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THE MULTIVARIATE GENERALIZED WARING DISTRIBUTION

Evdokia Xekalaki

Department of Mathematics
University of Crete
Heraklio,Greece

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ABSTRACT

A multivariate extension of the generalized Waring distribution is defined through a multivariate generalization of Waring's expansion and its properties are studied. It is proved that the marginal distributions (conditional or unconditional) and their convolution are generalized Waring distributions. The multivariate generalized Waring distribution is then shown to arise in accident theory as the joint distribution of accidents incurred by an accident prone population exposed to variable external accident risk over a series of non-overlapping time periods. It is further demonstrated that using this multivariate distribution one can "measure" the contributions of accident proneness, accident risk exposure and chance to a given accident situation.

1. INTRODUCTION

Among the various hypotheses that have been developed in

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interpreting the accident experience of a population, accident proneness seems to have drawn a lot of attention. The main reason is that it provides some explanation as to why some individuals in the population tend to have more accidents than others. The origin of this hypothesis can be traced in Greenwood and Wood's (1919) work in which they obtained the negative binomial distribution as the distribution of accidents sustained by a group of industrial workers.

In the context of accident proneness accidents are considered to be the result of two kinds of factors, random and non-random. Till 1968 statistical workers took the implicit view that the non-random factors referred to the individual's psychology since in designing their investigations care was taken to ensure an equal-risk environment for all individuals. Irwin (1968), however, broke new ground by introducing an unequal-risk-environment hypothesis i.e. by assuming that non-random factors can be further split into psychological factors and environmental factors. On this assumption, he obtained the generalized Waring distribution as a mixture on λ of a Poisson (λ) distribution with a gamma $(\frac{1}{\nu},k)$ distribution for λ whose scale parameter ν was a beta random variable (r.v.). In this context the distribution of $\lambda|\nu$ described the fluctuations of the environmental risk (accident liability) while the distribution of ν described fluctuations in the person's idiosyncracy predisposition to accidents (accident proneness). Irwin showed that using this distribution it was possible to "measure" the effects of liability and proneness. It was not possible, however, to obtain distinguishable estimates for these effects unless subjective judgement was used. This difficulty was overcome with the introduction of a bivariate version of the generalized Waring distribution by Xekalaki (1984a). This allowed the study of the accident distribution of a population in two consecutive time periods while it provided distinguishable estimates of the degree to which liability and proneness influenced the particular accident situation.

Since accident statistics aims at reducing (if not preventing) the accident causation, the need for a multivariate version of the generalized Waring distribution becomes obvious. Such an extension will make possible the year-by-year study of the accident experience of the individuals under observation and will give an indication as to how the contributions of accident liability, accident proneness and chance vary through time.

In this paper a multivariate version of the generalized Waring distribution is defined and its relation to accident theory is studied. In particular, in section 2 a multivariate generalization of Waring's expansion is given in terms of a multivariate generalized hypergeometric series whose successive terms multiplied by a suitable constant are regarded as defining the multivariate version of the generalized Waring distribution. In section 3 the structure of this multivariate distribution is examined and it is shown among other results that the marginal distributions and their convolution are univariate generalized Waring distributions. Finally, section 4 examines how the distribution defined in section 2 can arise in accident theory as the joint accident distribution over n time periods of an accident prone population subjected to an unequal-risk environment. Further, it shows that it is possible to assess the effects of proneness and external risk for each time period.

Before presenting the main results some notation and terminology will be provided.

A non-negative, integer-valued r.v. X is said to have the univariate generalized Waring distribution with parameters a,k and ρ (UGWD($a,k;\rho$)) if its probability function (p.f.) is given by

$$p_r \equiv P(X=r) = \frac{\rho_{(k)}}{(a+\rho)_{(k)}} \frac{a_{(r)}\, k_{(r)}}{(a+k+\rho)_{(r)}} \frac{1}{r!}, \qquad (1.1)$$

$$a,k,\rho>0;\ r = 0,1,2,\ldots$$

where $a_{(\beta)}=\Gamma(a+\beta)/\Gamma(a)$, $a>0$ $\beta \in R$.

The probability generating function (p.g.f.) of X is given by

$$G(s)=\frac{\rho_{(k)}}{(a+\rho)_{(k)}}\ {}_2F_1(a,k;a+k+\rho;s) \tag{1.2}$$

where ${}_2F_1$ is the Gauss hypergeometric function defined by

$${}_2F_1(\alpha,\beta;\gamma;s)=\sum_{r=0}^{\infty}\frac{\alpha_{(r)}\beta_{(r)}}{\gamma_{(r)}}\frac{s^r}{r!},\quad |s|\leq 1.$$

The mean and variance of X are given by

$$\mu=\frac{ak}{\rho-1}\text{ and }\sigma^2=\frac{ak(a+\rho-1)(k+\rho-1)}{(\rho-1)^2(\rho-2)} \tag{1.3}$$

and exist only if $\rho>1$ and $\rho>2$ respectively. For a detailed account of the properties and applications of the UGWD see Irwin (1975) and Xekalaki ((1981), (1983a,b)). Some of the more recent applications of certain special cases of it can be found in Xekalaki (1983c,d, 1984c).

