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# An Extension of the Damage Model

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## 1. Introduction

Let $X$ be an original observation subjected to a destructive process. Then, what is observed is the undamaged part of $X$, say $Y$. This is usually called the resulting random variable (r.v.). The destruction process (or the survival distribution) can be represented by the conditional distribution of $Y$ given $X$ $(Y \mid X)$. This model first considered by *Rao* [1963] is called a damage model.

In the simple case where the distribution of $Y \mid X$ is Binomial with parameters $n, p$ we have

$$G_Y(t) = G_X(q + pt), \; G_{Y|X=Y}(t) = \frac{G_X(pt)}{G_X(p)}$$

$$0 < p < 1, \; q = 1 - p$$

where $G_X(t)$, $G_Y(t)$, $G_{Y|X=Y}(t)$ are the probability generating functions (p.g.f.'s) of the original r.v., the resulting r.v. and the resulting r.v. when no damage has occurred.

In section 2 of this paper we consider the problem of obtaining the p.g.f. of the resulting distribution when the parameter $p$ of the Binomial survival is a r.v. with d.f. $F_2(p)$. Section 3 deals with the same problem when the parameter $\lambda$ of the original distribution is a r.v. with d.f. $F_1(\lambda)$, and the survival distribution is Binomial. Several known distributions are derived for various forms of $F_1(\lambda)$ and $F_2(p)$. In section 4 the relation between the p.g.f.'s of the resulting distribution in general and the resulting distribution when no damage has occurred is studied. Using this relation we obtain a characterization of the Poisson distribution which gives *Rao/Rubin's* [1964] result as a special case.

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## 2. The Damage Model with the Survival Distribution Mixed Binomial

Let us consider the more general form of the damage model in which the parameter $p$ of the Binomial survival distribution is not a fixed number. Instead, suppose that $p$ is a r.v. with d.f. $F_2(p)$. In this case

$$P(Y=r \mid X=n)=\int_0^1 \binom{n}{r} p^r q^{n-r} dF_2(p), \qquad r=0,1,\ldots,n \quad n=0,1,\ldots \tag{2.1}$$

and hence

$$G_Y(t)=\int_0^1 G_X(q+pt)\, dF_2(p), \qquad q=1-p \tag{2.2}$$

and

$$G_{y\mid x=y}(t)=\frac{\int_0^1 G_X(pt)\, dF_2(p)}{\int_0^1 G_X(p)\, dF_2(p)}. \tag{2.3}$$

If we now assume that $X$ is Poisson with parameter $\lambda$, (2.2) and (2.3) respectively become

$$G_Y(t)=M_p(\lambda(t-1)) \tag{2.4}$$

$$G_{Y\mid X=Y}(t)=\frac{M_p(\lambda t)}{M_p(\lambda)} \tag{2.5}$$

where $M_Z(t)$ denotes the moment generating function of the r.v. $Z$.

Different forms of the mixing distribution $F_2(p)$ give rise to various distributions representing the resulting distribution when the original distribution is Poisson and the survival distribution is Binomial $(n, p) \underset{p}{\wedge} F_2(p)$. Here are some examples

a) $(Y \mid X) \sim$ Binomial $(n, p) \underset{p}{\wedge}$ Beta $(\alpha, \beta)$ (Negative Hypergeometric).

$$G_Y(t)={}_1F_1(\alpha; \alpha+\beta; \lambda(t-1)), \qquad \alpha>0, \beta>0 \tag{2.6}$$

where ${}_1F_1(a; b; t)$ is the confluent hypergeometric function given by

$${}_1F_1(a; b; t)=\frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 e^{tu} u^{a-1}(1-u)^{b-a-1} du. \tag{2.7}$$

The distribution with p.g.f (2.6) was first examined by *Gurland* [*1958*].

b) $(Y \mid X) \sim$ Binomial $(n, p) \underset{p}{\wedge}$ Gamma $(\alpha, \beta)$, truncated to the right at the point 1.

$$G_Y(t) = c\ {}_1F_1\left(\alpha; \alpha + 1; \lambda(t-1) - \frac{1}{\beta}\right), \quad \alpha > 0, \beta > 0 \tag{2.8}$$

where $c$ is the normalizing constant.
The distribution with p.g.f. (2.8) has been studied by *Kemp* [1968] as a limited risk Compound Poisson Process.

