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ON REGULARITY OF BANACH-VALUED PROCESSES

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For the study of the paths of stochastic processes, the upcrossing-downcrossing arguments are available in the real-valued case but not in the vector-valued case. We use in this article stopping time methods and the Kadec-Klee renorming theorem to obtain regularity properties for Banach-lattice valued submartingales and Banach-valued pramarts.

It is known that real-valued properly bounded processes $(X_t, t \in \mathbb{R}^+)$ that are martingales, submartingales, quasimartingales or amarts have almost surely left and right limits (see [17], [8]). The same is true if (X_t) is a martingale, quasimartingale [19] or a uniform amart [4], taking values in a Banach space with the Radon-Nikodym property (RNP). The proofs depend on convergence theorems for the analogous processes indexed by $\mathbb{N}$ and $-\mathbb{N}$, and thus there cannot be any regularity results for nonnecessarily positive submartingales (X_t) taking values in a Banach lattice E with RNP because L_1-bounded nonpositive E-valued submartingales $(X_n, n \in \mathbb{N})$ need not converge (see [10]). However, Heinich [11] proved that there is convergence if $X_n \geq 0$, $n \in \mathbb{N}$. Here the same is proved for *reversed* positive submartingales $(X_n, n \in -\mathbb{N})$ assuming only order-continuity instead of RNP (Proposition 3.3). The stage is thus set for obtaining regularity theorems for vector-valued positive submartingales $(X_t, t \in \mathbb{R}^+)$ (Theorem 3.4). We accomplish the translation of convergence results into regularity properties by using stopping time methods, passing to the real valued case, and returning to the vector-valued processes via the Kadec-Klee renorming theorem. The method, embodied in Theorems 2.2 and 2.3. below, is quite general and can be applied to other processes; thus we prove regularity properties for continuous parameter vector-valued pramarts (Theorem 3.7). The class of pramarts includes the class of martingales, quasimartingales and uniform amarts. Exact definitions are given below.

1. Definitions and basic notions. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. For each $t \in \mathbb{R}^+ = [0, \infty)$, let $\mathscr{F}_t$ be a sub-σ-field of $\mathscr{F}$ which contains all the P-null sets. The collection $(\mathscr{F}_t, t \in \mathbb{R}^+)$ is assumed increasing and right continuous (i.e., $\mathscr{F}_t = \cap_{s>t} \mathscr{F}_s$ for all $t \in \mathbb{R}^+$).

A function $\tau: \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ is a *stopping time* for $(\mathscr{F}_t)$ if $\{\omega: \tau(\omega) \leq t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}^+$. An increasing sequence $\tau_1 \leq \tau_2 \leq \cdots$ of stopping times is said to *announce* τ if $\lim_{n\to\infty} \tau_n = \tau$ and $\tau_n < \tau$ (except on $\{\tau = 0\}$). Similarly a decreasing sequence of stopping times is said to *recall* τ if $\lim_{n\to\infty} \tau_n = \tau$ and $\tau_n > \tau$ (except on $\{\tau = \infty\}$). A *predictable time* is a stopping time that is announced by some

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sequence (τ_n). A stopping time is called *simple* if it takes finitely many finite values. Let T denote the set of all simple stopping times and $T(S)$ the set of all simple stopping times with values in S, where S is a countable dense subset of $\mathbb{R}^+$ containing 0.

Let $(E, \|\cdot\|)$ be a Banach space. Let $(X_t, t \in \mathbb{R}^+)$ be a family of E-valued, Bochner integrable random variables adapted to $(\mathscr{F}_t)$. The process (X_t) is called *separable* S if there exists a P-null set N such that for each ω outside N the sample path, $t \to X_t(\omega)$ of ω, is separable with respect to S in the strong topology of E. The process (Y_t) is a *modification* of (X_t) if $X_t = Y_t$ a.s. for all $t \in \mathbb{R}^+$.

2. Convergence.

We start with the following Proposition which can be found in [8].

PROPOSITION 2.1. *Let τ be a bounded stopping time (not necessarily simple) and let $(X_t, t \in \mathbb{R}^+)$ be a real-valued process adapted to $(\mathscr{F}_t)$.*

(a) *There is a sequence $\tau_{-1} \geq \tau_{-2} \geq \cdots$ in $T(S)$ which recalls τ and such that, for almost all $\omega \in \Omega$, the two nets $(X_{\tau_n}(\omega))_{n\downarrow -\infty}$ and $(X_t(\omega))_{t\downarrow\tau(\omega), t\in S}$ have the same cluster points in $[-\infty, \infty]$.*