A random vector (X_1,X_2) with non-negative, integer-valued components is said to have the bivariate generalized Waring distribution with parameters a,k,m and ρ (BGWD(a;k,m;ρ) if its p.f. is given by

$$p_{r,\ell}\equiv P(X_1=r,X_2=\ell)=\frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}}\frac{a_{(r+\ell)}k_{(r)}m_{(\ell)}}{(a+k+m+\rho)_{(r+\ell)}}\frac{1}{r!}\frac{1}{\ell!}$$

$$a,k,m,\rho>0;\ r,\ell=0,1,2,\ldots \tag{1.4}$$

The p.g.f. of (X_1,X_2) is given by

$$G(s,t)=\frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}}F_1(a;k,m;a+k+m+\rho;s,t) \tag{1.5}$$

where F_1 represents the Appell hypergeometric function of type 1 defined by

$$F_1(\alpha;\beta,\beta';\gamma;s,t)=\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\alpha_{(r+\ell)}\beta_{(r)}\beta'_{(\ell)}}{\gamma_{(r+\ell)}}\frac{s^r}{r!}\frac{t^\ell}{\ell!}$$

$$(s,\ell)\in[-1,1]\times[-1,1]$$

For information on the chance mechanisms that give rise to this distribution and on the estimation of its parameters see Xekalaki (1984b,1985).

2. THE MULTIVARIATE GENERALIZED WARING DISTRIBUTION

Irwin's(1963) original derivation of the UGWD was in the context of a biological problem. He obtained it as an inverse factorial distribution starting from Waring's expansion for the function $(x-a)^{-1}$, $x>a>0$, i.e. from the formula

$$\frac{1}{x-a}=\sum_{r=0}^{+\infty}\frac{a_{(r)}}{x_{(r+1)}} \tag{2.1}$$

which he generalized for the function $\frac{1}{(x-a)_{(k)}}$, $(k>0)$.

Xekalaki(1984a) used a similar approach in deriving the BGWD by further generalizing (2.1) for the function $\frac{1}{(x-a)_{(k+m)}}$ where k and m are distinct positive integers.

In the sequel, we will provide a further generalization of (2.1) which will lead to a multivariate version of the generalized Waring distribution.

For k_i, $i=1,2,\ldots,n$, distinct positive real numbers, define a real function of x by

$$f(x)=\frac{1}{x_{(\sum_i k_i)}}=\prod_{i=1}^{n}\frac{1}{(x+\sum_{j=0}^{i-1}k_j)_{(k_i)}},\quad k_o=0. \tag{2.2}$$

It is known that, for any real function g(x) and a>0

$$g(x-a)=(1+\Delta)^{-a}g(x) \tag{2.3}$$

and

$$\Delta^r g(x)=\sum_{\ell=0}^{r}(-r)_{(\ell)}g(x+\ell)/\ell!,\quad r=0,1,2,\ldots \tag{2.4}$$

where $\Delta g(x)=g(x+1)-g(x)$. Moreover, if $g_i(x)$, $i=1,2,\ldots n$ are n functions of x

$$\Delta^m[\prod_{i=1}^{n}g_i(x)]=\sum_{\Sigma r_i=m}m!\prod_{i=1}^{n}\Delta^{r_i}g_i(x+\sum_{j=0}^{i-1}r_j),\quad r_o=0 \tag{2.5}$$

and it can be shown using (2.4) that

$$\Delta^r \frac{1}{x_{(\ell)}} = \frac{(-1)^r \ell_{(r)}}{x_{(\ell+r)}}, r=0,1,2,\ldots \tag{2.6}$$

