A METHOD FOR OBTAINING THE PROBABILITY DISTRIBUTION OF \( m \) COMPONENTS CONDITIONAL ON \( l \) COMPONENTS OF A RANDOM VECTOR

EVDOKIA NEKALAKI

1. THE RESULT

Let \( \mathbf{X} = (X_1, X_2, \ldots, X_s) \) be a random vector of nonnegative and integer valued components and let \( p_s \) and \( G_X(u) \equiv G_{X_1, X_2, \ldots, X_s}(u_1, u_2, \ldots, u_s) \) denote its probability function (p.f.) and its probability generating function (p.g.f.), respectively, i.e., \( p_s = P(X_1 = x_1, X_2 = x_2, \ldots, X_s = x_s) \) and \( G_X(u) = \sum_{x_1} \ldots \sum_{x_s} p_s \prod_{i=1}^{s} u_i^{x_i} \). Here \( \sum \) denotes the multiple summation \( \sum_{x_1 \geq 0} \ldots \sum_{x_s \geq 0} \sum_{k_1, k_2} \ldots \sum_{k_s \in I^+ \cup \{+\infty\}, i = 1, 2, \ldots, s} \). Obtaining the p.f. of the conditional distribution of a random vector consisting of \( m, m < s \) components of \( \mathbf{X} \) given some of the remaining \( s - m \) components from \( p_s \) can sometimes be tedious, especially when \( p_s \) is of a complicated form. However, utilizing \( G_s(u) \) the task of obtaining the conditional p.g.f. of the particular vector can become easier. Subrahmanian (1966) showed that, for \( s = 2 \), the p.g.f. of \( X_1 \) conditional on \( X_2 \) can be obtained by

\[
G_{X_1|X_2}(u) = \frac{\partial^{x_2}}{\partial u^{x_2}} G_{X_1,X_2}(u, 0) \div \frac{\partial^{x_1}}{\partial u^{x_1}} G_{X_1,X_2}(1, 0).
\]

Steyn and Roux (1970), based on a restatement of the theorem of compound probability distributions, gave the following formula

\[
G_s(u) = \sum_{x_{m+1} \ldots x_s} p_{x_{m+1} \ldots x_s} u_{m+1}^{x_{m+1}} \ldots u_{s}^{x_{s}} G_{X_{m+1} \ldots X_s}(u_{m+1}, \ldots, u_s)
\]

from which the p.g.f. of the conditional distribution follows if the joint p.g.f. \( G_s(u) \) and the marginal p.f. \( p_{x_{m+1} \ldots x_s} \) are known.

We now prove the following theorem which provides a multivariate version of Subrahmanian's result.

**Theorem.** Let \( X_1, \ldots, X_s \) be discrete r.v.'s with joint p.g.f. \( G_s(u) \).

Then,

\[
G_{X_1, \ldots, X_m\mid X_{m+1}, \ldots, X_s}(u_1, \ldots, u_m) = \frac{\partial^x}{\partial u_{m+1}^{x_{m+1}} \ldots \partial u_{s}^{x_{s}}} G_s(u_1, \ldots, u_m, 0, \ldots, 0) \div \frac{\partial^x}{\partial u_{m+1}^{x_{m+1}} \ldots \partial u_{s}^{x_{s}}} G_s(1, \ldots, 1, 0, \ldots, 0)
\]

REV. ROUMAINE MATH. PURES APPL. 32(1987), 6, 584–583
where \( x' = x_{m+1} + \cdots + x_{s} \).

**Proof.** We have

\[
G_{x_{1}, x_{2}, \ldots, x_{s}}(u_{1}, \ldots, u_{m}) = \sum_{x_{1}, x_{2}, \ldots, x_{s}} p_{x_{1}, x_{2}, \ldots, x_{s}} u_{1}^{x_{1}} u_{2}^{x_{2}} \cdots u_{m}^{x_{s}}
\]

(1.2)

\[
\sum_{x_{1}, \ldots, x_{m}} p_{x_{1}, \ldots, x_{m}} u_{1}^{x_{1}} u_{2}^{x_{2}} \cdots u_{m}^{x_{m}} = \prod_{i=j+1}^{s} \frac{1}{p_{x_{i+1}, \ldots, x_{s}} u_{i}^{x_{i}} u_{j}^{x_{j}} \cdots u_{m}^{x_{m}}}
\]

We will show that

\[
\sum_{x_{1}, \ldots, x_{m}} p_{x_{1}, \ldots, x_{m}} u_{1}^{x_{1}} \cdots u_{m}^{x_{m}} = \prod_{i=j+1}^{s} \frac{\partial^{x_{i}}}{\partial u_{i}^{x_{i}}} G_{x}(u_{j}, \ldots, u_{m}, 0, \ldots, 0),
\]

(1.3)

where \( x = x_{m+1} + \cdots + x_{s} \). We have

\[
\frac{\partial^{x_{i}}}{\partial u_{i}^{x_{i}}} \frac{\partial^{x_{j}}}{\partial u_{j}^{x_{j}}} G_{x}(u_{j}, \ldots, u_{m}, 0, \ldots, 0) =
\]

\[
\frac{\partial^{x_{i}}}{\partial u_{i}^{x_{i}}} \left[ \frac{\partial^{x_{j}}}{\partial u_{j}^{x_{j}}} \left[ \cdots \left[ \frac{\partial^{x_{m+1}}}{\partial u_{m+1}^{x_{m+1}}} G_{x}(u_{j}, \ldots, u_{m}) \right] \ldots \right] \right]_{u_{m+1} = \cdots = u_{m} = 0}.
\]

