THE GENERALIZED WARING PROCESS AND ITS APPLICATION

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ABSTRACT

The generalized Waring distribution is a discrete distribution with a wide spectrum of applications in areas such as accident statistics, income analysis, environmental statistics, etc. It has been used as a model that better describes such practical situations as opposed to the Poisson distribution or the negative binomial distribution. Associated to both the Poisson and the negative binomial distributions are the well-known Poisson and Pólya processes. In this paper, the generalized Waring process is defined. Two models have been shown to lead to the generalized Waring process. One is related to a Cox process, while the other is a compound Poisson process. The defined generalized Waring process is shown to be a stationary, but non-homogenous Markov process. Several properties are studied and the intensity, the individual intensity and the Chapman-Kolmogorov differential equations of it are obtained. Moreover, the Poisson and the Pólya processes are shown to arise as special cases of the generalized Waring process. Using this fact, some known results and some properties of them are obtained.

Keywords and phrases: Pólya process, accident proneness, accident liability, Markovian property, stationary increments, Cox process, transition probabilities, Chapman-Kolmogorov equations, individual intensity.

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1. Introduction - Basic concepts

The Poisson and the Pólya processes have been used in accident theory to describe the accident pattern. Under the hypotheses of pure chance, the Poisson process with intensity λ has been proposed as a model that can describe the number of accidents sustained by an individual during several years. The Pólya process, which is of negative binomial form, is defined by starting from a Poisson process, which then, is mixed with a gamma distribution. It has been obtained as a model, which can describe the accident pattern of a population of individuals during several years, under the hypotheses of *"accident proneness"*, i.e. that individuals differ in their probabilities of having an accident, which remain constant in time (Newbold, 1927). Both of these processes satisfy the Markovian property as this is a property of the accident pattern, i.e. the number of accidents during the 'next' period $(t, t+h]$ depends only on the number of accidents at the present time t.

In this paper, a new process is defined and studied. This process is associated with a discrete distribution with a wide spectrum of applications known in the literature as the generalized Waring distribution (see, e.g. Irwin, 1975; Xekalaki, 1983b). Analogously to the case of Poisson and Pólya process, this new process, termed in the sequel as the *generalized Waring process,* is postulated to be a Markov process, as shown in section 2. The starting point is a process of negative binomial form, but different from a Pólya process. This process is then mixed with a beta distribution of the second type (beta II). Further, an alternative genesis scheme referring to Cresswell and Froggatt's (1963) *spells model* is proposed in the framework considered by Xekalaki (1983b). Section 3 indicates how the above considerations formulate the framework for the definition of the generalized Waring process as a stationary, but non-homogenous Markov process. Expressions for the first two moments of this process, as well as results on the intensity and the individual intensity of it, are also given in section 3 and its transition probabilities and the associated forward and backward Chapman-Kolmogorov differential equations are derived. In section 4, the Poisson and Pólya processes are obtained as limiting cases of the generalized Waring process. Using this fact, some known theoretical results concerning these processes are presented and their transition probabilities and associated Chapman-Kolmogorov differential equations are derived in this context. Some inferential aspects connected with the mixed negative binomial derivation of the generalized Waring process are discussed in section 5. An application in a web access modeling context is provided in section 6. Finally, two further genesis schemes considered by Zografi and Xekalaki (2001) are presented in section 7.

The results obtained in the paper are in the context of models that have widely been considered for the interpretation of accident data. However, the concepts and terminology used can easily be modified so that the obtained results can be applied in several other fields ranging from economics, inventory control and insurance through to demometry, biometry, psychometry and web access modeling as the case is with the application discussed in section 6.

2. The basic hypotheses of the generalized Waring process

2.1 The description of the accident pattern by a Cox process

In this section, we consider first the assumptions of a Pólya process, developed by Newbold (1927). This model considers several individuals exposed to the same external risk (e.g. drivers all driving about the same distance within a similar traffic environment) and that there are intrinsic differences among different individuals (e.g. differences in accident proneness). Supposing that, the number of accidents up to time t, for each individual, conforms with a Poisson process with a "personal rate λ" (λ stands for the respective accident proneness), and regarding λ as the outcome of a random variable Λ with a gamma distribution with parameters k and ν, the number of accidents $N(t)$ at time t, $t = 0, 1, 2, \ldots$ defines the Pólya process with parameters k and ν as follows:

(i) $N(0) = 0$,

(ii) $N(t)$ is a birth process,

(iii) $N(t+h) - N(t)$ has a distribution defined by the probability function

$$P\{N(t+h) - N(t) = m\} = E\left[\frac{(\Lambda h)^n}{n!} e^{-\Lambda h}\right]$$

$$= \binom{k+m-1}{m} \left(\frac{1}{1+vh}\right)^k \left(\frac{vh}{1+vh}\right)^m \quad , \quad m = 0,\ 1, \ldots, \tag{2.1.1}$$

where Λ is a random variable with density u given by

$$u(l) = \frac{\nu^{-k}}{\Gamma(k)} l^{k-1} e^{-\left(l\nu^{-1}\right)},\ l > 0,$$

It is clear that $N(t)$ has a negative binomial distribution with parameters k and $\frac{1}{1+\nu t}$, i.e. $N(t) \sim NB\left(k, \frac{1}{1+\nu t}\right)$.

The distribution of the random variable Λ explains here the variation of the accident proneness from individual to individual. As noted by Irwin (1968) and Xekalaki (1984), the term accident proneness here refers to both, the external and the internal risk of accident. It seems more natural to assume that this variation in an interval of time $(t,\ t+h]$ depends on the length h of the interval, while, in two non-overlapping time periods, the respective variations are independent. So, now, a personal λ, in an interval of time $(t,\ t+h]$, is regarded as the outcome of a random variable $\Lambda(h)$ with distribution $U(h)$, which depends on the interval length h. If $U(h)$ is assumed to be $\Gamma(k(h), 1/\nu(h))$, where $k(h)$ and $\nu(h)$ are in general some functions of h, then, clearly, the number of accidents $N(t)$ forms a stochastic process of a negative binomial form satisfying the assumptions

(i) $N(0) = 0$

and

(ii) $N(t+h) - N(t)$ has the distribution:

$$P\{N(t+h) - N(t) = n\} = \int_0^{+\infty} \frac{(\lambda h)^n}{n!} e^{-\lambda h} \frac{\nu(h)^{-k(h)}}{\Gamma(k(h))} \lambda^{(k(h)-1)} e^{(-\lambda/\nu(h))} d\lambda, \quad n = 0,1,\ldots \tag{2.1.2}$$

It can be shown that

$$P\{N(t+h) - N(t) = n\} = \binom{k(h)+n-1}{n} \left(\frac{1}{1+\nu(h)h}\right)^{k(h)} \left(\frac{\nu(h)h}{1+\nu(h)h}\right)^n .$$

Then, using the first assumption, it follows that for any t, $N(t)$ has a negative binomial distribution with parameters $k(t)$ and $\frac{1}{1+\nu(t)t}$. Hence, one can verify that

$$P\{N(t) = n\} = \int_0^{+\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \frac{\nu(t)^{-k(t)}}{\Gamma(k(t))} \lambda^{(k(t)-1)} e^{-\lambda/\nu(t)} d\lambda, \ n = 0,1,...$$

This tells us precisely that $N(t)$ is a *Cox Process* (see e.g. Grandell, J. 1997, p. 83).

Assume that the accident proneness varies from individual to individual with a mean that does not depend on time. This is equivalent to considering a parameter pair $(k(h),\ \nu(h))$ with $k(h)\cdot\nu(h) = \text{constant}$. So, letting $\nu(h) = \nu/h$, and $k(h) = kh$, i.e., allowing $\Lambda(h)$ having a gamma distribution that changes with time so that its expectation remains constantly equal to νk, we obtain

$$P\{N(t+h)-N(t)=n\}=\binom{kh+n-1}{n}\left(\frac{1}{1+\nu}\right)^{kh}\left(\frac{\nu}{1+\nu}\right)^{n}, \qquad n=0,1,\ldots \qquad (2.1.3)$$

and that $N(t)$ is $NB\left(kt,\frac{1}{1+\nu}\right)$-distributed.

2.2 An extension of Irwin's accident model

This model considers a population which is not homogeneous with respect to personal and environmental attributes that affect the occurrence of accidents. In his model, Irwin (1968, 1975) used the term "accident proneness" ν to refer to a person's predisposition to accidents, and the term "accident liability" $(\lambda|\nu, \text{i.e. } \lambda \text{ for given } \nu)$ to refer to a person's exposure to external risk of accident.