Therefore, combining (2.3),(2.2),(2.5) and (2.6) we obtain

$$\frac{1}{(x-a)_{(\Sigma k_i)}} = (1+\Delta)^{-a} \frac{1}{x_{(\Sigma k_i)}} = (1+\Delta)^{-a} \prod_{i=1}^{n} \frac{1}{(x+\sum_{j=0}^{i-1} k_j)_{(k_i)}}$$

$$= \sum_{r=0}^{+\infty} \frac{a_{(r)}(-1)^r}{r!} \Delta^r \prod_{i=1}^{n} \frac{1}{(x+k_0+k_1+\ldots+k_{i-1})_{(k_i)}}$$

$$= \sum_{r_1,\ldots,r_n} \frac{a_{(\Sigma r_i)}(-1)^{\Sigma r_i}}{r_1! r_2! \ldots r_n!} \prod_{i=1}^{n} \Delta^{r_i} \frac{1}{(x+\sum_{j=0}^{i-1}(r_j+k_j))_{(k_i)}}$$

$$= \sum_{r_1,\ldots,r_n} \frac{a_{(\Sigma r_i)}(k_1)_{(r_1)}\cdots(k_n)_{(r_n)}}{x_{(\Sigma r_i+\Sigma k_i)} r_1! \ldots r_n!}$$

i.e.

$$1 = \frac{(x-a)_{(\Sigma k_i)}}{x_{(\Sigma k_i)}} \sum_{r_1,\ldots,r_n} \frac{a_{(\Sigma r_i)}(k_1)_{(r_1)}\cdots(k_n)_{(r_n)}}{(x+\Sigma k_i)_{(\Sigma r_i)}(r_1)! \ldots (r_n)!} \tag{2.7}$$

The multiple series in the right hand side is convergent for $x>a$.

Hence we obtained a multiple series which converges to 1 and therefore its successive terms can be considered as defining a multivariate discrete distribution. In the sequel, we set $x=a+\rho$, $\rho>0$ and refer to the distribution defined by the resulting expressions of the successive terms of the series in (2.7) as the multivariate generalized Waring distribution with parameters $a, \underset{\sim}{k} \doteq (k_1,\ldots,k_n)$ and ρ (MGWD$(a;k_1,k_2,\ldots,k_n;\rho) \equiv$ MGWD$(a;\underset{\sim}{k};\rho)$), i.e. the following definition may be given.

A random vector $\underset{\sim}{X}=(X_1,X_2,\ldots,X_n)$ with non-negative, integer-valued components will be said to have the MGWD$(a;\underset{\sim}{k};\rho)$ $a,\rho,k_i>0$, $i=1,\ldots,n$ if its p.f. is given by

$$p_{\underset{\sim}{r}} \equiv P(\underset{\sim}{X}=\underset{\sim}{r}) = \frac{\rho_{(\Sigma k_i)}}{(a+\rho)_{(\Sigma k_i)}} \frac{a_{(\Sigma r_i)}(k_1)_{(r_1)}\cdots(k_n)_{(r_n)}}{(a+\rho+\Sigma k_i)_{(\Sigma r_i)}(r_1)!\cdots(r_n)!}, \quad \begin{matrix} r_i=0,1,2,\ldots \\ i=1,\ldots,n \end{matrix} \tag{2.8}$$

It should be mentioned at this point that the name multivariate generalized Waring distribution for the distribution in (2.8) does not reflect only the fact that its derivation was based on a multivariate generalization of Waring's expansion. As shown in the sequel, the structural properties of (2.8) present further justification for the choice of the name.

The successive probabilities of the MGWD($a;\underset{\sim}{k};\rho$) are related by first order recurrence relationships,thus

$$\frac{p_{\ell_1,\ell_2,\ldots,\ell_{h-1},\ell_h+1,\ell_{h+1},\ldots,\ell_n}}{p_{\ell_1,\ell_2,\ldots,\ell_n}} = \frac{[a+\sum_{i=1}^{n}\ell_i](k_h+\ell_h)}{[a+\sum_{i=1}^{n}k_i+\sum_{i=1}^{n}\ell_i](\ell_h+1)}$$

$\ell=0,1,2,\ldots;\ i=1,2,\ldots,n.$

It is a generalized hypergeometric distribution since its p.g.f. can be expressed in terms of Lauricella's hypergeometric function of type D, i.e.

$$G(\underset{\sim}{t}) = \frac{\rho_{(\Sigma k_i)}}{(a+\rho)_{(\Sigma k_i)}} F_D(a;k_1,\ldots,k_n;a+\sum_{i=1}^{n}k_i+\rho;\underset{\sim}{t}) \tag{2.9}$$

where

$$F_D(\alpha;\beta_1,\beta_2,\ldots,\beta_n;\gamma;\underset{\sim}{t}) = \sum_{r_1,\ldots,r_n} \frac{\alpha_{(\Sigma r_i)}}{\gamma_{(\Sigma r_i)}} \prod_{i=1}^{n} \frac{(\beta_i)_{(r_i)} t_i^{r_i}}{(r_i)!}$$

The fact that $\rho>0$ ensures the convergence of (2.9) for all the values of the parameters $a,k_1,\ldots,k_n$, and ρ for $t_i \theta [-1,1]$, $i=1,2,\ldots,n$.