(b) *Suppose that τ is a predictable stopping time, announced by a sequence in $T(S)$. Then there is a sequence (τ_n) in $T(S)$ announcing τ and such that, for almost all $\omega \in \Omega$, the two nets $(X_{\tau_n}(\omega))_{n\uparrow\infty}$ and $(X_t(\omega))_{t\uparrow\tau(\omega), t\in S}$ have the same cluster points in $[-\infty, \infty]$.*

THEOREM 2.2. *Let E be a separable Banach space. Let $(X_t, t \in \mathbb{R}^+)$ be an E-valued process with $E\|X_t\| < \infty$ for each $t \in S$.*

(a) *Let τ be a bounded stopping time. Suppose that for every sequence (τ_{-n}) in $T(S)$ recalling τ, $\lim_{n\to+\infty} X_{\tau_{-n}}$ exists strongly almost surely. Then $\lim_{t\downarrow\tau(\omega), t\in S} X_t(\omega)$ exists strongly for almost all ω (the exceptional null set depends on τ).*

(b) *Let τ be a bounded predictable stopping time which is announced by a sequence in $T(S)$. Suppose that, for every sequence (τ_n) in $T(S)$ announcing τ, $\lim_{n\to\infty} X_{\tau_n}$ exists strongly almost surely. Then $\lim_{t\uparrow\tau(\omega), t\in S} X_t(\omega)$ exists strongly for almost all ω (the exceptional set depends on τ).*

PROOF. (a) Let $X = \lim_{n\to+\infty} X_{\tau_{-n}}$. It is easy to see that X does not depend on the choice of the sequence (τ_{-n}) which recalls τ. Consequently $\lim_{n\to\infty} \chi(X_{\tau_n}) = \chi(X)$ a.s. for all $\chi \in E'$ and $\lim_{n\to\infty} \|X_{\tau_{-n}}\| = \|X\|$ a.s. Now using Proposition 2.1, we can choose sequences (τ_{-n}) and (τ'_{-n}) in $T(S)$ so that both (τ_{-n}) and (τ'_{-n}) recall τ and the two nets $(\chi(X_{\tau_{-n}}(\omega)))_{n\uparrow\infty}$ and $(\chi(X_t(\omega)))_{t\downarrow\tau(\omega), t\in S}$, $(\|(X_{\tau'_{-n}}(\omega))\|)_{n\uparrow\infty}$ and $(\|X_t(\omega)\|)_{t\downarrow\tau(\omega), t\in S}$, have the same cluster points in $[-\infty, \infty]$. But if a net has only one cluster point in $[-\infty, \infty]$, then it converges; so

$$\lim_{t\downarrow\tau(\omega), t\in S} \chi(X_t(\omega)) = \chi(X(\omega)) \text{ a.s.} \quad \text{and } \lim_{t\downarrow\tau(\omega), t\in S} \|X_t(\omega)\| = \|X(\omega)\| \text{ a.s.}$$

Since E is a separable Banach space it admits an equivalent norm $\|\cdot\|^*$ which is Kadec-Klee with respect to a countable norming subset D of E' (whenever $\chi(x_n) \to \chi(x)$ for all $\chi \in D$ and $\|x_n\|^* \to \|x\|^*$ then $x_n \to x$ strongly) [2, page

176]). Thus

(i) for all $\chi \in D$, $\lim_{t\downarrow\tau(\omega),t\in S}\chi(X_t) = \chi(X)$ a.s.

(ii) $\lim_{t\downarrow\tau(\omega),t\in S} \| X_t \|^* = \| X \|^*$ a.s.

The Kadec-Klee property of the norm gives the strong convergence of $(X_t)_{t\downarrow\tau}$ a.s. to X.

(b) The proof of (b) is identical.

THEOREM 2.3. *Let E be a Banach space, and $(X_t, t \in \mathbb{R}^+)$ an E-valued process.*

(a) *Suppose that, for every bounded stopping time τ, $\lim_{t\downarrow\tau(\omega),t\in S} X_t(\omega)$ exists strongly for all $\omega \in \Omega_\tau$ with $P(\Omega_\tau) = 1$. Then for almost all ω, the limit $X_{t^+}(\omega) = \lim_{s\downarrow t,s\in S} X_s(\omega)$ exists strongly for all $t \in \mathbb{R}^+$.*

(b) *Suppose that, for every bounded predictable stopping time τ, announced by a sequence (τ_n) in $T(S)$, $\lim_{t\uparrow\tau(\omega),t\in S} X_t(\omega)$ exists strongly for all $\omega \in \Omega_\tau$ with $P(\Omega_\tau)$ = 1. Assume also that (X_t) has right limits. Then for almost all ω, the limit $X_{t^-} = \lim_{s\uparrow t,s\in S} X_s(\omega)$ exists strongly for all $t \in \mathbb{R}^+$.*

The proof (but not the statement) appears in [4] pages 90 and 93. See also [7].