## 3. The Damage Model with the Original Distribution Mixed Poisson

Let us now turn to the situation where the parameter $\lambda$ of the original distribution is a r.v. with d.f. $F_1(\lambda)$ $(\lambda > 0)$. Denote by $G_{X|\lambda}(t)$ the p.g.f. of the conditional distribution of $X \mid \lambda$, i.e. of $X$ for given $\lambda$. Then on the assumption that the conditional distribution $Y \mid X$ (i.e. the survival distribution) is Binomial with parameters $n, p$ we have

$$G_Y^*(t) = \int_0^\infty G_{X|\lambda}(q + pt)\, dF_1(\lambda) \tag{3.1}$$

and

$$G_{Y|X=Y}^*(t) = \frac{\int_0^\infty G_{X|\lambda}(pt)\, dF_1(\lambda)}{\int_0^\infty G_{X|\lambda}(p)\, dF_1(\lambda)}. \tag{3.2}$$

(We use the notation $G_Y^*(t)$ to indicate that this time, the mixing is taking place in the original distribution.) If, in particular $X \mid \lambda$ is Poisson $(\lambda)$ (3.1), (3.2) become respectively

$$G_Y^*(t) = M_\lambda(p(t-1)) \tag{3.3}$$

and

$$G_{Y|X=Y}^*(t) = \frac{M_\lambda(pt-1)}{M_\lambda(p-1)}. \tag{3.4}$$

By making use of (3.3) one can obtain the form of the p.g.f of the resulting distribution for different forms of $F_1(\lambda)$. Here are two interesting examples.

a) $X \sim \text{Poisson}(\lambda) \underset{\lambda}{\wedge} \text{Beta}(\alpha, \beta)$.

$$G_Y^*(t) = {}_1F_1(\alpha; \alpha + \beta; p(t-1)) \tag{3.5}$$

b) $X \sim \text{Poisson}(\lambda) \underset{\lambda}{\wedge} \text{Gamma}(\alpha, \beta)$ (Negative Binomial)

$$G_Y^*(t) = \left(\frac{1}{1 + p\beta}\right)^\alpha \left(1 - \frac{p\beta t}{1 + p\beta}\right)^{-\alpha}, \quad \alpha, \beta > 0. \tag{3.6}$$

Clearly (3.6) is again the p.g.f. of a Negative Binomial distribution.

*Remark 1.* It is obvious that one can obtain the p.g.f of the resulting distribution when no damage has occurred for the examples given in sections 2 and 3 by using formulae (2.5) and (3.4) respectively.

*Remark 2.* Results similar to those obtained in sections 2 and 3 can be derived for discrete forms of $F_1(\lambda)$ and $F_2(p)$.

## 4. Relations Between $G_Y(t)$ and $G_{Y|X=Y}(t)$ in the Extended Damage Model

As *Rao* [1963] pointed out, in the simple damage model where the original distribution is Poisson and the survival distribution is Binomial the following relation holds.

$$P(Y=r)=P(Y=r\mid X=Y),\quad r=0,1,\ldots$$

which, in terms of p.g.f.'s can be written as

$$G_Y(t)=G_{Y|X=Y}(t). \tag{4.1}$$

(This condition has come to be known as the Rao-Rubin condition.)

For our extended form of the damage model the following two theorems can be established.

*Theorem 1.* If $X$ is Poisson with parameter $\lambda$ and $Y\mid X$ is Mixed Binomial then

$$G_Y(t+1)=c\,G_{Y|X=Y}(t) \tag{4.2}$$

where $c^{-1}=G_{Y|X=Y}(0)$ is a constant.