Obviously, the MGWD($a,\underset{\sim}{k};\rho$) reduces to the BGWD($a;k_1,k_2;\rho$) when n=2 and to the UGWD($a,k_1;\rho$)when n=1.

Note that the MGWD($a;\underset{\sim}{k};\rho$) as defined by (2.9) is a member of Sibuya and Shimizu's(1981) and of Janardan and Patil's(1972) families of multivariate generalized hypergeometric distributions.

3. STRUCTURAL PROPERTIES

Consider a random vector $\underset{\sim}{X}=(X_1,X_2,\ldots,X_n)$ of non-negative, integer-valued components that has the MGWD$(a;\underset{\sim}{k};\rho)$($\underset{\sim}{X}\sim$MGWD$(a;\underset{\sim}{k};\rho)$). Then the following properties hold.

(i) $X_i\sim\text{UGWD}(a,k_i;\rho), i=1,2,\ldots,n.$

(ii) $\sum_{j=1}^{s} X_{i_j}\sim\text{UGWD}(a,\sum_{j=1}^{s} k_{i_j};\rho), \{i_1,i_2,\ldots,i_s\}\subseteq\{1,2,\ldots,n\},\ s\leq n.$

(iii) $(X_{i_1},X_{i_2},\ldots,X_{i_s})\sim\text{MGWD}(a;k_{i_1},k_{i_2},\ldots,k_{i_s};\rho),$

$\{i_1,i_2,\ldots,i_s\}\subseteq\{1,2,\ldots,n\};\ s\leq n.$

(iv) $(X_{i_1},X_{i_2},\ldots,X_{i_s})|(X_{i_{s+1}},X_{i_{s+2}},\ldots,X_{i_n})\sim$

$\text{MGWD}(a+\sum_{j=s+1}^{n} x_{i_j};k_{i_1},k_{i_2},\ldots,k_{i_s};\rho+\sum_{j=s+1}^{n} k_{i_j}),$

$\{i_1,i_2,\ldots,i_s\}\cup\{i_{s+1},i_{s+2},\ldots,i_n\}=\{1,2,\ldots,n\},\ s\leq n.$

(v) $(X_i,\sum_{j\neq i} X_j)\sim\text{BGWD}(a;k_i,\sum_{j\neq i} k_j;\rho), i=1,\ldots,n.$

(vi) $X_i|(\sum_{j\neq i} X_j)\sim\text{UGWD}(a+\sum_{j\neq i} x_j,k_i;\rho+\sum_{j\neq i} k_j), i=1,\ldots,n.$

Proof. Let $G_Z(s)$ denote the p.g.f. of a r.v. Z.

(i) $$G_{X_i}(t_i)=\frac{\rho_{(\Sigma k_i)}}{(a+\rho)_{(\Sigma k_i)}}F_D(a;k_1,\ldots,k_n;a+\Sigma k_i+\rho;1,\ldots,1,t_i,1,\ldots,1)$$

$$=\frac{\rho_{(k_i)}}{(a+\rho)_{(k_i)}}{}_2F_1(a,k_i;a+k_i+\rho;t_i)\sim\text{UGWD}(a,k_i;\rho).$$

(ii) Without loss of generality assume $i_j=j, j=1,2,\ldots,s$. Then

$$G_{\Sigma_j X_j}(t)=\frac{\rho_{(\sum_{i=1}^{n} k_i)}}{(a+\rho)_{(\sum_{i=1}^{n} k_i)}}F_D(a;k_1,\ldots,k_n;a+\Sigma k_i+\rho;\underbrace{t,\ldots,t}_{s},\underbrace{1,\ldots,1}_{n-s})$$

$$=\frac{\rho_{(\sum_{i=1}^{s} k_i)}}{(a+\rho)_{(\sum_{i=1}^{s} k_i)}}\,{}_2F_1(a,\sum_{i=1}^{s} k_i;\rho;t)\sim UGWD(a,\sum_{i=1}^{s} k_i;\rho).$$

(iii) Putting $i_j=j, j=1,2,\ldots,s$ we have $G_{X_1,\ldots,X_s}(t)=$

$$=\frac{\rho_{(\sum_{i=1}^{n} k_i)}}{(a+\rho)_{(\sum_{i=1}^{n} k_i)}}\,F_D(a;k_1,\ldots,k_n;a+\sum_{i=1}^{n} k_i+\rho;t_1,\ldots,t_s,1,\ldots,1)$$

$$=\frac{\rho_{(\sum_{i=1}^{s} k_i)}}{(a+\rho)_{(\sum_{i=1}^{s} k_i)}}\,F_D(a;k_1,\ldots,k_s;a+\sum_{i=1}^{s} k_i+\rho;t_1,\ldots,t_s)$$