The conditional distribution of the random variable Λ given ν describes differences in external risk factors among individuals. As before, liability fluctuations over a time interval $(t, t+h)$ depend on the length h of the interval and are described by a $\Gamma(kh, 1/\nu h)$ distribution for $\Lambda|\nu$. Moreover, assuming independence in two non-overlapping time periods, the number of accidents $N(t)$ given ν will be a stochastic process of a negative binomial form with parameters kt and $\frac{1}{1+\nu}$. This starts at 0 and has stationary increments with a distribution given by (2.1.3). Let us further allow the parameter ν of the negative binomial to follow a beta distribution of the second kind with parameters a and ρ, i.e. ν is a random variable with density φ given by

$$\varphi(\nu)=\frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)}\nu^{(a-1)}(1+\nu)^{-(a+\rho)} \qquad a,\rho \geq 0$$

obtaining thus for the distribution of the number of accidents $N(t)$:

$$P\big(N(t+h)-N(t)=n\big)=\frac{\rho_{(kh)}}{(a+\rho)_{(kh)}}\frac{a_{(n)}(kh)_{(n)}}{(a+\rho+kh)_{(n)}}\frac{1}{n!}$$

and

$$P(N(t)=n)=P_n(t)=\frac{\rho_{(kt)}}{(a+\rho)_{(kt)}}\frac{a_{(n)}(kt)_{(n)}}{(a+\rho+kt)_{(n)}}\frac{1}{n!}, \qquad n=0,\ 1,\ \ldots \qquad (2.2.1)$$

In the sequel, we refer to the process defined by $N(t)$ as the *Generalized Waring Process.*

Remark 2.1

If we consider individuals of proneness ν and liability $\lambda_i \mid \nu \quad i=1,2$ respectively in each of two non-overlapping intervals of time, it follows by the model's assumptions that the numbers $N_1,\ N_2$ of accidents incurred by these individuals are jointly distributed according to a double Poisson distribution with parameter $(\lambda_1 \mid \nu,\ \lambda_2 \mid \nu)$. Then, for individuals with the same proneness but varying liabilities, the joint distribution of accidents over the two intervals, is the double negative binomial with parameters $\left(\left(kh_1, \frac{1}{1+\nu}\right);\left(kh_2, \frac{1}{1+\nu}\right)\right)$, where $h_1,\ h_2$ are the respective sizes of these intervals. If, further, the proneness parameter ν is allowed to follow a *beta* distribution of the second kind with parameters a and ρ, the joint distribution of the numbers of accidents over the two intervals is a bivariate generalized Waring distribution with parameter $\left((a, kh_1, \rho);(a, kh_2, \rho)\right)$ (Xekalaki, 1984). Now, it is clear that, if a number of non-overlapping intervals greater than two is considered, the joint distribution of the numbers of accidents over those intervals, will follow a multivariate Generalized Waring distribution (Xekalaki, 1986).

In the sequel, we use the above remark to show that the Generalized Waring process resulting from the above generating scheme is a Markov process, i.e. that $P(N(t+h)=n \mid N(t)=m,\ N(s)=n_s,\ 0 \le s < t)$ coincides with $P(N(t+h)=n \mid N(t)=m)$ for every non-negative integer $n, m, n_s \ 0 \le s < t$.

For a proof of this, observe that

$$P\{N(t+h)=n \mid N(t)=m, N(s)=n_s,\ 0 \le s < t\} =$$

$$P\{N(t+h)-N(t)=n-m \mid N(t)-N(s)=m-n,\ N(s)-N(0)=n_s, 0 \le s < t\}$$

and consider the random vector

$\left(N(t+h)-N(t), N(t)-N(s), N(s)-N(0)\right)$, $0 \le s < t$.

It follows from Remark 2.1 that this vector has a trivariate generalized Waring distribution with parameters $\alpha, \underline{k},$ and ρ, where $\underline{k} = (kh, k(t-s), ks)$. This is a three dimensional special case of Xekalaki's (1986) multivariate generalized Waring distribution whose structural properties imply that the random vector $\left(N(t+h)-N(t) \mid N(t)-N(s), N(s)-N(0)\right)$ has a univariate generalized Waring distribution with parameters $\alpha + n(t), kh$ and $\rho + kt$, where *n(t)* is the value of *N(t).* Hence,

$$P\{N(t+h)-N(t)=n-m \mid N(t)-N(s)=m-n_s, N(s)-N(0)=n_s\}$$

$$=\frac{(\rho+kt)_{(a+m)}}{(\rho+kt+kh)_{(a+m)}}\frac{(a+m)_{(n-m)}(kh)_{(n-m)}}{(\rho+kt+kh+a+m)_{(n-m)}}\frac{1}{(n-m)!}$$

$$=\frac{\Gamma(a+n)}{\Gamma(a+m)}\frac{(kh)_{(n-m)}}{(n-m)!}\frac{(\rho+kt)_{(a+m)}}{(\rho+kt+kh)_{(a+n)}} \tag{2.2.2}$$

$$=P\{N(t+h)-N(t)=n-m \mid N(t)-N(0)=m\}$$

$$=P\left[N(t+h)=n \mid N(t)=m\right],$$

which proves that the generalized Waring process has the Markovian property, i.e. the conditional distribution of the future state $N(t+h)$ given the present state $N(t)$ and the past state $N(s)$, $0 \le s \le t$, depends only on the present state.

2.3 The *spells* model

In the sequel, an alternative scheme generating a process of a generalized Waring form is considered. This is a variant of Cresswell and Froggatt's (1963) *spells* model that has been considered in the paper of Xekalaki (1984). According to this model, each person is liable to spells. For each person, no accidents can occur outside spells. Let $S(t)$ denote the number of spells up to a given moment t. It is assumed that $S(t)$, $t = 0, 1, 2, \ldots$ is a homogeneous Poisson process with rate k/m, $k > 0$, the number of accidents within a spell is a random variable with a given distribution F and that the number of accidents arising out of different spells are independent and also independent of the number of spells. So, the total number of accidents at time t

is $X(t)=\sum_{k=1}^{S(t)} X_k$, where $S(t)$ is a homogenous Poisson process with rate k/m and $\{X_k\}_1^{\infty}$ are identically and independently distributed (i.i.d.) random variables from the distribution F.

When $\{X_k\}_1^{\infty}$ is a logarithmic series distribution with parameters (m,ν), i.e. $P(X_i=0)=1-m\log(1+v)$ and $P(X_i=n)=\frac{m}{n}\left(\frac{v}{1+v}\right)^n$, $n\geq 1,\ m>0,\ v>0$, the random variable $X(t)$, is a negative binomial random variable with parameters $\left(kt,\frac{1}{1+v}\right)$ for each t (Chatfield and Theobald, 1973). Here ν is regarded as the external risk parameter, too. Then, if the differences in the external risk can be described by a $beta(a,\rho)$ distribution of the second kind, the resulting accident distribution is of a generalized Waring form with parameters a, kt, and ρ.

Let us consider, now, the counting process $\{N(t),t\geq 0\}$ with $N(t)$ represented, for $t\geq 0$, by $\sum_{k=1}^{S(t)} X_k$, $\left(\sum_{k=1}^{0} X_k=0\right)$, where $S(t)$ is a homogenous Poisson process with rate k/m, $\{X_k\}_1^{\infty}$ has a logarithmic series distribution with parameters (m,ν) and is independent of the process $S(t)$, and ν is a non-negative random variable with a $Beta(a,\rho)$ distribution of the second kind.

Theorem 2.3.1

For the process $\{N(t),\ t\geq 0\}$ defined as above the following conditions hold:

(i) $N(0)=0$

(ii) $\{N(t),\ t\geq 0\}$ possesses stationary increments

(iii) $\{N(t),\ t\geq 0\}$ is a Markov process.