*Theorem 2.* If $X$ is mixed Poisson and $Y\mid X$ is Binomial then

$$G_Y^*(t)=c^*G_{Y|X=Y}^*\left(t+\frac{q}{p}\right) \tag{4.3}$$

where $(c^*)^{-1}=G_{Y|X=Y}^*(1/p)$ is a constant.

The proof of these theorems can be easily obtained using relations (2.4), (2.5) for theorem 1 and (3.3), (3.4) for theorem 2.

*Rao/Rubin* [1964] used a Binomial survival distribution to show that (4.1) holds if and only if (iff) the distribution of $X$ is Poisson.

In the sequel we extend the Rao-Rubin characterization of the Poisson distribution to the case where the survival distribution is mixed Binomial.

*Theorem 3.* Let us consider the random vector $(X, Y)$ with non-negative real components such that $P(X=n)=P_n$, $n=0,1,\ldots$, with $P_0\neq 0$ and

$$Y\mid X\sim \text{Binomial}(n,p)\underset{p}{\wedge}F_2(p),\, p\in(0,1),\, r=0,1,\ldots,n. \tag{4.4}$$

Then condition (4.2) holds iff $P_n$ is Poisson.

*Proof*. Necessity follows by theorem 1. To prove sufficiency we first observe that (2.2) can be written as

$$G_Y(t+1) = \int_0^1 G_X(pt+1)\, dF_2(p). \tag{4.5}$$

We also have

$$G_{Y|X=Y}(t) = \frac{\int_0^1 G_X(pt)\, dF_2(p)}{\int_0^1 G_X(p)\, dF_2(p)}. \tag{4.6}$$

Substituting (4.5), (4.6) in (4.2) gives

$$\int_0^1 G_X(pt+1)\, dF_2(p) = c_0 \int_0^1 G_X(pt)\, dF_2(p), \; (c_0^{-1} = \int_0^1 G_X(p)\, dF_2(p)).$$

Hence

$$\int_0^1 \sum_{n=0}^{\infty} P_n (pt+1)^n\, dF_2(p) = c_0 \int_0^1 \sum_{n=0}^{\infty} P_n (pt)^n dF_2(p) \Rightarrow$$

$$\sum_{n=0}^{\infty} \{\sum_{r=0}^{n} P_n \int_0^1 \binom{n}{r} p^r\, dF_2(p)\, t^r\} = c_0 \sum_{n=0}^{\infty} P_n \{\int_0^1 p^n\, dF_2(p)\}\, t^n \Rightarrow$$

$$\sum_{r=0}^{\infty} \{\sum_{n=r}^{\infty} P_n \int_0^1 \binom{n}{r} p^r\, dF_2(p)\}\, t^r = c_0 \sum_{r=0}^{\infty} P_r \{\int_0^1 p^r\, dF_2(p)\}\, t^r.$$

Consequently

$$\sum_{n=r}^{\infty} \binom{n}{r} P_n \int_0^1 p^r\, dF_2(p) = c_0 P_r \int_0^1 p^r\, dF_2(p) \text{ i.e.}$$

$$\sum_{n=r}^{\infty} P_n \binom{n}{r} = c_0 P_r. \tag{4.7}$$

Taking the p.g.f.'s for both sides of (4.7) we find that

$$G_X(t+1) = c_0 G_X(t) \qquad 0 \leqslant t \leqslant 1. \tag{4.8}$$

But *Shanbhag* [1974] using an elementary approach showed that the unique solution of the functional equation

$$G(q+pt) = \frac{G(pt)}{G(p)} \qquad |t| \leqslant 1 \tag{4.9}$$

where $G(t)$ is a p.g.f., is

$$G(t) = e^{\lambda(t-1)} \qquad \text{for some } \lambda > 0.$$

Since our functional equation (4.8) is a particular case of (4.9) the result is established.

*Remark 3.* Clearly if the survival distribution is Binomial, i.e. if $F_2(p)$ is degenerate, then condition (4.2) reduces to (4.1) and hence theorem 3 reduces to the Rao-Rubin characterization of the Poisson distribution.

## References

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