$$\sim MGWD(a;k_1,\ldots,k_s;\rho).$$

(vi) Set again $i_j=j$, $j=1,2,\ldots,n$. Xekalaki(1983e) has shown that

$G_{X_1,\ldots,X_s|X_{s+1},\ldots,X_n}(t_1,\ldots,t_s)=$

$$\frac{\partial^x}{\partial t_{s+1}^{x_{s+1}}\ldots\partial t_n^{x_n}}G_{\underset{\sim}{X}}(t_1,\ldots,t_s,0,\ldots,0)\div\frac{\partial^x}{\partial t_{s+1}^{x_{s+1}}\ldots\partial t_n^{x_n}}G_{\underset{\sim}{X}}(1,\ldots,1,,0,\ldots,0)$$

where $x=x_{s+1}+\ldots+x_n$. Then the result follows immediately if one observes that

$$\frac{\partial^{r_1+\ldots+r_n}}{\partial t_1^{r_1}\ldots\partial t_n^{r_n}}F_D(a;b_1,\ldots,b_n;c;\underset{\sim}{t})=$$

$$\frac{a_{(\sum r_i)}\prod_{i=1}^{n}(b_i)_{(r_i)}}{c_{(\sum r_i)}}F_D(a+\sum r_i;b_1+r_1,\ldots,b_n+r_n;c+\sum r_i;\underset{\sim}{t}) \qquad (3.1)$$

(v) Let $Y=\sum_{j\neq i} X_j$. Then

$$G_{X_i,Y}(t_i,t)=\frac{\rho_{(\Sigma k_j)}}{(a+\rho)_{(\Sigma k_j)}}F_D(a;k_1,\ldots,k_n;a+\Sigma k_j+\rho;\underbrace{t,\ldots,t}_{i-1},t_i,\underbrace{t,\ldots,t}_{n-i})$$

$$= \frac{\rho_{(\Sigma k_j)}}{(a+\rho)_{(\Sigma k_j)}} F_1(a;k_i,\sum_{j\neq i} k_j;a+\Sigma k_j+\rho;t_i,t)\sim BGWD(a;k_i,\sum_{j\neq i} k_j;\rho)$$

(vi) The proof of this property in an immediate consequence of (v) (see Xekalaki, 1984a).

Hence, as implied by properties (i), (ii) and (iv), the marginal distributions (conditional and unconditional) as well as their convolution are of the same form (UGWD's). This fact is very interesting from both the theoretical and the practical viewpoints. Theoretically, these properties exhibit a symmetry analogous to that existing in the case of the multivariate normal distribution. To appreciate the practical value of them consider an accident situation. Then properties (i), (ii) and (iv) imply that the distribution of accidents within the entire period of observation and the distributions of accidents within non-overlapping subperiods are of the same form and yet the individual's performances in the various subperiods are correlated.

Another interesting fact is that propositions(iv) and(vi) are equivalent in the case s=1 and this implies that the conditional distribution of X_i does not depend on the individual values of the conditioning r.v.'s but on their sum,i.e. $X_i|\{X_j,j\neq i\} \overset{d}{=} X_i|\sum_{j\neq i} X_j$. This in turn implies(taking into account relationship(1.3)) that

$$E(X_i|\{X_j=x_j,j\neq i\})=E(X_i|\sum_{j\neq i} X_j=\sum_{j\neq i} x_j)=\frac{(a+\sum_{j\neq i} x_j)k_i}{\rho+\sum_{j\neq i} k_j-1}$$

and $V(X_i|\{X_j=x_j,j\neq i\})=V(X_i|\sum_{j\neq i} X_j=\sum_{j\neq i} x_j)$

$$=\frac{k_i(a+\sum_{j\neq i} x_j)(\rho+\sum_j k_j-1)(a+\sum_{j\neq i}(k_j+x_j)+\rho-1)}{(\rho+\sum_{j\neq i} k_j-1)^2(\rho+\sum_{j\neq i} k_j-2)}$$

From proposition(i) it follows that the marginal means and variances are for $\rho>1$ and $\rho>2$ respectively

$$\mu_{X_i}=\frac{ak_i}{\rho-1},\sigma^2_{X_i}=\frac{ak_i(\rho+a-1)(\rho+k_i-1)}{(\rho-1)^2(\rho-2)},\ i=1,2,\ldots,n. \tag{3.2}$$

Also from proposition (iv), for s=2 the covariances are (see Xekalaki, 1984a)

$$\sigma_{X_iX_j}=\frac{a(a+\rho-1)k_ik_j}{(\rho-1)^2(\rho-2)},i,j=1,2,\ldots,n;i\neq j,\rho>2 \tag{3.3}$$

The fact that $\sigma_{X_iX_j}$ exists only if $\rho>2$ implies that the components of $\underset{\sim}{X}$ are always positively correlated.