Proof: The proof of (i) is straightforward. To prove condition (ii), denote by φ the probability distribution function (p.d.f.) of the random variable ν. Then, we can write:

$$P\big(N(t+h)-N(t)=n\big)=\int_0^{+\infty} P\big(N(t+h)-N(t)=n/v\big)\varphi(v)dv$$

$$= \int_0^{+\infty} P\left(\sum_{k=S(t)}^{S(t+h)} X_k = n\right)\varphi(v)dv = \int_0^{+\infty}\left[\sum_{i=0}^{+\infty} P\left(\sum_{k=1}^{i} X_k = n\right) p\big(S(t+h)-S(t)=i\big)\right]\varphi(v)dv$$

$$= \int_0^{+\infty}\left[\sum_{i=0}^{+\infty} P\left(\sum_{k=1}^{i} X_k = n\right)\frac{1}{i!}\exp\left(-\frac{kh}{m}\right)\left(\frac{kh}{m}\right)^i\right]\varphi(v)dv = \frac{\rho_{(kh)}}{(\rho+a)_{(kh)}}\frac{a_{(n)}(kh)_{(n)}}{(a+\rho+kh)_{(n)}}\frac{1}{n!}.$$

To prove the Markovian property, let $N_\nu(t) = \sum_{k=1}^{S(t)} X_k$ for a given ν. The process $N_\nu = \{N_\nu(t),\ t \ge 0\}$ is a compound Poisson process. Hence, it is a Markov process.

We now note that:

$$P\big(N(t+h) = n \,|N(t) = m, N(s) = n_s \ for\ \ 0 \le s \le t\big) =$$

$$= \frac{\int_0^{+\infty} P_\nu\big(N(t+h) = n, N(t) = m, N(s) = n_s \ for\ \ 0 \le s \le t\big)\varphi(\nu)d\nu}{\int_0^{+\infty} P_\nu\big(N(t) = m, N(s) = n_s \ for\ \ 0 \le s \le t\big)\varphi(\nu)d\nu},$$

where $P_\nu(A)$ stands for the conditional probability of an event A given the value ν of the random variable ν. Then, $P_\nu\big(N(t+h) = n,\ N(t) = m,\ N(s) = n(s),\ 0 \le s \le t\big)$ is equal to $p_h(m-n)\cdot p_{t-s}(m-n_s)\cdot p_s(n(s))$ and $P_\nu\big(N(t) = m,\ N(s) = n(s),\ 0 \le s \le t\big)$ is equal to $p_{t-s}(m-n_s)\cdot p_s(n(s))$. Therefore,

$$P\big(N(t+h) = n,\ N(t) = m,\ N(s) = n(s),\ 0 \le s \le t\big)$$

$$= \frac{\Gamma(a+n)}{\Gamma(a+m)}\frac{(kh)_{(n-m)}}{(n-m)!}\frac{(\rho+kt)_{(a+m)}}{(\rho+kt+kh)_{(a+n)}} \tag{2.3.1}$$

The last result proves the Markovian property of the process and provides its transition probabilities.

3. The generalized Waring process

The generalized Waring process can, now, be defined in the following way:

Definition 3.1 The counting process $\{N(t),\ t \ge 0\}$ is said to be a *generalized Waring process* with parameters $(a,\ k,\ \rho)$, $a > 0,\ k > 0,\ \rho > 0$ if (i) $N(0) = 0$, (ii) $N(t)$ is a

Markov process, (iii) $N(t+h)-N(t)$ is $GW(a,\ kh,\ \rho)$-distributed for each $h>0,\ t\geq 0$.

Conditions (i), (ii) and (iii) tell us that this process starts at 0, it has stationary increments and $P\big(N(t)=n\big)=\dfrac{\rho_{(kt)}}{(\rho+a)_{(kt)}}\dfrac{a_{(n)}(kt)_{(n)}}{(a+\rho+kt)_{(n)}}\dfrac{1}{n!}$, i.e. $N(t)$ is $GW(a,\ kt,\ \rho)$-distributed.

Given that the defined generalized Waring process is a Markov process, the relations (2.2.2) and (2.3.1) can lead to its transition probabilities. Their explicit form is

$$P\{N(s+t)=n \mid N(s)=m\}=\frac{\Gamma(a+n)}{\Gamma(a+m)}\frac{(kt)_{(n-m)}}{(n-m)!}\frac{(\rho+ks)_{(a+m)}}{(\rho+ks+kt)_{(a+n)}} .$$

The above probability denoted in the sequel by $p_{m,n}(s,s+t)$, represents the probability that a process presently in state m will be in state n a later time t. This probability in this case depends on the present time so the defined generalized Waring process is a non-homogenous Markov process.

It is clear that $p_{0,n}(0,t)=P\big(N(t)=n \mid N(0)=0\big)=P\big(N(t)=n\big)=p_n(t)$.

In order to show that such a process does exist, it is sufficient to prove that the transition probabilities satisfy the Chapman-Kolmogorov equations, i.e.

$$p_{m,n}(s,t)=\sum_{i=m}^{n}p_{m,i}(s,\tau)p_{i,n}(\tau,t), \qquad \text{for } s\leq\tau\leq t,\ m\leq n . \quad (3.1)$$

To this aim, observe that

$$\sum_{i=m}^{n}p_{m,i}(s,\tau)p_{i,n}(\tau,t)=$$

$$=\left[\sum_{i=m}^{n}\frac{\Gamma(a+i)}{\Gamma(a+m)}\frac{(k(\tau-s))_{(n-i)}}{(i-m)!}\frac{(\rho+ks)_{(a+m)}}{(\rho+k\tau)_{(a+i)}}\cdot\frac{\Gamma(a+n)}{\Gamma(a+i)}\frac{(k(t-\tau))_{(i-m)}}{(n-i)!}\frac{(\rho+k\tau)_{(a+i)}}{(\rho+kt)_{(a+n)}}\right]$$

$$=\frac{\Gamma(a+n)}{\Gamma(a+m)}\frac{(\rho+ks)_{(a+m)}}{(\rho+kt)_{(a+n)}}\sum_{i=m}^{n}\frac{(k(t-\tau))_{(n-i)}}{(n-i)!}\frac{(k(\tau-s))_{(i-m)}}{(i-m)!} .$$

Then, using the identity

$$\binom{k(t-s)+(n-m)}{(n-m)}=\sum_{i=m}^{n}\binom{k(t-\tau)+(n-i)}{(n-i)}\cdot\binom{k(\tau-s)+(i-m)}{(i-m)} ,$$

we obtain

$$\sum_{i=m}^{n} p_{m,i}(s,\tau)p_{i,n}(\tau,t) = \frac{\Gamma(a+n)}{\Gamma(a+m)}\frac{(\rho+ks)_{(a+m)}}{(\rho+kt)_{(a+n)}}\binom{k(t-s)+(n-m)}{(n-m)} = p_{m,n}(s,t),$$

which proves (3.1).

3.1 The moments and some other properties

Let $N(t)$ define a generalized Waring process with parameters $(a,\ k,\ \rho)$. Then, for any t, $E[N(t)] = \frac{akt}{\rho - 1}$, $Var[N(t)] = \frac{akt(\rho + kt - 1)(\rho + a - 1)}{(\rho-1)^2(\rho-2)}$.

Following Irwin (1975), one may show that the variance can be divided into three additive components, thus

$$Var[N(t)] = \sigma^2_{\Lambda(t)} + (kt)^2\sigma^2_{\nu} + \sigma^2_R,$$

where

$$\sigma^2_{\Lambda(t)} = akt(a+1)(\rho-1)^{-1}(\rho-2)^{-1} \text{ is the component due to liability}$$

$$\sigma^2_{\nu} = a(a+\rho-1)(\rho-1)^{-2}(\rho-2)^{-1} \text{ is the component due to proneness}$$

and $\sigma^2_R = akt(\rho-1)^{-1}$ is the component due to randomness.

The generalized Waring process is a stationary process. For a stationary process N, $E[N(t)] = \eta \cdot t$, where η is termed the *intensity of* N (see e.g. Grandell, 1997, p.53). It is clear that the intensity of the generalized Waring process is $\eta = \frac{ak}{\rho - 1}$. For this process (like for all stationary processes), there always exists, a random variable $\overline{N}$ with $E(\overline{N}) = \eta$, called the *individual intensity*, such that $\frac{N(t)}{t} \xrightarrow[t\to\infty]{p} \overline{N}$ (see, e.g. Grandell, 1997, p.53). The intensity η is finite. Hence, it follows that the individual intensity $\overline{N}$ is finite with probability 1.

In what follows, we give an equivalent definition of the Generalized Waring process.

Definition 3.1.1 The counting process $\{N(t),\ t \geq 0\}$ is said to be a negative binomial process with parameters $\left(k, \frac{1}{1+\nu}\right)$ $k > 0,\ \nu > 0$, if (i) $N(0) = 0$, (ii) $N(t)$ is a Markov process, (iii) $N(t+h) - N(t)$ is $NB\left(kh, \frac{1}{1+\nu}\right)$-distributed for each $h > 0,\ t \geq 0$.