3. Applications. As a first application, we show that almost all the paths of a positive submartingale in a Banach lattice E with the Radon-Nikodym property have left and right limits. We begin with a discrete parameter convergence result. The following Lemma can be found in [1].

LEMMA 3.1. *Let E be an order continuous Banach lattice. If $(x_n) \subset E$ is such that $0 \le x_n \le x$ and (x_n) converges weakly to 0 then (x_n) converges strongly to 0.*

REMARK. If $0 \le x_n \le y_n$ and (x_n) converges weakly to 0 and (y_n) converges strongly to $y \in E$ then (x_n) converges strongly to 0. Indeed $0 \le x_n \wedge y \le x_n$ implies that $(x_n \wedge y)$ converges weakly to 0 and therefore strongly. The inequality $y \le x_n \vee y \le y_n \vee y$ implies that $(x_n \vee y)$ converges strongly to y. The equality $x_n \wedge y + x_n \vee y = x_n + y$ finishes the proof.

The next Proposition, a result of Brunel and Sucheston [3], is stated for easy reference.

PROPOSITION 3.2. *Let (X_n) be an E-valued sequence of random variables such that for almost all ω, $(X_n(\omega))$ is weakly sequentially compact. If for every $\chi \in E'$, $(\chi(X_n(\omega)))$ converges to a finite limit for all ω in some set Ω_χ with $P(\Omega_\chi) = 1$ then (X_n) converges weakly* a.s.

PROPOSITION 3.3. *Let E be an order continuous Banach lattice. Let $(X_n, n \in -\mathbb{N})$ be an E-valued, positive, integrable reversed submartingale. Then $(X_n, n \in -\mathbb{N})$ converges strongly* a.s. *to an E-valued integrable random variable.*

PROOF. By the definition of reversed submartingale [18] $0 \le X_n \le E^{\mathcal{F}_n}(X_{n+1}) \le E^{\mathcal{F}_n}(X_{-1})$, $n \in -\mathbb{N}$. Thus $0 \le E^{\mathcal{F}_n}(X_{n+1}) - X_n \le E^{\mathcal{F}_n}(X_{-1})$, $n \in -\mathbb{N}$. It is well known that the reversed martingale $(E^{\mathscr{F}_n}(X_{-1}), n \in -\mathbb{N})$ converges strongly a.s. and in L_1 to $E^{\mathscr{F}_{-\infty}}(X_{-1})$ (see, e.g., [16]).

If $Z_n = E^{\mathscr{F}_n}(X_{n+1}) - X_n$ then $E^{\mathscr{F}_{-\infty}}(X_{-1}) \leq Z_n \vee E^{\mathscr{F}_{-\infty}}(X_{-1}) \leq E^{\mathscr{F}_n}(X_{-1}) \vee E^{\mathscr{F}_{-\infty}}(X_{-1})$ thus $\lim_{n\to-\infty} Z_n \vee E^{\mathscr{F}_{-\infty}}(X_{-1}) = E^{\mathscr{F}_{-\infty}}(X_{-1})$ a.s. in the strong topology of E. On the other hand, the equality $Z_n \vee E^{\mathscr{F}_{-\infty}}(X_{-1}) + Z_n \wedge E^{\mathscr{F}_{-\infty}}(X_{-1}) = Z_n + E^{\mathscr{F}_{-\infty}}(X_{-1})$, $n \in -\mathbb{N}$ and the fact that for each $\chi \in E'(\chi(Z_n), n \in -\mathbb{N})$ converges a.s. to 0, imply that for each $\chi \in E'(\chi(Z_n \wedge E^{\mathscr{F}_{-\infty}}(X_{-1}))(\omega))$ converges to 0 for all ω in some set Ω_χ with $P(\Omega_\chi) = 1$. Since order intervals in order continuous Banach lattices are weakly compact, we deduce from Proposition 3.2 that $(Z_n \wedge E^{\mathscr{F}_{-\infty}}(X_{-1}), n \in -\mathbb{N}$ converges to 0 weakly a.s. Thus $(Z_n, n \in -\mathbb{N})$ converges weakly to 0 a.s. and by the Remark strongly a.s. Since submartingales have the optional sampling property, we have that $(Z_{\tau_n}, n \in -\mathbb{N})$ converges strongly to 0 a.s. for every increasing sequence $(\tau_n, n \in -\mathbb{N})$ of simple stopping times. But then the stochastic limit of the net $(\| E^{\mathscr{F}_\sigma}(X_\tau) - X_\sigma \|, \sigma \leq \tau)$ is 0 as $\tau \to -\infty$ [18] page 96. This is just the definition of reversed pramart [15, 9]. It has been proved in [9] Theorem 3.3. that if E is any Banach space then E-valued integrable reversed pramarts $(X_n, n \in -\mathbb{N})$ converge strongly a.s.