The factorial moments of the MGWD($a;\underset{\sim}{k};\rho$) can be obtained, for $r_i=0,1,2,\ldots;i=1,2,\ldots,n$ by the formula

$$\mu_{(r_1,r_2,\ldots,r_n)}=\frac{a_{(\sum r_i)}\prod_{i=1}^{n}(k_i)_{(r_i)}}{(\rho-1)(\rho-2)\ldots(\rho-\sum r_i)} \tag{3.4}$$

Obviously, they are finite only for $\rho>\sum_{i=1}^{n}r_i$ which is the necessary condition for the series

$$F_D(a+\sum_{i=1}^{n}r_i;k_1+r_1,\ldots,k_n+r_n;a+\rho+\sum_{i=1}^{n}(k_i+r_i);\underset{\sim}{1})$$

to be convergent.

The mode of the MGWD($a;\underset{\sim}{k};\rho$) will now be located using Janardan and Patil's (1970) technique. They followed an approach due to Moran to locate the mode $\underset{\sim}{x}$ for various multivariate discrete distributions with the assumption that $a\leq\sum_i x_i\leq\beta$ for some $\alpha,\beta>0$, to reduce the number of modes.

Moran's condition for $\underset{\sim}{x}$ to be a mode of a multivariate discrete distribution with p.f. $p_{\underset{\sim}{x}}$ is

$$\frac{p_{x_1,\ldots,x_{i-1},x_i+1,x_{i+1},\ldots,x_{j-1},x_j-1,x_{j+1},\ldots,x_n}}{p_{x_1,x_2,\ldots,x_n}}\leq 1,\ i,j=1,2,\ldots,n.$$

Following this definition we have for the MGWD$(a;\underset{\sim}{k};\rho)$ that $\underset{\sim}{x}=(x_1,\ldots,x_n)$ is a mode if and only if

$$\frac{(k_i)_{(x_i+1)}(k_j)_{(x_j-1)}x_i!x_j!}{(k_i)_{(x_i)}(k_j)_{(x_j)}(x_i+1)!(x_j-1)!}\leq 1$$

or equivalently if and only if

$$x_j(k_i-1)\leq(1+x_i)(k_j-1),\ i,j=1,2,\ldots,n. \tag{3.5}$$

If we also impose the restriction $\alpha\leq\sum_{i=1}^{n}x_i\leq\beta$ we have from (3.5) by summing over all i that

$$x_j\left(\sum_{i=1}^{n}k_i-n\right)\leq\left(n+\sum_{i=1}^{n}x_i\right)(k_j-1)\leq(n+\beta)(k_j-1)$$

which implies that

$$x_j\leq(n+\beta)(k_j-1)/\left(\sum_{i=1}^{n}k_i-n\right),\ j=1,2,\ldots,n. \tag{3.6}$$

On the other hand, summing both sides of (3.5) over j we obtain

$$\left(\sum_{j=1}^{n}k_j-n\right)(1+x_i)\geq(k_i-1)\sum_{j=1}^{n}x_j\geq(k_i-1)\alpha.$$

Hence,

$$x_i\geq-1+\alpha(k_i-1)/\left(\sum_j k_j-n\right),\ i=1,2,\ldots,n \tag{3.7}$$

Thus, from (3.6) and (3.7) it follows that the mode $\underset{\sim}{x}$ of the MGWD$(a;\underset{\sim}{k};\rho)$ satisfies the inequalities

$$-1+\alpha(k_i-1)/\left(\sum_j k_j-n\right)\leq x_i\leq(n+\beta)(k_i-1)/\left(\sum_j k_j-n\right),\ i=1,2,\ldots,n$$

where α,β are positive constants satisfying $\alpha\leq\sum_{i=1}^{n}x_i\leq\beta$.

Finally, using (3.1) it can be easily shown that the p.g.f. of the MGWD$(a;\underset{\sim}{k};\rho)$ satisfies the second order partial differential equations

$$t_i(1-t_i)\frac{\partial^2}{\partial t_i^2} G_{\underset{\sim}{X}}(\underset{\sim}{t})+(1-t_i)\sum_{j\neq i} t_j\frac{\partial^2}{\partial t_i \partial t_j} G_{\underset{\sim}{X}}(\underset{\sim}{t})+[a+\sum_{\ell=1}^{n} k_\ell+\rho-$$

$$(a+k_i+1)t_i]\ \frac{\partial}{\partial t_i} G_{\underset{\sim}{X}}(\underset{\sim}{t})-k_i\sum_{j\neq i} t_j\frac{\partial}{\partial t_j} G_{\underset{\sim}{X}}(\underset{\sim}{t})-ak_i G_{\underset{\sim}{X}}(\underset{\sim}{t})=0, i=1,2,\dots,n.$$

4. THE MGWD IN RELATION TO ACCIDENT THEORY

Let us now examine how the multivariate distribution defined in section 2 may arise in the context of accident theory.