The first condition together with condition (iii) leads to the conclusion that $N(t)$ is $NB\left(kt, \frac{1}{1+\nu}\right)$-distributed.

Then, if we use a negative binomial process with parameters $k=1$ and $\nu = 1$ (a *standard negative binomial process)* as a starting point, we can define the Generalized Waring Process in the following way.

Definition 3.1.2 Let ν be a $betaII(a,\rho)$-distributed random variable and consider a standard negative binomial process $\widetilde{N}$ independent of it. Let $k>0$ be a constant. The point process $N = \widetilde{N} \circ \left(k, \frac{1}{1+\nu}\right)$, where $\widetilde{N} \circ \left(kt, \frac{1}{1+\nu}\right) \overset{def}{=} \widetilde{N}\left(kt, \frac{1}{1+\nu}\right)$ and, for every t, $\widetilde{N}\left(kt, \frac{1}{1+\nu}\right) \sim NB\left(kt, \frac{1}{1+\nu}\right)$, is called the *Generalized Waring Process.*

It is already clear that definition 3.1.2 is equivalent to definition 3.1.

By definition 3.1.2, one can prove the following property.

Theorem 3.1.1 Let N be a generalized Waring process. Then,

$$\frac{1}{t}N(t)\xrightarrow[t\to\infty]{p}\nu k \ .$$

Proof

$$\lim_{t\to\infty}\frac{1}{t}N(t) = \nu k \lim_{t\to\infty}\frac{\widetilde{N}\left(kt, \frac{1}{1+\nu}\right)}{\nu kt} \ .$$

Taking into account that

$$E\left[\widetilde{N}\left(kt, \frac{1}{1+\nu}\right)\right] = \nu kt \ \text{ and } \ \operatorname{var}\left\{\frac{\widetilde{N}\left(kt, \frac{1}{1+\nu}\right)}{\nu kt}\right\} = \frac{1+\nu}{\nu kt}\xrightarrow[t\to\infty]{}0,$$

and using Chebyshev's inequality, we have that $\frac{\widetilde{N}\left(kt, \frac{1}{1+\nu}\right)}{\nu kt} \xrightarrow[t\to\infty]{p} 1$, which implies that $\frac{1}{t}N(t) \xrightarrow[t\to\infty]{p} \nu k$.

Combining this result and the fact that, since ν is $betaII(a,\rho)$-distributed, $E(\nu)=\frac{a}{\rho-1}$, we obtain $E(\nu k)=\frac{ak}{\rho-1}$. Hence, the random variable $\overline{N}=\nu k$ is the individual intensity of the generalized Waring process.

3.2 The transition probabilities and the Chapman-Kolmogorov equations of the generalized Waring process

Using (2.2.2) and (2.3.1), we obtain for the transition probabilities of the generalized Waring process

$$p_{m,n}(s,t)=P(N(s+t)=n\mid N(s)=m)=\frac{\Gamma(a+n)}{\Gamma(a+m)}\frac{(kt)_{(n-m)}}{(n-m)!}\frac{(\rho+ks)_{(a+m)}}{(\rho+ks+kt)_{(a+n)}}$$

The transition probabilities of a Markov process satisfy the Chapman-Kolmogorov equations

$$p_{m,\,n}(s,\ t)=\sum_{i=m}^{n}p_{m,\,i}(s,\ \tau)p_{i,\,n}(\tau,\ t)\ \text{ for }\ s\le\tau\le t,\ m\le n.$$

Then, for the *forward* Kolmogorov differential equations, starting from $p_{m,\,n}(s,\ t+h)=\sum_{i=m}^{n}p_{m,\,i}(s,\ \tau)p_{i,\,n}(\tau,\ t+h)$ for $s\le\tau\le t,\ m\le n,\ h\ge 0,$ we obtain

$$p'_{m,\,n}(s,t)=\sum_{i=m}^{n}p_{m,\,i}(s,\ t)\lim_{h\to 0}\frac{p_{i,\,n}(t,\ t+h)}{h}-\lim_{h\to 0}\left(1-\frac{p_{nn}(t,\ t+h)}{h}\right)p_{m,\ n}(s,\ t),$$

$$\lim_{h\to 0}\frac{p_{i,\,n}(t,\ t+h)}{h}=\begin{cases}q_{n-1,n}(t)=\dfrac{k(a+n-1)}{(a+\rho+kt+n-1)}\ , & n-i=1\\[2ex] q_{i,n}(t)\dfrac{\Gamma(a+n)}{\Gamma(a+i)}\dfrac{k}{(n-i)(n-i-1)}\dfrac{(\rho+kt)_{(a+i)}}{(\rho+kt)_{(a+n)}}\ , & n-i>1\end{cases}$$

and $\quad\lim_{h\to 0}\left(\frac{1-p_{n,\,n}(t,t+h)}{h}\right)=\nu_n(t)=k\cdot\sum_{i=0}^{a+n-1}\frac{1}{\rho+kt+i}.$

Hence, the *forward* Chapman-Kolmogorov equations for the generalized Waring process are:

$$\frac{\partial p_{n,\,n}(s,\ t)}{\partial t}=-\nu_n(t)p_{n,\,n}(s,\ t)$$

$$\frac{\partial p_{m,\,n}(s,\ t)}{\partial t}=-\nu_n(t)p_{m,\,n}(s,\ t)+\sum_{i=m}^{n-1}q_{i,\,n}(t)p_{m,\,i}(s,\ t),\quad m<n.$$

The *backward* equations follow from the Chapman-Kolmogorov equations with $\tau = s + h$. Then, the *backward* equations for the generalized Waring process are:

$$\frac{\partial p_{m,\,m}(s,\ t)}{\partial t} = \nu_m(t) p_{m,\,m}(s,\ t)$$

$$\frac{\partial p_{m,\,n}(s,\ t)}{\partial t} = \nu_m(t) p_{m,\,n}(s,\ t) - \sum_{i=m+1}^{n} q_{m,\,i}(t) p_{i,\,n}(s,\ t), \quad m < n,$$

where

$$q_{m,\,m+1}(s) = \frac{k(a+m)}{(a+\rho+ks+m)}, \qquad q_{m,i}(s) = \frac{\Gamma(a+i)}{\Gamma(a+m)} \frac{k}{(i-m)(i-m-1)} \frac{(\rho+ks)_{(a+m)}}{(\rho+ks)_{(a+i)}}, \quad i > m$$

and $v_m(s) = k \cdot \sum_{i=0}^{a+m-1} \frac{1}{\rho + ks + i}$.

4. The Poisson and the Pólya processes as limiting cases of the generalized Waring process

It can be shown that the Poisson and the Pólya processes are limiting cases of the Generalized Waring process (in the sense of weak convergence).

Theorem 4.1 If $\rho = c \cdot k$, where $c > 0$ is a constant, the generalized Waring process tends to a Pólya process with parameters a and $1/c$, i.e. $\{N_k(t)\} \xrightarrow[k\to\infty]{d} \{N_c(t)\}$, where $\{N_k(t), t \geq 0\}$ is the generalized Waring process indexed by the parameter $k > 0$ and $\{N_c(t), t \geq 0\}$ is the Polya process indexed by the parameter $c > 0$ and with $P\{N_c(t) = n\} = \binom{\alpha + n - 1}{n} \left(\frac{c}{t+c}\right)^{\alpha} \left(\frac{t}{t+c}\right)^{n}, \ n = 0,1,...$

Theorem 4.2 Consider now the Pólya process $\{N_c(t), t \geq 0\}$ defined as in the previous theorem. Then, if $a = \lambda \cdot c$, where $\lambda > 0$ is a constant, $\{N_c(t)\} \xrightarrow[c\to\infty]{d} \{N_\lambda(t)\}$, where $\{N_\lambda(t)_\lambda, t \geq 0\}$ is a homogeneous Poisson process indexed by the parameter $\lambda > 0$ and with $P\{N_\lambda(t) = n\} = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}$, $n = 0,1,...$

The results of these theorems tell us that the the Pólya and the Poisson processes are limiting forms of the generalized Waring process. Thus, utilizing results holding for the generalized Waring process, one may obtain the following results for a Pólya process $\{X(t), t \geq 0\}$ with parameters $(a, 1/c)$ and for a Poisson process $\{Y(t), t \geq 0\}$ with parameter λ defined as in Theorems 4.1 and 4.2, respectively.:

- For any $t \geq 0$, $E\left[X(t)\right] = \frac{a}{c}t$, $Var\left[X(t)\right] = \frac{a}{c}t + \frac{a}{c^2}t^2$ and $E\left[Y(t)\right] = Var\left[Y(t)\right] = t\lambda$
- The Pólya and the Poisson processes are both stationary Markov processes. Their respective transition probabilities are:

$$P\left(X(t+h) \mid X(t) = m\right) = \begin{cases} \left(\frac{c+t}{c+t+h}\right)^{(a+n)} & n = m \\ (a+m)\frac{h(c+t)^{(a+m)}}{(c+t+h)^{(a+n1)}} & n = m+1 \\ \frac{\Gamma(a+n)}{\Gamma(a+m)(n-m)!}\frac{h^{(n-m)}(c+t)^{(a+m)}}{(c+t+h)^{(a+n)}} & n > m+1 \end{cases}$$

and

$$P\left(Y(t+h) \mid Y(t) = m\right) = \begin{cases} \exp(\text{-}\lambda h) & n = m \\ \lambda h \exp(\text{-}\lambda h) & n = m+1 \\ \frac{(\lambda h)^{(n-m)}}{(n-m)!}\exp(\text{-}\lambda h) & n > m+1 \end{cases}$$

- The Pólya process is a stationary non-homogenous birth process with transition intensities $k_n(t) = \frac{a+n}{c+t}$ and **t**he Poisson process is a stationary homogenous birth process with transition intensities $k_n(t) = \lambda$.

5. Some inferential aspects connected with the mixed negative binomial derivation of the generalized Waring process

Let $M(t)$ be associated with a negative binomial process specified by (2.1.1) and $N(t)$ be associated with a generalized Waring process as defined in section 2.2. The derivation of the latter implies that regarding the parameter ν in

$$P\{M(t)=n\}=\binom{kt+n-1}{n}\left(\frac{1}{\nu+1}\right)^{kt}\left(\frac{\nu}{1+\nu}\right)^{n},\quad n=0.1,... \tag{5.1}$$

as the outcome of a random variable having the $beta(a,\rho)$ distribution of the second kind, we can interpret $\{P(M(t)=n); n=0,1,...\}$ as the conditional distribution of $N(t)$ given the value ν. Hence, the unconditional distribution of $N(t)$ can be represented by

$$P_n(t)=P\{N(t)=n\}=E\left[\binom{kt+n-1}{n}\left(\frac{1}{\nu+1}\right)^{kt}\left(\frac{\nu}{1+\nu}\right)^{n}\right]$$

$$=\frac{\rho_{(kt)}}{(\rho+a)_{(kt)}}\frac{a_{(n)}(kt)_{(n)}}{(\rho+a+kt)_{(n)}}\frac{1}{n!}\qquad n=0,1,\ldots \tag{5.2}$$

Using this interpretation of the generalized Waring distribution we can, for any event B, regard the probability

$$U^B(x)=P\{\nu\le x|N(s)\in B\}=\frac{\int_0^x P(N(s)\in B|\nu)dU(\nu)}{\int_0^{+\infty} P(N(s)\in B|\nu)dU(\nu)},$$

with U denoting the probability function of the random variable ν, as the posterior distribution of ν given B or, more precisely, given $\{N(s)\in B\}$, provided that $P\{N(s)\in B\}=\int_0^{+\infty}P(N(s)\in B|\,l)dU(l)>0$.

Proposition 5.1 Let $N(s)$ be defined as above. Then

$$P\{\nu\le x|\,N(s)=n\}=\frac{\int_0^x \nu^{n+a-1}(1+\nu)^{-(n+a+\rho+ks)}d\nu}{\int_0^{+\infty} \nu^{n+a-1}(1+\nu)^{-(n+a+\rho+ks)}d\nu}.$$

Proof

Using an argument similar to that used by Xekalaki (1983b) for the case of the generalized Waring distribution, we obtain

$$P\{\nu \le x \mid N(s)=n\} = \frac{P\{\nu \le x,\ N(s)=n\}}{P\{\ N(s)=n\}}$$

$$= \frac{\int_0^x \frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)} \nu^{a-1}(1+\nu)^{-(a+\rho)} d\nu \int_0^x \frac{(ls)^n \exp(-ls)}{n!} \frac{(\nu/s)^{-ks}}{\Gamma(ks)} l^{ks-1} \exp\left(-\frac{ls}{\nu}\right) dl.}{\int_0^{+\infty} \frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)} \nu^{a-1}(1+\nu)^{-(a+\rho)} d\nu \int_0^{+\infty} \frac{(ls)^n \exp(-ls)}{n!} \frac{(\nu/s)^{-ks}}{\Gamma(ks)} l^{ks-1} \exp\left(-\frac{ls}{\nu}\right) dl.}$$

$$= \frac{\binom{ks+n-1}{n} \int_0^x \nu^{n+a-1}(1+\nu)^{-(a+n+\rho+ks)} d\nu}{\binom{ks+n-1}{n} \int_0^{+\infty} \nu^{n+a-1}(1+\nu)^{-(a+n+\rho+ks)} d\nu} = \frac{\int_0^x \nu^{n+a-1}(1+\nu)^{-(a+n+\rho+ks)} d\nu}{\int_0^{+\infty} \nu^{n+a-1}(1+\nu)^{-(a+n+\rho+ks)} d\nu}.$$

This proposition implies that $\left(\frac{\rho+ks}{a+n}\right)\nu \,\Big|\, (N(s)=n)$ has the F distribution with $2(a+n)$ and $2(\rho+ks)$ degrees of freedom. Following Xekalaki (1983b), this result can be used to construct confidence intervals for $\nu \mid (N(s)=n)$, 'estimating' in this way a person's proneness on the basis of the incurred number of accidents.

Corollary

If $N(s)$ is defined as above, then

$$E\{\nu \mid N(s)=n\} = \frac{\int_0^{+\infty} \nu^{n+a}(1+\nu)^{-(n+a+\rho+ks)} d\nu}{\int_0^{+\infty} \nu^{n+a-1}(1+\nu)^{-(n+a+\rho+ks)} d\nu} = \frac{a+n}{\rho+ks}. \tag{5.3}$$

Proof

Using the result of the proposition and the relation

$$E\{\nu \mid N(s)=n\} = \int_0^{+\infty} x\,dP\{\nu \le x \mid N(s)=n\},$$

we obtain

$$E\left\{\nu \mid N(s)=n\right\} = \frac{\int_0^{+\infty} \nu^{n+a}(1+\nu)^{-(n+a+\rho+ks)}\,d\nu}{\int_0^{+\infty} \nu^{n+a-1}(1+\nu)^{-(n+a+\rho+ks)}\,d\nu}$$

$$= \frac{\Gamma(n+a+1)\Gamma(\rho+ks-1)}{\Gamma(a+n+\rho+ks)} \frac{\Gamma(a+n+\rho+ks)}{\Gamma(n+a)\Gamma(\rho+ks)},$$

which leads to the result.

Remark 5.1: From (5.3), it would seem natural to interpret $\nu_B^* = \dfrac{a+n}{\rho+ks} = E\{\nu \mid N(s)\}$ as a Bayes estimate of ν.

6. An Application

As mentioned in the introduction, the concepts and terminology used in this paper can easily be modified so that the obtained results can be implemented in several other fields. As an example, we present here an application of the generalized Waring process in the context of modeling web access patterns.

Consider in particular, modeling the whole counting process $\{N(s),\ s>0\}$ associated with the access pattern of a web site, where, for any $t>0$, the variable $N(t)$ denotes the number of visits that the web pages on this particular site get within the interval $(0,t)$. (Note that the generalized Waring distribution has been cited in Ajiferuke et al.(2004) as used by them to fit an observed website visitation frequency distribution for a given period, i.e, to model counts $N(t_0)$ of web visits on a given fixed time interval $(0,t_0)$)

Except for chance, visits to a web site can be regarded as affected by the intrinsic appeal of the particular site to web users (corresponding to proneness) as well as by exogenous factors (corresponding to external factors) such as, links provided by other sites to the particular site, how well the site is advertised etc.