But

\[
\frac{\partial^{x_{i}}}{\partial u_{i}^{x_{i}}} G_{x}(u_{j}, \ldots, u_{m}, 0, u_{m+2}, \ldots, u_{s}) =
\]

\[
\sum_{x_{1}, \ldots, x_{m+1}, \ldots, x_{s}} p_{x_{1}, x_{2}, \ldots, x_{m+1}} u_{1}^{x_{1}} \cdots u_{m}^{x_{m}} (x_{m+1})! u_{m+2}^{x_{m+2}} \cdots u_{s}^{x_{s}}.
\]

So

\[
\frac{\partial^{x_{m+1} + x_{m+2}}}{\partial u_{m+1}^{x_{m+1}} \partial u_{m+2}^{x_{m+2}}} G_{x}(u_{j}, \ldots, u_{m}, 0, u_{m+3}, \ldots, u_{s}) =
\]

\[
\sum_{x_{1}, \ldots, x_{m+1}, \ldots, x_{s}} p_{x_{1}, \ldots, x_{m+1}} u_{1}^{x_{1}} \cdots u_{m}^{x_{m}} (x_{m+1})! (x_{m+2})! u_{m+3}^{x_{m+3}} \cdots u_{s}^{x_{s}}.
\]

By induction we obtain (1.3). We notice that

\[
\frac{\partial^{x}}{\partial u_{1}^{x_{1}} \cdots \partial u_{s}^{x_{s}}} G_{x}(1, \ldots, 1, 0, \ldots, 0) =
\]

\[
\prod_{i=j+1}^{s} x_{i}! \sum_{x_{1}, \ldots, x_{m}} p_{x_{1}, \ldots, x_{m}} = \prod_{i=j+1}^{s} x_{i}! p_{x_{m+1} \cdots x_{s}}.
\]

So that

\[
(1.4) \quad \sum_{x_{1}, \ldots, x_{m}} p_{x_{1}, \ldots, x_{m}} \frac{1}{x_{1}! \cdots x_{s}!} \frac{\partial^{x}}{\partial u_{1}^{x_{1}} \cdots \partial u_{s}^{x_{s}}} G_{x}(1, \ldots, 1, 0, \ldots, 0).
\]

Using (1.2), (1.3) and (1.4) we obtain the result.
Note. Obvious modifications to the proof of the theorem lead to a more general formula concerning the p.g.f. of \((X_{i_1}, X_{i_2}, \ldots, X_{i_m})\) conditional on \((X_{i_{m+1}}, X_{i_{m+2}}, \ldots, X_{i_s})\), where \(i_1, i_2, \ldots, i_m\) and \(i_{m+1}, i_{m+2}, \ldots, i_s\) are permutations of the subscripts \((1, 2, \ldots, s)\) taking \(m\) and \(l\) at a time respectively \((m, l \geq 1; m + l \leq s)\).

2. SOME EXAMPLES

• As an illustration of the result shown in the previous section we consider obtaining the p.g.f. of the conditional distribution of \((X_1, X_2, \ldots, X_m)\) \((X_{m+1}, X_{m+2}, \ldots, X_s)\), \(m < s\) in two cases.

Example 1. Let \((X_1, X_2, \ldots, X_s)\) have an \(s\)-variate binomial distribution with p.g.f. \(G_X(u) = (p_0 + p_1 u_1 + p_2 u_2 + \ldots + p_s u_s)^s,\) \(n \in I^*,\)

\[ p_i > 0, \; i = 0, 1, 2, \ldots, s; \sum_{i=1}^{s} p_i = 1 - p_0. \]

Then

\[
G_{X_1, \ldots, X_m | X_{m+1}, \ldots, X_s}(u_1, \ldots, u_m) = \frac{n^{(c)}}{\prod_{i=m+1}^{s} p_i^{n^{(c)}}(p_0 + p_1 u_1 + \ldots + p_s u_s)^{s-1}} \prod_{i=m+1}^{s} p_i^{\left(1 - \sum_{i=m+1}^{s} p_i\right)}^{n^{(c)}} = (p_0 + p_1 u_1 + \ldots + p_s u_s)^{s-1},
\]

where \(c\) is defined as in the theorem, \(n^{(c)} = \prod_{i=1}^{s} (n - i + 1),\) and \(p_i = \prod_{i=m+1}^{s} p_i\), \(i = 0, 1, 2, \ldots, s.\)

Example 2. Let \((X_1, X_2, \ldots, X_s)\) have an \(s\)-variate negative binomial distribution with p.g.f. \(G_X(u) = (1 + \theta_1 (1 - u_1) + \ldots + \theta_s (1 - u_s))^{-1},\)

\(k > 0, \theta_i > 0, i = 1, 2, \ldots, s.\)

Then

\[
G_{X_1, \ldots, X_m | X_{m+1}, \ldots, X_s}(u_1, \ldots, u_m) = k_{(c)} \prod_{i=m+1}^{s} \theta_i^{n^{(c)}}\left(1 + \theta_1 (1 - u_1) + \ldots + \theta_m (1 - u_m)\right)^{s-k-1} = k_{(c)} \prod_{i=m+1}^{s} \theta_i^{n^{(c)}} = (1 + \theta_1 (1 - u_1) + \ldots + \theta_m (1 - u_m))^{-k-1},
\]

where \(x\) is defined as before.

Received July 27, 1984

Department of Statistics and Actuarial Science
The University of Iowa,
Iowa City, Iowa 52242, USA

REFERENCES