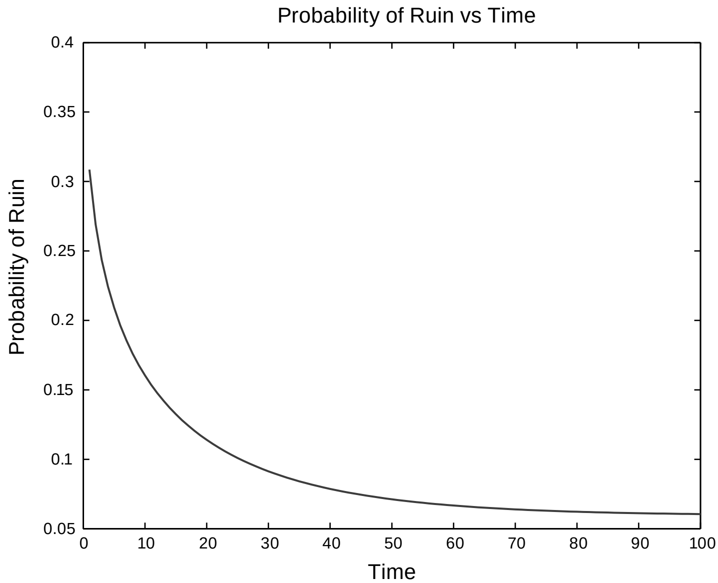

Figure 3. Probability of ruin as a function of time for $H = 0.7$.

probability of ruin in a finite time is also increasing in general. For the case of $H = 0.5$, we solved the pde derived by Norberg using an implicit finite difference scheme and we found that for $X_0 = 0.25$ $P^* = 0.239$, while with Monte Carlo we find 0.228, and for $X_0 = 0.50$ we find $P^* = 0.049$, while with Monte Carlo we find 0.047.

Table 5. The probability of ruin $P^* = P[\inf_{s \in [0, T]} X_s < 0 \mid X_0 = x]$ for different values of H and x.

H	X_0	P^*
0.5	0	0.941
0.6	0	0.915
0.7	0	0.871
0.8	0	0.798
0.9	0	0.671
1	0	0.310
0.5	0.25	0.228
0.6	0.25	0.232
0.7	0.25	0.263
0.8	0.25	0.292
0.9	0.25	0.314
1	0.25	0.285
0.5	0.50	0.047
0.6	0.50	0.078
0.7	0.50	0.126
0.8	0.50	0.189
0.9	0.50	0.244
1	0.50	0.287

7. Conclusions and extensions

In this paper we have derived a linear parabolic partial differential equation for the ruin probability at a given date of an insurance firm with long correlated claims, modelled by a fractional Brownian motion with Hurst exponent $1/2 < H < 1$. The equation has been solved and an explicit expression for the ruin probability at a given date has been derived in terms of error functions. Alternatively, this viewpoint offers a convenient way of calculating the ruin probability using standard finite difference schemes for the solution of partial differential equations.

The derivation of a differential equation is interesting in its own right for the following reason. If in the model the liability was driven by a Brownian motion then using the martingale properties and the Markov properties of Brownan motion the celebrated Feynman-Kac representation of the solution of partial differential equations can be obtained (see e.g. [22]). Through the use of the Feynman-Kac formula one may derive a PDE for the ruin probability. This was the original approach of Norberg [13]. However, in the case where liabilities are driven by a fractional Brownian motion the validity of a Feynman-Kac representation is no longer straightforward, since the fractional Brownian motion is neither a semimartingale, nor a Markov process.

The model may be generalized along the following directions:

An obvious generalization would be to include stochasticity in the interest force as well. This would lead to a more complicated linear stochastic differential equation driven by fractional Brownian motions. In this case the derivation of an equation for the ruin probability is possible but its form will be different and most probably it will not be expressible in local form. Also, since the fractional Brownian motions driving the liabilities and the interest force will, in principle, have different Hurst exponents the fractional stochastic calculus set up proposed by Duncan *et al*. may be insufficient and we may have to resort to the theory of stochastic integration on fractional Brownian motion proposed by Elliot and Van der Hoek [4], where mixtures of fractional Brwonian motions of different Hurst exponents may be used. This problem is under active consideration.

Another interesting direction towards the generalization of the model is the inclusion of Poisson jumps in the liability process. This will turn the partial differential equation for the ruin probability at a given date into a partial integrodifferential equation which may be treated using standard techniques for such problems.

Appendix

In this Appendix we include the parameters used for the implementation of the numerical schemes.

For the Monte Carlo we have used M = 30000 paths and L = 14.

For the Implicit method for $H = 0.5$ we took initial capital steps = 5000, time steps = 50000, X_min = −10, X_max = 10, and for the Crank–Nicolson, for $H = 0.5$, we took initial capital steps = 5000, time steps = 10000, $X_{min} = -10$, $X_{max} = 10$.

For the Implicit method for $H \in (0.5, 1)$ we took initial capital steps $= 1000$, time steps $= 1000$, $X_{min} = -10$, $X_{max} = 10$.

For the Crank–Nicolson method for $H = 0.5$, we used initial capital steps $= 5000$, time steps $= 10000$, Xmin $= -10$, Xmax $= 10$.

For the Implicit method for $H \in (0.5, 1)$ we used initial capital steps $= 1000$, time steps $= 1000$, Xmin $= -10$, Xmax $= 10$.

For the Crank–Nicolson method for $H \in (0.5, 1)$, we have some some singularities because of the discontinuity of the probability of ruin at $x = 0$, and thus we used for the first five steps the implicit finite difference method and for the rest steps the Crank–Nicolson method. We have used initial capital steps $= 1000$, time steps $= 1000$, Xmin $= -10$, Xmax $= 10$.

For the Implicit method for $H = 1$ we used initial capital steps $= 2000$, time steps $=$ 2000, Xmin $= -10$, Xmax $= 10$.

For the Crank–Nicolson method for $H = 1$, we used for the first five steps the implicit finite difference method and for the rest steps the Crank–Nicolson method. We have used initial capital steps $= 1000$, time steps $= 1000$, Xmin $= -10$, Xmax $= 10$.

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