Let us denote by ν the intrinsic factors and by $\lambda|\nu$ the exogenous factors. Assume that $N(t)|\lambda$ follows a $Poisson(\Lambda(t))$ distribution, where $\Lambda(t)=\lambda t$ with $\lambda|\nu$ following a $Gamma\left(kt,\frac{1}{\nu t}\right)$ distribution. Then, the conditional distribution of $N(t)|\nu$ is a $NB\left(kt,\frac{\nu}{1+\nu}\right)$ distribution with ν following a $Beta(\alpha,\rho)$ distribution of the second kind, while the unconditional distribution of $N(t)$ is the $GWD(a,kt;\rho)$ distribution, i.e. $\{N(t),\ t\geq 0\}$ is a generalized Waring process.

The log files representing the hits on an e-shop site for the period 31/03/2006-30/04/2006, have been used to fit this model. A log file typically contains information on the times of visits per IP address per day. On the basis of such log files, the visits per day made by each of 468 IP addresses to the particular site have been enumerated for the above-mentioned one-month period yielding the corresponding observed paths $\{N_i(t_j), i = 1,2,...,468, j = 1,2,...,31\}$ of the numbers of visits $N_i(t_j)$ made by IP address i up to and including time t_j. A sample of one thus obtained path corresponding to one of the IP addresses considered is presented in Table 6.1.

Table 6.1

Visits made by a given IP address to an e-shop site per date and time

Date	***Day***	***Hours***	***Minutes***	***Seconds***	***Times of visits in days***
12/04/2006:16:15:27	12	16	15	27	0
13/04/2006:01:30:57	13	1	30	57	0.385763889
13/04/2006:09:38:4	13	9	38	4	0.724039352
13/04/2006:14:44:41	13	14	44	41	0.936967593
13/04/2006:20:39:53	13	20	39	53	1.183634259
15/04/2006:21:28:53	15	21	28	53	3.217662037
16/04/2006:11:59:50	16	11	59	50	3.822488426
16/04/2006:19:27:24	16	19	27	24	4.133298611
17/04/2006:02:13:47	17	2	13	47	4.415509259
18/04/2006:17:41:12	18	17	41	12	6.059548611
24/04/2006:06:00:26	24	6	0	26	11.57290509
24/04/2006:12:38:52	24	12	38	52	11.84959491
24/04/2006:18:27:59	24	18	27	59	12.09203704
25/04/2006:00:17:51	25	0	17	51	12.335
25/04/2006:06:35:20	25	6	35	20	12.5971412
26/04/2006:21:05:30	26	21	5	30	14.20142361
29/04/2006:09:09:02	29	9	9	2	16.70387731
29/04/2006:09:17:15	29	9	17	15	16.70958333
29/04/2006:10:33:00	29	10	33	0	16.7621875
29/04/2006:15:06:17	29	15	6	17	16.95196759

The observed paths were compared to the corresponding time series of simulated realizations of the generalized Waring process over the same time segment.

Estimates of the parameters of the generalized Waring process have been obtained employing the centered reduced moment estimation procedure for spatial point process data (see, e.g., Ripley (1988), Daley and Vere-Jones(1988), Diggle and Chetwynd (1991), and Chetwynd and P.J. Diggle (1998) among others). This procedure utilizes the moment estimators

$$E\left(\hat{N}(s)\right) = \hat{\mu}_1 = \hat{\eta} \cdot s = \frac{n \cdot s}{h},$$

$$E\left(\hat{N}^2(s)\right) = \hat{\mu}_2 = \frac{X}{n_{(2)}}$$

$$E\left(\hat{N}^3(s)\right) = \hat{\mu}_3 = \frac{Z - X}{n_{(3)}}$$

with $X = \sum_{i=1}^{n} \sum_{i \neq j} \phi_s^2\left(x_i, x_j\right)$, $Z = \sum_{i=1}^{n} \left(\sum_{j \neq i} \phi_s\left(x_i, x_j\right) \right) \left(\sum_{k \neq i} \phi_s\left(x_i, x_k\right) \right)$, where the quantities involved in the above equations represent weights defined, for each value x_i in the collection of points $\{x_i : i = 1,2,...,n\}$of the process within a time interval of length *h*, defined as follows: For each x_i in $\{x_i : i = 1,2,...,n\}$ and a given $s > 0$, consider the interval of center x_i and length *s* and assign to every point $x_j, j \neq i$ in this interval the weight $\phi_s\left(x_i, x_j\right) = \omega\left(x_i, x_j\right)^{-1}$, where $\omega\left(x_i, x_j\right)$ is the number of other points $\{x_k, k \neq i, k \neq j\}$ of the process that are included in the interval of length $\left|x_i - x_j\right|$ and center x_i. Within the setting of our example, the set $\{x_i : i = 1,2,...,n\}$represents, for each IP address, the visits made by the particular IP address for the entire duration of the period of time $h = x_n - x_1$ considered, *n* = 31, $\hat{\eta}$ denotes an estimator of the process intensity η, i.e. of the expected number of visits in an interval of unit length, while the value set for the constants s was $s = 0.5$

For each of the IP addresses, one hundred simulated realizations of the generalized Waring process with the above estimated parameter values were obtained and each of the observed time series paths was compared to the corresponding simulated ones. The comparison showed that, on average, the realizations of a generalized Waring process with the obtained parameter values notably ‘resembled’ the observed paths of the observed time series, in the sense that they had recognizable similar structural characteristics.

For illustration purposes, the paths of the observed time series associated with a sample of three of the IP addresses are presented (Figures 1-3). Each of these paths is superimposed by a sample of three of the 100 corresponding simulated realizations of the generalized Waring process with parameter estimates obtained as above and given in Table 2.

Table 6.2

Centered reduced moment estimates for the parameters of the $GWD(a, kt; \rho)$

IP address	$\hat{\alpha}$	$\hat{k}$	$\hat{\rho}$
1	5,6888256	0,594154	3.86463
2	4,139105	0,929841	3,521098
3	3.8695139	0.8293397	4.2061137

Inspection of the graphs depicted by Figures 6.1-6.3 provides a visual appreciation of the degree of similarity in the structural characteristics of the paths of the observed and the realized time series.