Consider a population of individuals whose accident proneness over a period of observation is represented be the r.v. ν. Assume that the period of observation is split into n non-overlapping subperiods and let $\lambda_i|\nu$ represent the accident liability of the individuals for subperiod i $(i=1,2,\dots,n)$. Then, the numbers $X_1,X_2,\dots,X_n$ of accidents incurred by individuals of the same proneness (ν) and liabilities $(\lambda_i|\nu,\ i=1,2,\dots,n)$ over the n subperiods can reasonably by regarded as independent Poisson $(\lambda_i|\nu)$ r.v. 's with joint p.g.f.

$$G_{\underset{\sim}{X}|\underset{\sim}{\lambda},\nu}(\underset{\sim}{t})=e^{\lambda_1(t_1-1)+\dots+\lambda_n(t_n-1)}, \lambda_i>0,\ i=1,2,\dots,n \qquad (4.1)$$

Suppose that the liability parameters $\lambda_i|\nu$, $i=1,2,\dots,n$ vary independently among individuals of the same accident proneness ν and let their joint probability density function (p.d.f.) be

$$f(\underset{\sim}{\lambda}|\nu)=\prod_{i=1}^{n}\frac{\nu^{-k_i}}{\Gamma(k_i)}\, e^{-\frac{1}{\nu}\lambda_i}\,\lambda_i^{k_i-1}, k_i>0,\ i=1,2,\dots,n \qquad (4.2)$$

Then, the distribution of accidents for individuals with the same proneness is the multiple negative binomial with parameters $\underset{\sim}{k}=$ $(k_1,\dots,k_n)$ and $\underset{\sim}{p}=\underset{\sim}{1}\ \nu/(1+\nu)$, i.e.

$$G_{\underset{\sim}{X}|\nu}(\underset{\sim}{t})=\prod_{i=1}^{n}(1+\nu(1-t_i))^{-k_i}\ . \qquad (4.3)$$

Suppose further that the accident proneness parameter ν varies from individual to individual according to a beta distribution of the second kind(Pearson type VI) with p.d.f.

$$g(\nu)=\frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)}\nu^{a-1}(1+\nu)^{-(a+\rho)},a>0,\ \rho>0\ .$$

Then, the resulting accident distribution over the n subperiods of observation will have p.g.f. $G_{\underset{\sim}{X}}(\underset{\sim}{t})=E_\nu(G_{\underset{\sim}{X}|\nu}(\underset{\sim}{t}))=$

$$\frac{\rho_{(\sum k_i)}}{(a+\rho)_{(\sum k_i)}}F_D(a;k_1,\dots,k_n;a+\rho+\sum k_i;\underset{\sim}{t})\sim MGWD(a;\underset{\sim}{k};\rho). \qquad (4.4)$$

That is, the joint distribution of the numbers $X_1,X_2,\dots,X_n$ of accidents incurred by individuals with varying accident proneness and exposed to variable external risk of accident over n non-overlapping time periods is the MGWD$(a;\underset{\sim}{k};\rho)$.

Note that interchanging the forms of the distributions of the liability and proneness parameters leads to the same accident distribution. i.e. if

$$f(\underset{\sim}{\lambda}|\nu)=\frac{\Gamma(\rho+\sum k_i)}{\Gamma(\rho)\prod_i\Gamma(k_i)}\prod_{i=1}^{n}\left(\frac{\lambda_i}{\nu}\right)^{k_i-1}\left(1+\sum_{i=1}^{n}\frac{\lambda_i}{\nu}\right)^{-(\rho+\sum k_i)}\nu^{-n}$$

$$\rho,\lambda_i,k_i>0,\ i=1,2,\dots,n$$

and

$$g(\nu)=\frac{1}{\Gamma(a)}e^{-\nu}\nu^{a-1},a>0$$

the final accident distribution is again the MGWD$(a;\underset{\sim}{k};\rho)$.

At this point, it is worth mentioning that the accident model considered in this section allows for differences in the exposure to external risk from person to person within each period as well as for differences in the risk exposure of the same person between periods. Accident proneness is assumed to differ from person to person within each period but is regarded constant for a given person throughout the entire period of observation. Of course, the contribution of a person's accident proneness to the person's accident experience is not expected to be the same for each period of observation. In fact, as demonstrated in the sequel, using the MGWD one can assess the significance of the contribution that accident proneness, accident liability and chance have had in a given situation for each of the time intervals considered as well as for the whole time period.