Remark. The necessity of order continuity is trivial since any decreasing sequence is a deterministic reversed submartingale.

Theorem 3.4. (a) *Let E be an order continuous Banach lattice. If $(X_t, t \in \mathbb{R}^+)$ is an E-valued separable, positive, integrable submartingale, then almost all paths have right limits.*

(b) *Let E be a Banach lattice with the Radon-Nikodym property. If $(X_t, t \in \mathbb{R}^+)$ is an E-valued, separable, positive, L_1-bounded submartingale, then almost all paths have left limits.*

Proof. (a) Let τ be a bounded stopping time and $(\tau_n, n \in -\mathbb{N})$ a sequence in $T(S)$ recalling τ. Then $(X_{\tau_n}, n \in -\mathbb{N})$ is a reversed submartingale. The assertion now follows from Proposition 3.3, Theorem 2.2(a) and Theorem 2.3(a).

(b) Let τ be a bounded predictable stopping time and $(\tau_n, n \in \mathbb{N})$ a sequence in $T(S)$ announcing τ. Then $(X_{\tau_n}, n \in \mathbb{N})$ is an L_1-bounded positive submartingale which converges by Heinich's Theorem [11], thus Theorem 2.2(b) applies. Since E has the Radon-Nikodym property, it does not contain c_0 ([6], pages 60 and 81]) and therefore E is weakly sequentially complete ([13], page 34), hence order continuous. Because order continuity implies the existence of right limits the result now follows from Theorem 2.3(b).

The following result about continuous modifications is known for the real-valued case [17], and the proof (from Theorem 3.4.) is the same.

Theorem 3.5. *Let E be an order continuous Banach lattice. Let $(X_t, t \in \mathbb{R}^+)$ be an E-valued, separable, positive integrable submartingale. Then (X_t) admits a right continuous modification if and only if the function $E(X_t)$ of t is right continuous in the norm topology of E.*

The second application is that pramarts taking values in a Banach space have right limits whereas pramarts taking values in a Banach space with the Radon-Nikodym property have both right and left limits.

DEFINITION. An adapted family $(X_t, t \in \mathbb{R}^+)$ of Bochner integrable random variables is an *ascending* (*reversed or descending*) *pramart* at a stopping time τ if for each increasing sequence $(\tau_n, n \in \mathbb{N})$ $[(\tau_n, n \in -\mathbb{N})]$ in T, converging to τ one has

$$s - \lim_n \| X_{\tau_n} - E^{\mathcal{F}_{\tau_n}}(X_{\tau_{n+1}}) \| = 0$$

(s-lim means limit in probability) [15].

$(X_t, t \in \mathbb{R})$ is an ascending [reversed] pramart if it is an ascending [reversed] pramart at each stopping time.

$(X_t, t \in \mathbb{R}^+)$ is a *pramart* if it is both an ascending and reversed pramart.

If the stochastic limit above is replaced by the L_1 limit, then the process is called uniform amart [4].

It is clear that the class of pramarts contains the class of uniform amarts and hence also the class of quasimartingales [4]. We say that

(i) $(X_t, t \in \mathbb{R}^+)$ is of *class* (B) if, $\sup_{\tau \in T} E \| X_\tau \| < \infty$

(ii) $(X_t, t \in \mathbb{R}^+)$ is of *class* (AL) if, for all uniformly bounded increasing sequences $(\tau_n) \subset T$, we have $\sup_n E \| X_{\tau_n} \| < \infty$ [8]. Now we observe the following Lemma, the proof of which uses the optional sampling theorem for pramarts [15] and Theorem 2.4. of [12].

LEMMA 3.6. *Let $(X_t, t \in \mathbb{R}^+)$ be a pramart of class (AL) then $(X_{\tau_n}, n \in \mathbb{N})$ is a discrete parameter pramart of class (B), for every increasing sequence (τ_n) in T converging to a bounded stopping time τ.*

THEOREM 3.7. (a) *Let E be an arbitrary Banach space. If $(X_t, t \in \mathbb{R}^+)$ is an E-valued, separable, integrable, reversed pramart, then almost all paths have right limits.*

(b) *Let E be a Banach space with the Radon-Nikodym property. If $(X_t, t \in \mathbb{R}^+)$ is an E-valued, separable, pramart of class* (AL), *then almost all paths have left limits.*

The proof is similar to that of Theorem 3.4. We use discrete parameter results proved in [9] (reversed pramarts), [14] (pramarts) and Lemma 3.6.

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