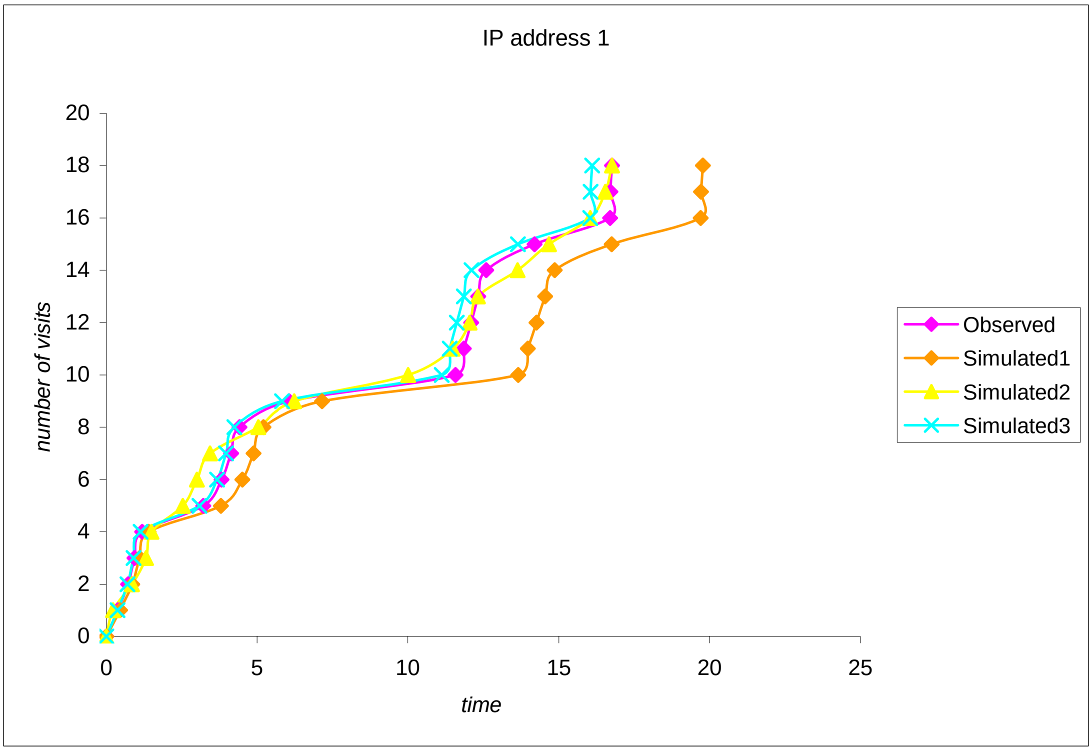

Figure 6.1. Observed and simulated paths corresponding to IP address 1

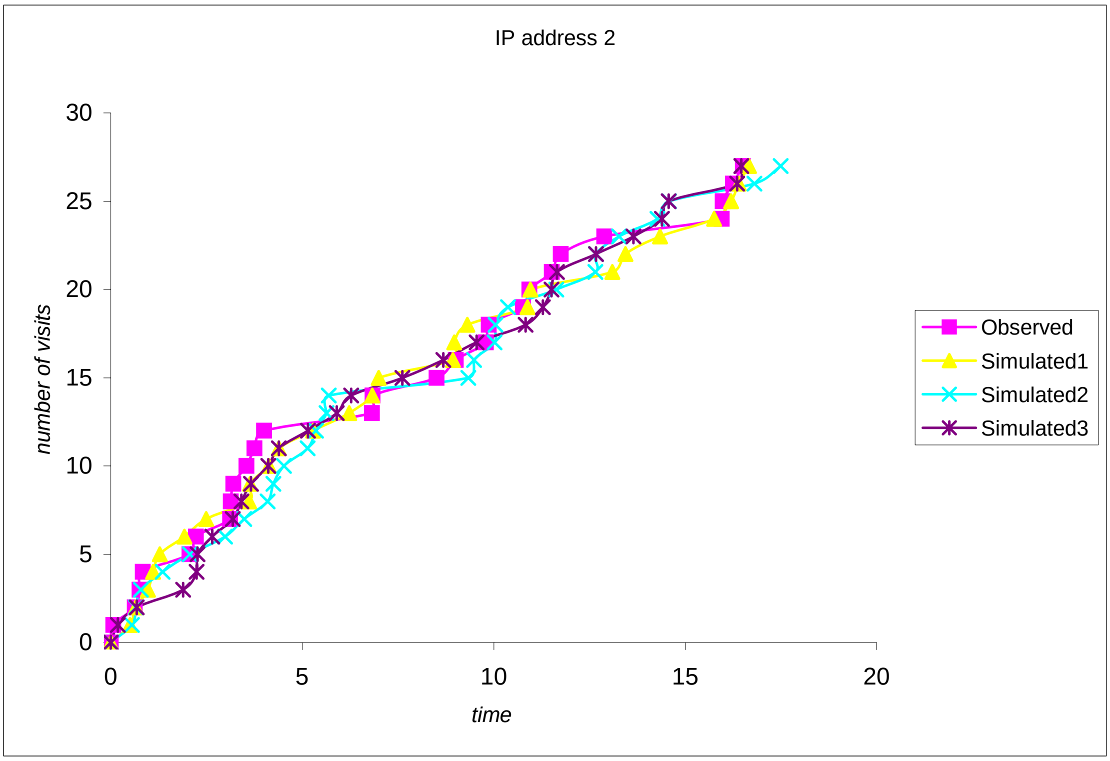

Figure 6.2. Observed and simulated paths corresponding to IP address 2

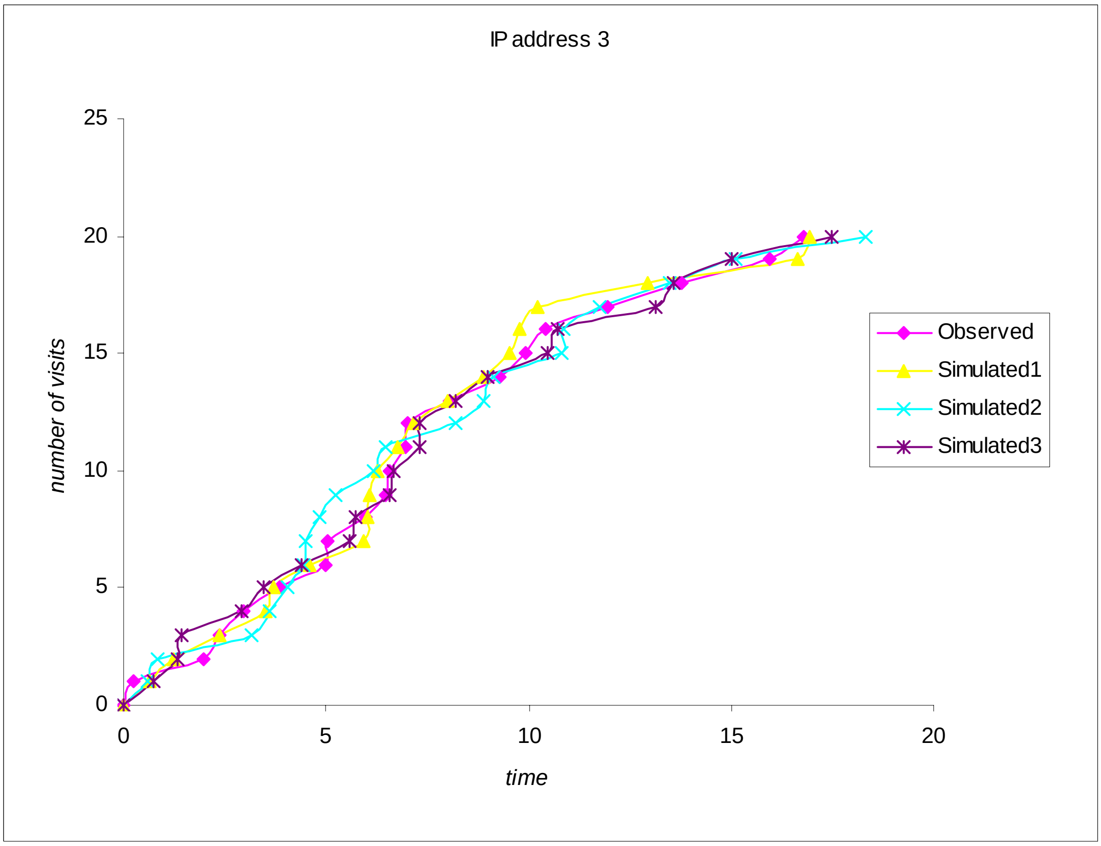

Figure 6.3. Observed and simulated paths corresponding to IP address 3

The closeness of the observed and realized time series was also checked using diagnostic plots based on the inverse-intensity residuals (used for point processes in time, e.g. Lewis (1972), Brillinger (1978, 1994) and Andersen et al.(1993)) These have been computed for each value x_j in the collection of points $\{x_j : j = 1,2,...,n\}$of the process by:

$$R_{\hat{\theta}}\left(B_j, \eta^{-1}\right) = \sum_{x_i \in B_j} \hat{\eta}(x_i) - \int_{B_j}^{j} I\{\hat{\eta}(x) > 0\} dx ,$$

where $B_j = (0, x_j)$, $\hat{\theta} = \left(\hat{a}, \hat{k}, \hat{\rho}\right)^{-1}$ and $\hat{\eta}(x) = \eta\left(x, \hat{\theta}\right)$ is the fitted intensity. Figures 6.4-6.6 exhibit similar results.