Suppose that a period of observation is divided into n non-overlapping sub-periods and that the MGWD($a;\underset{\sim}{k};\rho$) describes satisfactorily the resulting multivariate accident distribution. Then, since the accident distribution for the entire period is the UGWD($a,\Sigma k_i;\rho$) the variance σ^2 of the total number of accidents in the overall period can be partitioned into three additive components due to accident proneness (σ_ν^2), accident liability (σ_λ^2) and chance (σ_R^2) .i.e.,

$$\sigma^2=\sigma_\lambda^2+\sigma_\nu^2+\sigma_R^2 \tag{4.5}$$

(see Irwin(1968)) where

$$\sigma_\lambda^2=\frac{a(a+1)\sum_{i=1}^{n}k_i}{(\rho-1)(\rho-2)},\sigma_\nu^2=\frac{a(a+\rho-1)(\sum_{i=1}^{n}k_i)^2}{(\rho-1)^2(\rho-2)},\sigma_R^2=\frac{a\sum_{i=1}^{n}k_i}{\rho-1} \tag{4.6}$$

Inspecting these formulae one can observe that, for n=1 they cannot provide distinguishable estimates for σ_λ^2 and σ_ν^2. As Irwin (1968)pointed out, this is the result of the fact that the UGWD ($a,\underset{\sim}{k};\rho$) is symmetrical in a and k(UGWD($a,k;\rho$)~UGWD($k,a;\rho$)) which implies that any method of parameter estimation will lead to two solutions for a and k. So, Irwin(1968) had to decide judging from extra information he had concerning the individuals under observation. However,Xekalaki (1984a) showed that if the observations can be rearranged in a bivariate form, one does not have to resort to personal judgement by using the BGWD(case n=2) since then the question of symmetricity does not arise. This is also the case with the multivariate version of the generalized Waring distribution. Relation (2.8) indicates that the MGWD ($a;\underset{\sim}{k};\rho$) is not symmetrical in a and k_i($i=1,2,\ldots,n$) and hence distinguishable estimates can be obtained for the liability and proneness components of the variance through relationships (4.6). By an argument analogous to that used by Xekalaki (1984a) for the bivariate case one can have a further breakdown of the variance components due to liability, proneness and chance into components corresponding to the subperiods of time considered. Table I summarizes the results.

TABLE I

Estimators of the components of the variance of the MGWD*

Component due to	Marginal variance of X_i $(i=1,2,\ldots,n)$	Variance of $X_1+X_2+\ldots+X_n$
Chance	$\frac{\hat{a}\hat{k}_i}{\hat{\rho}-1}$	$\frac{\hat{a}\sum_{j=1}^{n}\hat{k}_j}{\hat{\rho}-1}$
Proneness	$\frac{\hat{a}(\hat{a}+\hat{\rho}-1)\hat{k}_i^2}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$	$\frac{\hat{a}(\hat{a}+\hat{\rho}-1)(\sum_{j=1}^{n}\hat{k}_j)^2}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$
Liability	$\frac{\hat{a}(\hat{a}+1)\hat{k}_i}{(\hat{\rho}-1)(\hat{\rho}-2)}$	$\frac{\hat{a}(\hat{a}+1)\sum_{j=1}^{n}\hat{k}_j}{(\hat{\rho}-1)(\hat{\rho}-2)}$
Total	$\frac{\hat{a}(\hat{a}+\hat{\rho}-1)\hat{k}_i(\hat{k}_i+\hat{\rho}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$	$\frac{\hat{a}(\hat{a}+\hat{\rho}-1)\sum_{j=1}^{n}\hat{k}_j(\sum_{j=1}^{n}\hat{k}_j+\hat{\rho}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$

*$\hat{\vartheta}$ represents an estimator of a parameter ϑ.

Hence, if for a population of individuals one can have the joint frequency distribution of accidents over n time periods, then one is in a position to "estimate" the contribution that each of the three kinds of factors has had in each period as well as in the entire period of observation.

Note : If we let $\underset{\sim}{k}^*$ denote the vector that arises by considering a given permutation of the components $k_1,k_2,\ldots,k_n$ of the vector $\underset{\sim}{k}$, then MGWD$(a;\underset{\sim}{k};\rho)$~MGWD$(a;\underset{\sim}{k}^*;\rho)$. This implies that some ambiguity may arise in estimating the components of the n marginal variances. However, in a manner analogous to that of the bivariate case, one can observe that the random component of the variance coincides with the marginal mean corresponding to the same time period and hence eliminate this ambiguity by selecting $\hat{k}_i$ so that $\hat{\sigma}^2_{R_i}=\bar{X}_i$ (i refers to period i).

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