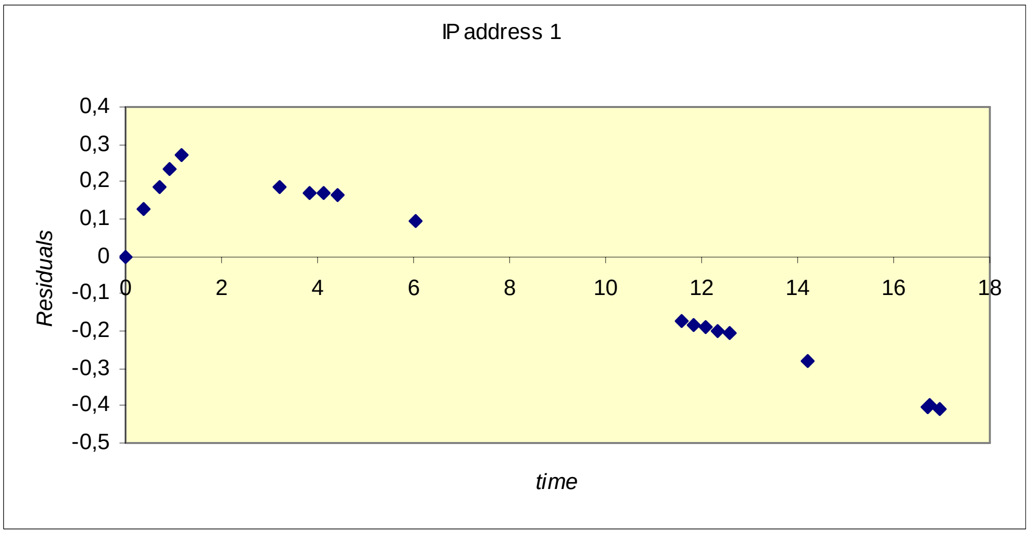

Figure 6. 4. Plot of inverse-intensity residuals corresponding to IP address 1

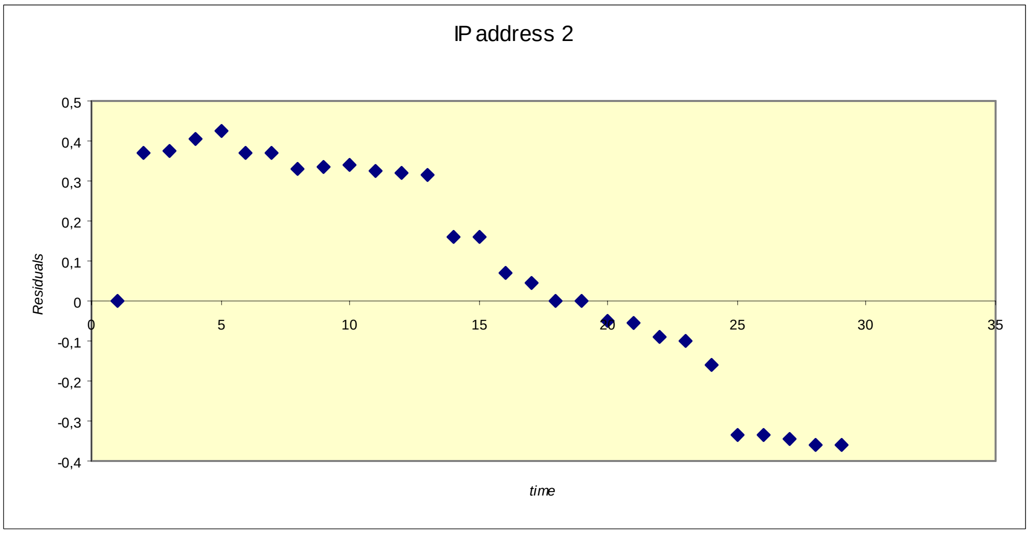

Figure 6.5. Plot of inverse-intensity residuals corresponding to IP address 2

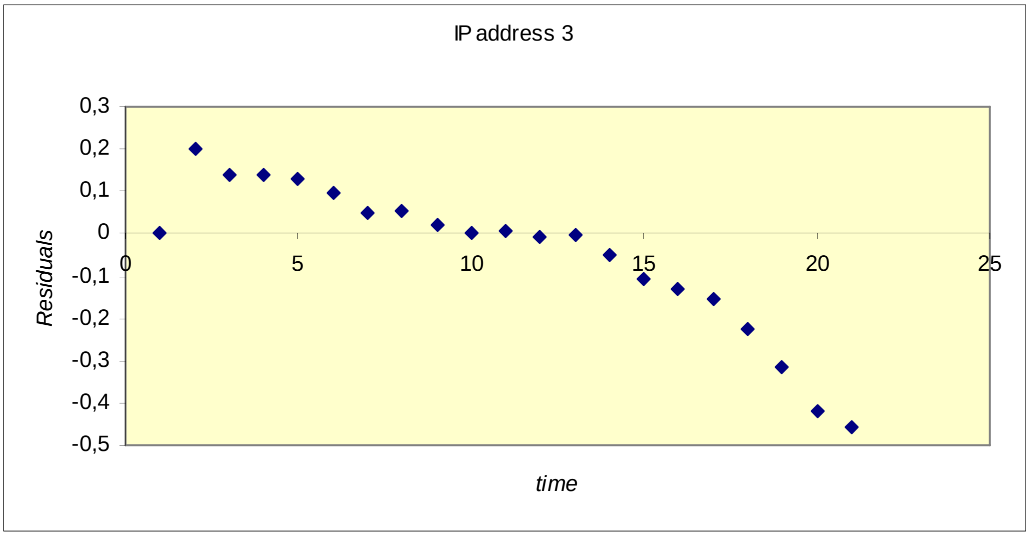

Figure 6.6. Plot of inverse-intensity residuals corresponding to IP address 3

7. Some alternative genesis schemes

The generalized Waring process has been defined as a non-homogenous stationary Markov Process arising as a beta mixture of the negative binomial process in a "*proneness*" context. In this section, we present two further genesis schemes as put forward by Zografi and Xekalaki (2001), where the underlying mechanism is indicative of *contagion* rather than proneness in the sense of Irwin (1941) and Xekalaki (1983b).

The "*contagion*" model assumes that, at time $t = 0$, the individuals have had no accidents and that, during a time period $(t, t + dt]$, the probability of a person having another accident depends on time t and on the number of accidents x sustained by the person by time t. So this probability is a function $f_\lambda(x,t)$, with λ referring to the individual's risk exposure.

A mixed Pólya process

Assuming that $f_\lambda(x,t) = \dfrac{k+x}{(1/\lambda)+t} = \lambda \cdot \dfrac{k+x}{1+\lambda t}$, the distribution of accidents for each t (λ fixed) is negative binomial with parameters $(k, 1/\lambda t)$(the accident pattern is described in this case by a Pólya process). When λ varies from individual to individual, according to an exponential distribution, i.e., $\lambda \sim ae^{-a\lambda}$, $a > 0$, the overall distribution of $N(t)$, for $t = 1$, is (following Xekalaki, 1983b) the generalized Waring with parameters (a, k, ρ). In the general case, however, its distribution is of a more general form. In particular, we have

$$P(N = n) = P_n(t)$$

$$= \frac{\rho_{(k)}}{(a+\rho)_{(k)}} \frac{a_{(n)} k_{(n)}}{(a+\rho+k)_{(n)}} \frac{(1/t)^a}{n!} F(a+\rho, a+n, a+\rho+k+n; 1-1/t)), \tag{6.1.1}$$

where

$$F(a,b,c;z) = \sum_{m=0}^{\infty} \frac{a_{(m)} b_{(m)}}{c_{(m)}} \frac{z^m}{m!} .$$

The above distribution is not of a generalized Waring form, but reduces to it for $t = 1$. It can be shown that there exists a birth process $Y = \{Y(t);\ t > 0;\ Y(0) = 0\}$, such that $Y(t)$, for each t, has the distribution given by (6.1.1).

A non-Markovian stochastic process of a generalized Waring form

Assuming that $f_\lambda(x,t) = \lambda(k + mx)$, the distribution of accidents for each t is negative binomial with parameters $\left(-\frac{k}{m},\ \frac{1}{1-e^{-\lambda mt}}\right)$, when λ is fixed (Irwin, 1941) and generalized Waring with parameters $\left(\frac{k}{m},\ 1,\ \frac{a}{mt}\right)$, when $\lambda \sim ae^{-a\lambda}$, $a > 0$ (Xekalaki, 1981).

Further, following Irwin (1941), one may be verify in this case that the distribution of the increment $Y_t(h) = N(t+h) - N(t)$ at time t, given that $N(t) = x$, has a negative binomial distribution with parameters $\left(-\frac{k}{m} + x,\ \frac{1}{1-e^{-\lambda mt}}\right)$ when λ is fixed, and a generalized Waring distribution with parameters $\left(\frac{k}{m} + x,\ 1,\ \frac{a}{mt}\right)$, when $\lambda \sim ae^{-a\lambda}$, $a > 0$. Hence, in this case,

$$P_{i,j}(s,\ t) = P\left(N(t+s) = i \mid N(t) = j\right) = P\left(N(t+s) - N(t) = i - j \mid N(t) = j\right)$$

$$= \frac{(a/ms)_{(1)}}{\left(\frac{k}{m} + j + \frac{a}{ms}\right)_{(1)}} \frac{\left(\frac{k}{m} + j\right)_{(i-j)}}{\left(\frac{k}{m} + j + \frac{a}{ms} + 1\right)_{(i-j)}}.$$

From the last relationship, one may easily verify that

$$p_{2,i}(s,\tau) \cdot p_{j,2}(\tau,t) + p_{3,i}(s,\tau) \cdot p_{j,3}(\tau,t) \neq p_{j,i}(s,t)$$

for some values of a, m, s, t, τ, i, j. This implies that this process does not satisfy the Chapman-Kolmogorov equations and thus is not a Markov process.

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